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PHILOSOPHICAL
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ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

TIONS.

*Overstrain.**

1 *Exhibition Research*

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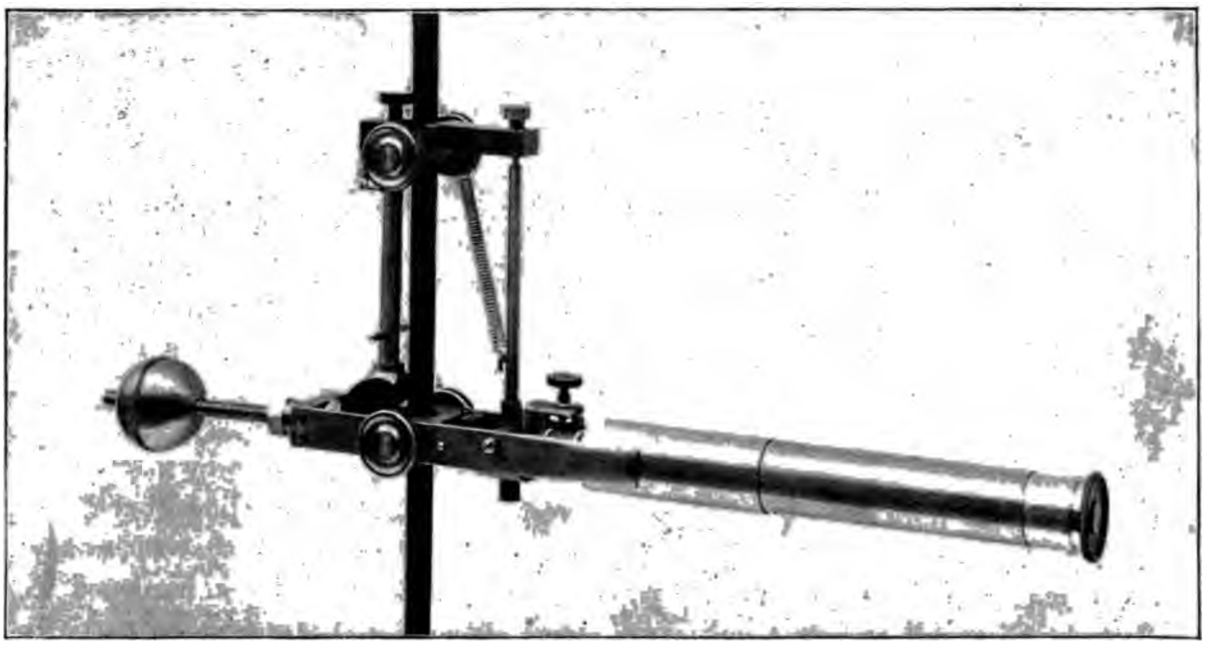
described in a paper by the present author, "On the Recovery of Iron from Overstrain," Phil. Trans., A, vol. 193, 1899.

slightly modified from that of the larger extensometer fully described by Professor **Ewing** in a paper " On Measurements of Small Strains in the Testing of Materials and Structures."*

For the purposes of tempering and annealing a gas furnace was employed, 2 feet in length. This furnace (manufactured by **Fletcher, Russell**, and Co., Warrington) is heated by means of a series of inclined bunsens entirely detached from the fire-clay portion of the furnace, so that regulation can be effected not only by altering the gas supply, but also by moving the bunsens nearer to or further from the orifices into which they play. The specimens were protected from direct contact with the flame by enclosing them in a thick porcelain tube. The temperature inside this tube was measured by means of a **Callendar's** direct-reading platinum-resistance pyrometer.† To ensure that the temperature recorded by the pyrometer was as accurately as possible that assumed by the specimen, readings were taken for both ends of the furnace, the specimen being moved from one end to the other to allow of the insertion of the pyrometer tube. In this reversal of positions the pyrometer tube in passing through the air was slightly cooled, so that the temperatures recorded immediaely after a change of positions were somewhat low. The following series of pyrometer readings is given by way of illustration :—



4-inch Extensometer.

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* 'Roy. Soc. Proc.,' vol. 58, April, 1895.

† "On the Construction of Platinum Thermometers," 'Phil. Mag.,' July, 1891 ; "On Platinum Thermometry," 'Phil. Mag.,' February, 1899. The instrument referred to above was made by the Cambridge Scientific Instrument Company, Limited.

Position of the Pyrometer.	Times in minutes.	Pyrometer readings.	
Pyrometer on the right	0	562° C.	—
„ „ „	2	582	—
„ „ „	3	592	—
„ „ „	4	599	Bunsens slightly removed.
Pyrometer on the left	7	574° C.	—
„ „ „	8	580	—
„ „ „	10	584	Bunsens slightly closer on the left.
„ „ „	12	595	—
Pyrometer on the right	13	597° C.	—
„ „ „	14	605	Bunsens slightly removed.
„ „ „	15	605	—
Pyrometer on the left	16	598° C.	—
„ „ „	17	607	Gas supply turned off.
„ „ „	18	604	—

From the above series of readings the specimen in the furnace would be said to have been heated to 605° C.

Preliminary Examination of the Material in the State as supplied.

The elastic properties of the materials to be employed were examined not only in the usual fashion by breaking a specimen in the testing machine, and recording the breaking load and the ultimate elongation, but also in a manner which has already been described in a paper by the present author on the recovery of iron from overstrain.* Diagram No. 1 of the present paper illustrates this method of examination, the material examined being a rod of steel rather under half an inch in diameter.

A specimen about a foot long was cut from this rod, and was turned down in the centre for a length of 5 inches to a diameter just over $\frac{3}{8}$ths of an inch. Sufficient length was left unturned at each end to enable the specimen to be securely gripped in the jaws of the testing machine. The diameter of the turned portion of the specimen was then accurately measured; a 4-inch length was marked off by means of a marking instrument of Professor EWING's design; the extensometer was attached to this 4-inch length, the specimen was put into the testing machine, and load was applied. Extensometer readings were taken after the addition of every ton per sq. inch of load until higher stresses were obtained, and then readings were taken after every half ton, and while each half ton was being slowly added the eye was kept at the microscope of the extensometer in order to detect as accurately as possible the load at which large plastic yielding commenced. Curve No. 1, Diagram 1 (p. 6), shows that such yielding began at the high load of 38 tons per sq. inch, a well-defined yield-point being obtained at that stress.

---

* The diagram in 'Phil. Trans.,' A, vol. 193, p. 31, 1899, shows this method of examining iron and steel.

In plotting the curves of the present example and of all other diagrams in this paper, the method of "shearing back" the curves, which was adopted on Professor EWING'S suggestion in the author's previous paper on Recovery from Overstrain, has again been employed.* This method consists in diminishing all extensions before plotting by an amount proportional to the loads producing them. By this means curves which would otherwise stretch far across the paper in the direction of extensions are brought more nearly into an upright position, so that a large scale for the measurement of extensions may be retained without an inconvenient amount of space being occupied. All the curves in this paper have been "sheared back" by the same amount; $\frac{110}{100000}$ths of an inch have been deducted from the extension of a 4-inch length for every 4 tons of stress. For example, the extensometer readings for Curve 1, Diagram 1, corresponding to the stresses of 4, 8, and 12 tons per sq. inch, were 120, 240, and 362 respectively. The numbers actually plotted were 10, 20, and 32.

It should be remarked that in all the diagrams of this paper, the origin for the measurement of extensions has been displaced by an arbitrary amount between each curve and the next in the series. This was merely to keep the various curves distinct, and to facilitate comparison.

The yielding which is shown by Curve No. 1, Diagram 1, to have begun at the stress of 38 tons per sq. inch soon became very rapid under this stress. The load was therefore slightly reduced† until a rate of extension convenient for observation was obtained, and it was then found that the stretching continued at a more or less constant speed (the lever of the testing machine being kept floating) for a considerable time, and then abruptly stopped; or, to be more accurate, rapid extension then abruptly changed into very slow creeping. The replacement of the full load of 38 tons was found to produce comparatively little further extension.

This yielding which takes place at the yield-point does not occur simultaneously throughout the length of the specimen. The material at some point in the bar yields and the yielding is observed to spread. The yielding at any point increases the intensity of the stress at neighbouring points of the bar, so that adjacent portions of material yield, and the action is transmitted piecemeal throughout the whole length of the specimen. With unturned material the progress of this yielding can be observed in the skin of oxide cracking and springing off as the action travels along the bar.* It was usually found with unturned specimens that the yielding started in the grips of the testing machine, and spread upwards or downwards as the case might be. Sometimes yielding was observed to begin at both ends and to travel towards the centre of the bar.

---

* In 'Phil. Trans.,' A, vol. 193, p. 12, 1899, a full account of "shearing back" is to be found.

† With Lowmoor iron it was found that the load at the yield-point could sometimes be reduced by almost two tons/in^2 without causing the yielding at the yield-point to cease.

The amount of stretching which occurred at the yield-point shown in Curve 1, Diagram 1 was very considerable, the specimen having been given a permanent extension of 0·13 of an inch on a 4-inch length. The horizontal part of Curve No. 1 would thus require to be continued for over 8 feet in order to represent this yielding to the scale employed in the diagram.

After Curve No. 1 had been determined and the load removed, the reduced diameter of the specimen was measured and the new area of section was calculated. A 4-inch length was again marked off by means of the marking instrument, the extensometer was readjusted, and the load was reapplied in tons per sq. inch of actual section.† Extensometer readings were taken after the addition of every 4 tons per sq. inch, and Curve No. 2 was plotted from these readings. This curve shows the semi-plastic nature of the material immediately after overstrain.

It was noticed in previous experiments on the recovery from overstrain, that different steels, after tensile overstrain, recovered their elasticity at very different rates; so it was decided to examine the rate at which the material in question recovered from the semi-plastic condition illustrated by Curve 2, Diagram 1. The specimen was simply allowed to rest and was tested at intervals. Curves Nos. 3 and 4 illustrate the progress made towards recovery of elasticity, one and three-quarter days, and two weeks after overstrain, respectively.

In order to effect perfect recovery of elasticity, the specimen was put in the gas furnace described above and was heated until the pyrometer recorded about 200° C. It is probable that a few minutes at the temperture of boiling water would have been nearly though perhaps not quite as effective in restoring the lost elasticity.‡ After cooling the specimen was tested by reloading and carefully increasing the load above its previous maximum amount. A well-defined yield-point was obtained (as is shown by Curve No. 5, Diagram 1) at the stress of 49 tons per sq. inch, the yield-point having been raised by the large step of 11 tons per sq. inch. The amount by which the material yielded at this second yield-point was, to the nearest $\frac{1}{100}$th of an inch, the same as that by which it yielded at the primary yield-point, namely, by 0·13 of an inch on the 4-inch length.

The material after this second overstrain was once more in the semi-plastic state, so to effect recovery of elasticity the specimen was, as before, heated to about 200° C. and slowly cooled. It was known as the result of earlier experiments that the yield-point had been raised by this process through a second step of 11 tons,*

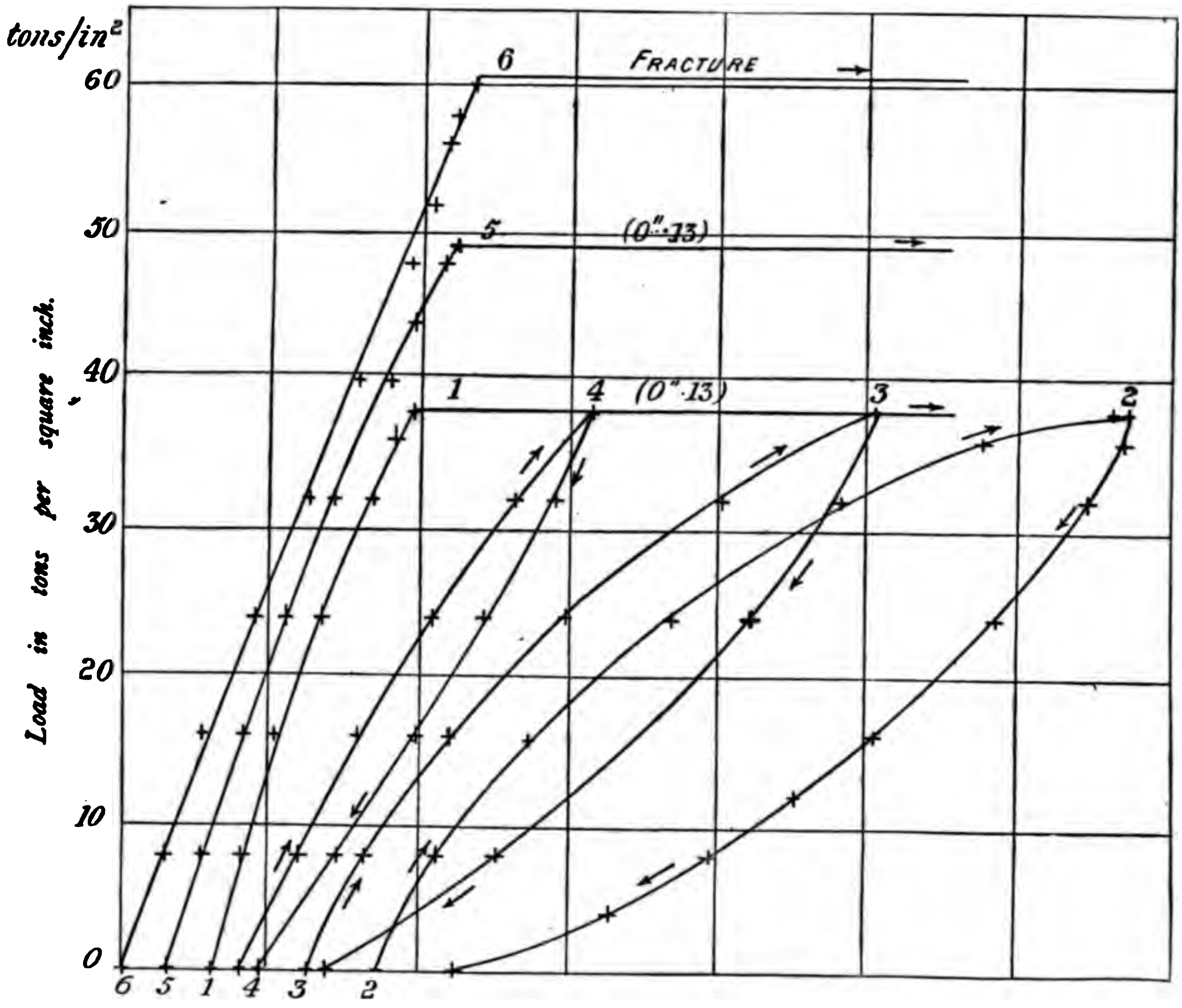
---

* Professor EWING, in the paper referred to above, "On Measurements of Small Strains," &c., has already observed that yielding may begin near one end of the specimen and gradually spread throughout the length, 'Roy. Soc. Proc.,' vol. 58, p. 135, April, 1895.

† This procedure was adopted throughout the course of the work. Whenever a yield-point had been passed, the specimen was re-measured and the loading altered to suit the new area of section.

‡ "On the Recovery of Iron from Overstrain," 'Phil. Trans.,' A, vol. 193, p. 22, 1899.

Diagram No. 1. (Steel as supplied.)



Extensions—diminished as explained on page 4.

Scale:—1 Unit = $\frac{1}{1000}$th of an inch. 0 1 2

Curve No. 1—Primary test.
" " 2—Shortly after No. 1.
" " 3—1¾ days " " 1.

Curve No. 4—2 weeks after No. 1.
" " 5—After heating to 200° C.
" " 6— " " 200° C.

so that the specimen should not yield until a stress of 60 tons per sq. inch had been applied. Curve No. 6, Diagram 1 shows that the specimen bore the stress of 60 tons, but that with 60½ tons per sq. inch a yield-point and fracture occurred.

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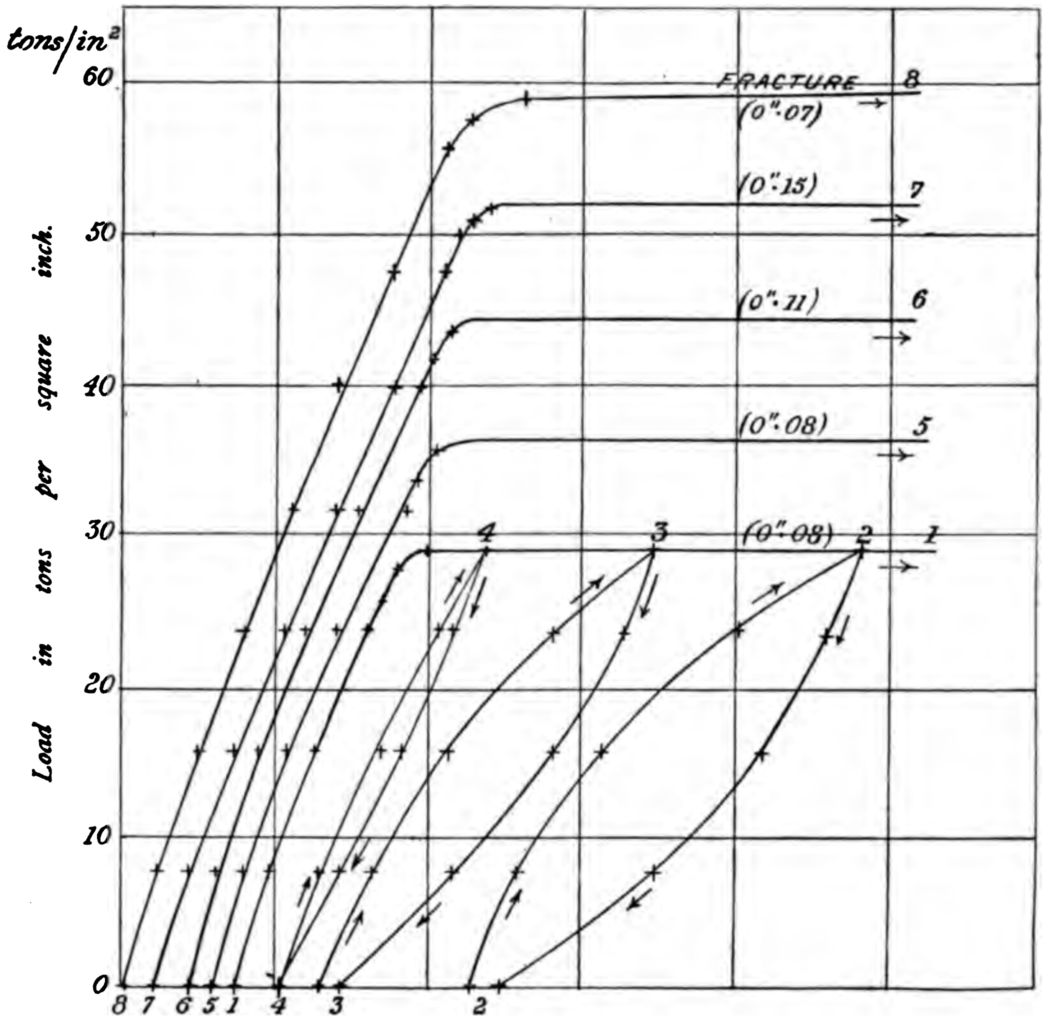
* "On the Recovery of Iron from Overstrain," 'Phil. Trans.,' A, vol. 193, p. 34, 1899.

Diagram No. 2. (Annealed Steel.)



Extensions—diminished as explained on page 4.

Scale:—1 Unit = $\frac{1}{1000}$th of an inch. 0 1 2

Curve No. 1—Primary test.
„ „ 2—Shortly after No. 1.
„ „ 3—1¾ days „ „ 1.
„ „ 4—2 weeks „ „ 1.

Curve No. 5—After heating to about 300° C.
„ „ 6— „ „ „ 300° C.
„ „ 7— „ „ „ 300° C.
„ „ 8— „ „ „ 300° C.

The Effect of Annealing.

It was found that the process of annealing altered in an interesting fashion the elastic properties of the material whose virgin properties are illustrated in Diagram No. 1. The primary yield-point was found to be considerably lowered by annealing, and the step by which the yield-point was raised in consequence of overstrain and recovery from overstrain was decidedly reduced. Diagram No. 2 gives the history of a specimen of this steel, which was first of all annealed by heating for a few minutes to about 750° C. with a slow cooling, and was then subjected to series of operations exactly similar to that described for Diagram No. 1.

A comparison of Diagrams No. 1 and No. 2, or an examination of the following tables of extensometer readings, clearly shows the effect produced on the elastic properties of the material by the process of annealing. Only a few of the readings taken to obtain the various curves of the two diagrams will be tabulated for the sake of comparison; and it should be mentioned that the curves were in many cases obtained from second loadings of the material at the varies stages of the experiments.

Readings for Diagram No. 1. (Steel as supplied.)

Load in tons/sq. inch.	Extensometer readings. (Unit = $\frac{1}{100000}$th of an inch.)					
	Curve 1.	Curve 2, zero time.	Curve 3, 1¾ days.	Curve 4, 2 weeks.	Curve 5, 200° C.	Curve 6, 200° C.
0	0	0	0	0	0	0
8	240	262	260	260	242	246
16	484	548	533	519	490	494
24	734	856	832	789	738	746
32	986	1192	1154	1061	990	1004
36	1112	1396	—	—	—	—
38	Yield-point	1540	1426	1281	—	—
40	—	—	—	—	1248	1256
44	—	—	—	—	1374	1386
48	—	—	—	—	1506	1513
49	—	—	—	—	Yield-point	
52	—	—	—	—	—	1638
56	—	—	—	—	—	1758
60	—	—	—	—	—	1884
60½	—	—	—	—	—	Fracture

---

* Curves 5 and 6 relate to tests made after exposing the material to a temperature of 200° C. for a few minutes.

Readings for Diagram No. 2. (Annealed Steel.)

Load in tons/sq. inch.	Extensometer readings. (Unit = $\frac{1}{1000000}$th of an inch.)							
	Curve 1.	Curve 2, zero time.	Curve 3, 1¾ days.	Curve 4, 2 weeks.	Curve 5, 300° C.	Curve 6, 300° C.	Curve 7, 300° C.	Curve 8, 300° C.
0	0	0	0	0	0	0	0	0
4	122	—	—	—	120	120	122	120
8	248	254	258	251	241	241	249	241
16	498	532	528	510	491	491	498	490
24	751	838	820	770	746	742	751	740
29	932 and yield-point	1060	1022	935				
32	—	—	—	—	1010	992	1010	991
36	—	—	—	—	1146	1122	1140	
36½	—	—	—	—	1250 and yield-point			
40	—	—	—	—	—	1256	1264	1242
44	—	—	—	—	—	1387	1391	
44½	—	—	—	—	—	1406 and yield-point		
48	—	—	—	—	—	—	1520	1499
50	—	—	—	—	—	—	1581	
52	—	—	—	—	—	—	1661 and yield-point	1627
56	—	—	—	—	—	—	—	1756
58	—	—	—	—	—	—	—	1828
59	—	—	—	—	—	—	—	1880
59½	—	—	—	—	—	—	—	Fracture

Curve No. 1 of Diagram 2 shows that annealing has had the effect of lowering the yield-point of the material by about 9 tons per sq. inch. With the virgin material the primary yield-point occurred at about 38 tons per sq. inch, with the annealed material at 29 tons per sq. inch. The stretching which occurred at the yield-point was also less in the case of the annealed material, the permanent extensions in the two cases being respectively 0·13 and 0·08 of an inch on the 4-inch lengths.

A comparison of Curves 2, 3, and 4 of Diagram No. 1, with Curves 2, 3, and 4 of Diagram No. 2, shows that immediately after overstrain annealed material exhibits rather less hysteresis than the same metal not previously annealed; and that recovery from the semi-plastic condition induced by overstrain takes place rather more rapidly in the case of the material which has been first of all annealed.

Curve No. 5 and the remaining three curves of Diagram No. 2 show that with an annealed specimen the step by which the yield-point is raised in consequence of tensile overstrain and recovery from overstrain is about 7½ tons per sq. inch, and that *four* such steps can be obtained before fracture occurs at about 59½ tons per sq. inch. With the material in the condition as supplied, fracture occurred at 60½ tons per sq. inch after *two* steps of about 11 tons.

It may be of interest to refer to the amount of stretching which occurred at the various yield-points shown in Diagram No. 2. At the first yield-point the permanent set was found to be 0·08 of an inch on the 4-inch length, at the second it was again 0·08 of an inch, at the third it was 0·11, at the fourth 0·15, and at the fifth fracture occurred and only local yielding of about 0·07 of an inch was obtained. Although the extensions at the third and fourth yield-points were thus greater than those at the first and second, it is probable that theoretically there need have been no such difference. Had it been possible to remove the load from each little portion of the specimen as soon as the yielding which occurs at a yield-point had spread throughout that portion of material, then probably the yielding at the third and fourth yield-points would not have been different from that at the first two. The extra extension at the higher yield-points was in all likelihood due to creeping, which continued after the break-down which occurs at a yield-point had taken place. In fact the elongation obtained at the fourth yield-point shown in Diagram No. 2 would have been greater still had not the load been removed shortly after the large yielding action had spread throughout the length of the specimen. Had this not been done, creeping would have continued, and probably fracture would have supervened, for previous experiments had shown that there was considerable danger of fracture occurring when a yield-point was passed at a high stress.*

The total elongation of the specimen whose history is given in Diagram 2 was thus 0·49 of an inch, or rather over 12 per cent. on a 4-inch length. The breaking load was 59½ tons per sq. inch. Another specimen of this material which was annealed and then broken by the testing machine in the usual fashion, that is, without allowing intermediate recoveries of elasticity to take place, gave an ultimate strength of slightly over 44 tons per sq. inch, with an elongation of about 26 per cent. on a 4-inch length.

Comparison of Two Materials.

The following comparison of the material whose elastic properties have just been described, with that employed previously in the work on recovery from overstrain, which has been referred to more than once already, may be of interest. The chemical analyses of these two materials were kindly supplied by Messrs. EDGAR ALLEN and Co., Limited, Sheffield. They are as follows:—

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* "On the Recovery of Iron from Overstrain," 'Phil. Trans.,' A, vol. 193, p. 35, 1899.

	Steel examined in the present paper (½-inch rod).	Steel used previously (1-inch rod).
Carbon	0·35	0·430
Silicon	0·102	0·112
Sulphur	0·063	0·010
Phosphorus	0·034	0·016
Maganese	1·16	0·450
Iron (by difference) ...	98·291	98·982
	100·000	100·000

The elastic properties of the two materials are compared in the table given below. In each case the results of an ordinary tensile test are given first, and then the data obtained by testing the material in the manner illustrated by Diagram 1 or Diagram 2 of this paper. In the second last column of the table there are tabulated the number of times the yield-point was raised in consequence of overstrain and recovery from overstrain when the material was tested in the manner just referred to, and in the last column the amount (in tons per sq. inch) by which the yield-point was raised each time. By way of illustration, in Diagram 1 the yield-point is shown to have been raised twice by a step of about 11 tons per sq. inch, while in Diagram 2 it is shown to have been raised four times by a step of 7½ tons per sq. inch. In an ordinary tensile test of course the specimen is not strained by steps.

Material.	Diameter of a turned specimen.	Yield-point.	Breaking stress.	Ultimate extension.	No. of steps.	"Step."
½-in. steel rod, as supplied	0·40 of an in.	37½ tons/sq. in.	47 tons/sq. in.*	23 per cent. on 4 ins.*	—	—
	0·40 " "	37½ " " "	60½ " " "	8 or 9 " " "	2	11 tons/sq. in.
½-in. steel rod, annealed	0·44 " "	29 " " "	44 " " "	26 " " "	—	—
	0·40 " "	29 " " "	59½ " " "	12 " " "	4	7½ tons/sq. in.
1-in. steel rod, as supplied	0·80 " "	23 " " "	36½ " " "	23 per cent. on 8 ins.	—	—
	0·79 " "	23 " " "	45½ " " "	10 " " "	4	5½ tons/sq. in.

From this table it will be seen that the 1-inch steel rod as supplied resembled in elastic properties the ½-inch rod in the annealed state, and not in the condition as supplied by the makers.

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* These two figures were not obtained by experiment, but were estimated from results obtained with material very similar to the above, but containing more silicon (0·6 per cent.) and ⅜ inch in diameter. This thinner material gave when in the condition as supplied a yield-point at 36½ tons/sq. inch and broke under an ordinary tensile test at 43½ tons with an elongation of 16 per cent. on 4 inches. After annealing, the yield-point occurred at 25 tons/sq. inch and fracture at 39½ tons/sq. inch, the elongation being 19½ per cent. on a 4-inch length.

Microscopic Examination of Steel.

The two steels whose elastic properties have just been compared, and whose chemical analyses are given above, were also examined by means of the microscope. Three methods of examination were adopted, the first being the ordinary method by etching a polished surface with dilute nitric acid.

A smooth surface was prepared by means of commercial emery paper, the final polishing being done by wet rouge contained on a piece of chamois leather stretched over a rotating disc. All or nearly all the fine scratches left on the surface of the steel by the emery paper having been removed by the rouge, the surface was washed and dried, and then etched with dilute nitric acid (0·1 per cent. strength).

Figs. A and B* (Plate 1) show, under a magnification of 150 diameters and with vertical illumination, the appearance (after polishing and etching in this manner) of transverse sections of the ½-inch steel rod used in the experiments described in this paper. Fig. A shows the structure of this steel when in the condition as supplied by the makers, that is when in the condition having the elastic properties illustrated by Diagram 1. Fig. B shows the structure of the same steel after annealing by heating for a few minutes to 750° C., that is, fig. B shows the structure of the steel after it had been brought by annealing into the condition having the elastic properties illustrated by Diagram No. 2 of this paper.

The photographs from which figs. A and B have been reproduced were taken three or four weeks after the specimens had been prepared and etched, so that the surfaces had become slightly tarnished. The tarnish, however, seemed only to emphasise the distinction between the two constituents, the "ferrite" and the "pearlite."

A comparison of figs. A and B shows that by annealing at 750° C. the structure of the ½-inch steel rod had been considerably altered. After annealing the steel was much coarser grained than when in the somewhat hardened condition as supplied by the makers. This change produced by annealing on the dimensions of the micro-structure of steel has been often observed before.† Figs. A and B are given here for the sake of comparison with the change in elastic properties illustrated by Diagrams 1 and 2 of this paper. It had been thought that this change in elastic properties could be entirely accounted for on the supposition that the bars left the rolling mills at a comparatively low temperature (say a dull red heat), and had so become hardened by a species of overstrain. The microscopic examination illustrated by figs. A and B seems, however, to indicate that the change in the properties of the material has to be attributed more to a thermal cause. It is unlikely that the structure of the material could be changed back from the annealed condition (fig. B) to the hardened condition (fig. A) by purely mechanical means. Mechanical hardening is accompanied by distortion of the grains, not by change in their dimensions.

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* Reproduced from photographs. The author is indebted to Mr. ROSENHAIN, of St. John's College, for photographing the micro-sections prepared for the paper, and also for information as to the methods of polishing and etching.

† For example, by OSMOND, "Méthode générale pour l'Analyse micrographique des Aciers au Carbone," 'Bulletin de la Soc. d'Encouragement,' Mai, 1895; by ARNOLD, "The Influence of Carbon on Iron," 'Proc. Inst. Civ. Eng.,' December, 1895; by STEAD, "The Crystalline Structure of Iron and Steel," 'Journ. of Iron and Steel Institute,' 1898.

The second method of microscopic examination which was adopted may be classified as what OSMOND calls "un polissage-attaque." The process was discovered accidentally, and consists simply in rubbing a surface of steel polished with wet rouge in the manner described above, with ordinary moistened cocoa. The cocoa stains the pearlite areas of the steel, which are thrown into relief by the polishing, the effect being probably very analogous to that produced by the infusion of liquorice root, which OSMOND and others usually employ. Fig. C (Plate 1) shows, under a magnification of 150 diameters and with vertical illumination, the structure of the 1-inch steel rod whose properties are given in the preceding section of this paper; the surface examined was polished with emery and rouge in the ordinary manner, and then rubbed for a little while on a piece of chamois leather which had been moistened and covered with VAN HOUTEN'S cocoa. The surface was also examined under a magnification of about 3000 diameters, and the beautiful laminated structure of the pearlite areas was thus very clearly shown. One series of the laminæ—the Sorbite series—were stained a brown colour, the other series remained bright. It was found that benzene removed the staining produced by the cocoa.

The specimens from which figs. A and B had been obtained were repolished to remove the effect of the etching by nitric acid, and were then stained with cocoa in the manner just described, and examined under magnifications of 150 and of 3000 diameters. The structures shown were much better defined than when produced by ordinary etching, and under the high magnification the laminated nature of the pearlite areas was clearly visible. The laminæ were more distinct in the annealed, that is in the coarser grained specimen.

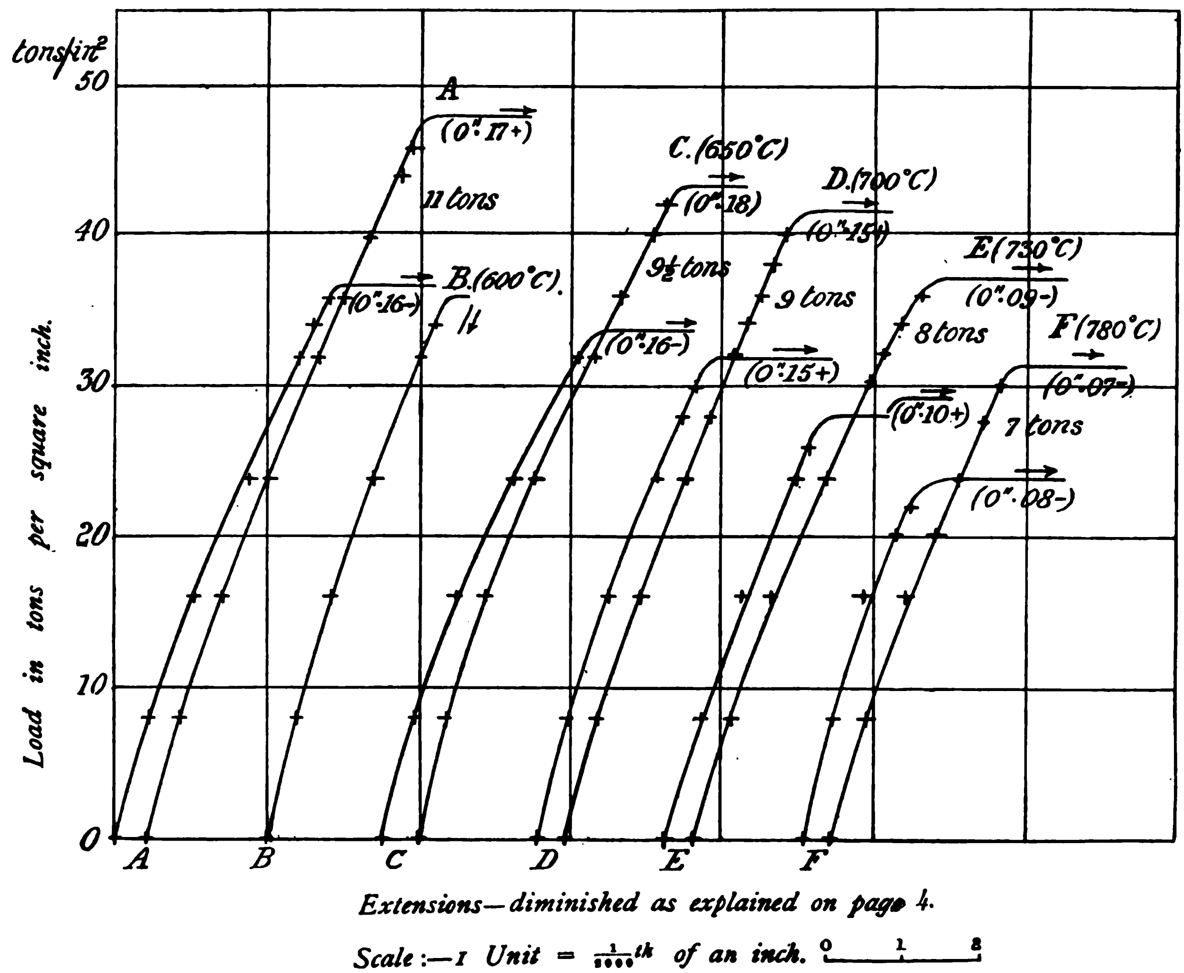
The third method of microscopic examination which was adopted consisted simply in polishing the steel in the ordinary way with emery and wet rouge, and then drying the surface and rubbing it on a leather pad coated with dry rouge. The polishing with wet rouge on chamois leather brought the surface into bas-relief, the rubbing on the dry pad filled the hollows with fine rouge, so that when examined under a magnification of 150 diameters the structure of the steel could readily be seen. The difference between the structures shown in figs. A and B was quite as clearly and accurately shown by this means as by etching or by staining, but of course under high magnification the nature of the constituents was not resolved.

The Tempering of Steel from the Condition as supplied.

It was found that the change in elastic properties shown by a comparison of Diagrams 1 and 2 to be produced by annealing, could be brought about gradually by heating the material to temperatures lower than that which is commonly known as an annealing temperature. Any temperature exceeding 600° C. was found sufficient to produce a distinct lowering of the primary yield-point and a reduction of the step by which the yield-point is raised in consequence of recovery from over-strain. Diagram No. 3 illustrates this gradual tempering of the material from the condition as supplied by the makers, that is, from the condition of high yield-point and large step.

Six specimens were employed to obtain the curves shown in Diagram 3, and these

Diagram No. 3. (Tempering of Steel from the condition as supplied.)



Curves A—Material as supplied.
„ B—After heating to 600° C.
„ C— „ „ „ 650° C.

Curves D—After heating to 700° C.
„ E— „ „ „ 730° C.
„ F— „ „ „ 780° C.

were all taken from a rod of steel similar in quality to that used for Diagrams 1 and 2, but of a smaller diameter.* This rod, in the rough, was about ⅜ths of an inch in diameter, but some of the specimens after being slightly turned down, except at the ends, measured only about 0·32 of an inch.

Specimen A was tested in the condition as supplied, and the two curves marked A in Diagram No. 3 show that the primary yield-point has occurred at 36½ tons per sq. inch, and that after recovery from overstrain the yield-point has been raised by 11 tons per sq. inch.

The second specimen, B, was heated to 600° C. for a few minutes, slowly cooled, and then tested, with the result that creeping set in at the stress of 36 tons per sq. inch. The load was, however, immediately reduced, no time being allowed for the extension of the yield-point to take place. It had thus been shown, however, that 600° C. was sufficient to produce a very slight annealing action. This specimen, B, was again used to obtain the curves marked D in Diagram No. 3.

Specimen C was heated to 650° C. and slowly cooled. On testing, a yield-point was obtained at 33½ tons per sq. inch, and the step by which the yield-point was raised after recovery from overstrain was found to be about 9½ tons per sq. inch.

The curves marked D in Diagram 3 show that material which had been heated to 700° C. gave a yield-point at about 31½ tons per sq. inch, and that the step by which the yield-point was raised by recovery from overstrain was about 9 tons per sq. inch.

Specimen E was raised to 730° C., and the yield-point was found to be thereby lowered to 28½ tons per sq. inch and the step reduced to 8 tons; while specimen F, which was tested after being heated to 780° and slowly cooled, showed that 780° C. brought the primary yield-point to 24 tons per sq. inch, and reduced the step to 7 tons per sq. inch.

The extensions which occurred at the various yield-points shown in Diagram 3 are all marked within brackets in the diagram, and it will be noticed that the extensions at the yield-points obtained with any one specimen are approximately equal, while for different specimens these extensions become less and less as the primary yield-point is lowered and the step diminished in consequence of tempering or annealing at higher and higher temperatures. In the following table there will be found the various data obtained from the experiments which have just been described.

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* Particulars of this material are given in a footnote on p. 11.

TABLE showing the Tempering of Steel from the condition as supplied.

Condition of the material.	Primary yield-point.	Extension at yield-points.	"Step."
A. As supplied	36½ tons/sq. inch	0″·16 on 4 inches	11 tons/sq. inch.
B. Annealed at 600° C. . .	36 ”		
C. ” 650° ” . .	33½ ”	0″·16 ”	9½ ”
D. ” 700° ” . .	31⅓ ”	0″·15 ”	9 ”
E. ” 730° ” . .	28½ ”	0″·09 ”	8 ”
F. ” 780° ” . .	24 ”	0″·07 ”	7 ”

There is thus a relation between the yielding at the yield-point and the step by which the yield-point is raised in consequence of recovery from overstrain. This step does not depend on the actual extension which the material receives, for it was shown in the author's paper on "Recovery from Overstrain"* that a specimen could be stretched by any amount (by increasing the overstraining load to any extent) without altering the step by which the yield-point was raised above the previous maximum stress. The amount by which the yield-point is raised after restoration of elasticity, and the extension which occurs just at a yield-point, are thus definite properties of the material—properties which can be altered, at least in some cases, by thermal treatment.

In the experiments which have just been described the specimens were only kept for a few minutes at the annealing temperatures, and the times taken to heat up and to cool down were in all cases practically the same. As it was thought probable that a prolonged exposure to any temperature would have a greater effect than a short exposure to the same temperature, the following experiment was tried:—

A specimen of the same material as was used to obtain Diagram 3 was raised to 700° C., slowly cooled, and loaded to 31 tons per sq. inch. No yield-point was passed, and the curves marked E in Diagram 3 show that no yield-point should be expected until a stress of 31½ tons had been applied. The specimen was then kept for four hours at a temperature of from 670° to 690° C., and after cooling it was found that a stress of 26 tons per sq. inch caused considerable creeping to occur, and that with 27 tons per sq. inch a yield-point was certainly passed. After recovery from the overstrain caused by the application of the load of 27 tons per sq. inch, it was found that the yield-point had been raised to between 35 and 36 tons per sq. inch. By comparing these figures with those given in the table in the preceding page, it will be seen that annealing for four hours at about 680° C. has produced a slightly greater effect than was produced by annealing for a few minutes at 730° C.

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* 'Phil. Trans.,' A, vol. 193, p. 34, 1899.

Experiments with Lowmoor Iron.

The Lowmoor iron, which was afterwards employed in order to show the phenomenon of tempering after hardening by tensile overstrain, was subjected to an examination exactly similar to that described above for steel. An iron rod ⅝ths of an inch in diameter, and somewhat over 6 feet long, was employed, and specimens were tested without being previously turned. The analysis of this material was kindly supplied by Messrs. EDGAR ALLEN & Co., Limited, Sheffield, and is as follows :—

Carbon.	0·12	per cent.
Silicon	0·149	„
Sulphur	0·011	„
Phosphorus	0·076	„
Manganese	trace	„
Iron (by difference). . . .	99·644	„
	100·000	

An ultimate strength of 23 tons per sq. inch was attained with an elongation, omitting all local extension, of 24½ per cent. on a 4-inch length, or, allowing for the local elongation at the neck which formed just outside the measured length, say an ultimate elongation of 27 or 28 per cent. on the 4-inch length.

Diagram No. 4 shows the elastic properties of this material when tested in the condition as obtained from the makers. The primary yield-point is shown to have occurred at about 17 tons per sq. inch,* and the step by which the yield-point is raised after recovery from overstrain was about 4 tons per sq. inch. After the primary overstrain the yield-point was raised (when elasticity had been restored) by about 4½ tons per sq. inch, after the second overstrain by 4 tons per sq. inch, after the third by less than 3½ tons, while fracture occurred (outside the length under examination, however) 2½ tons above the fourth yield-point. These figures seem to indicate a gradual and consistent diminution, as the load increases, of the step by which the yield-point is raised after recovery from overstrain. This is contrary to what was to be expected from the experiments with steel; but it may be remarked that Diagram 5, which gives the history of annealed specimens of the same Lowmoor iron, shows the step by which the yield-point is raised remaining practically constant.

Curves Nos. 2, 3, 4, 5, and 6 of Diagram 4 show the comparatively rapid rate at which Lowmoor iron recovers from tensile overstrain, the recovery only taking hours instead of days or weeks as in the case of steel. Curve No. 9 illustrates the semi-plastic nature of the material immediately after the fourth overstrain. All these curves were obtained from second loadings at the various times.

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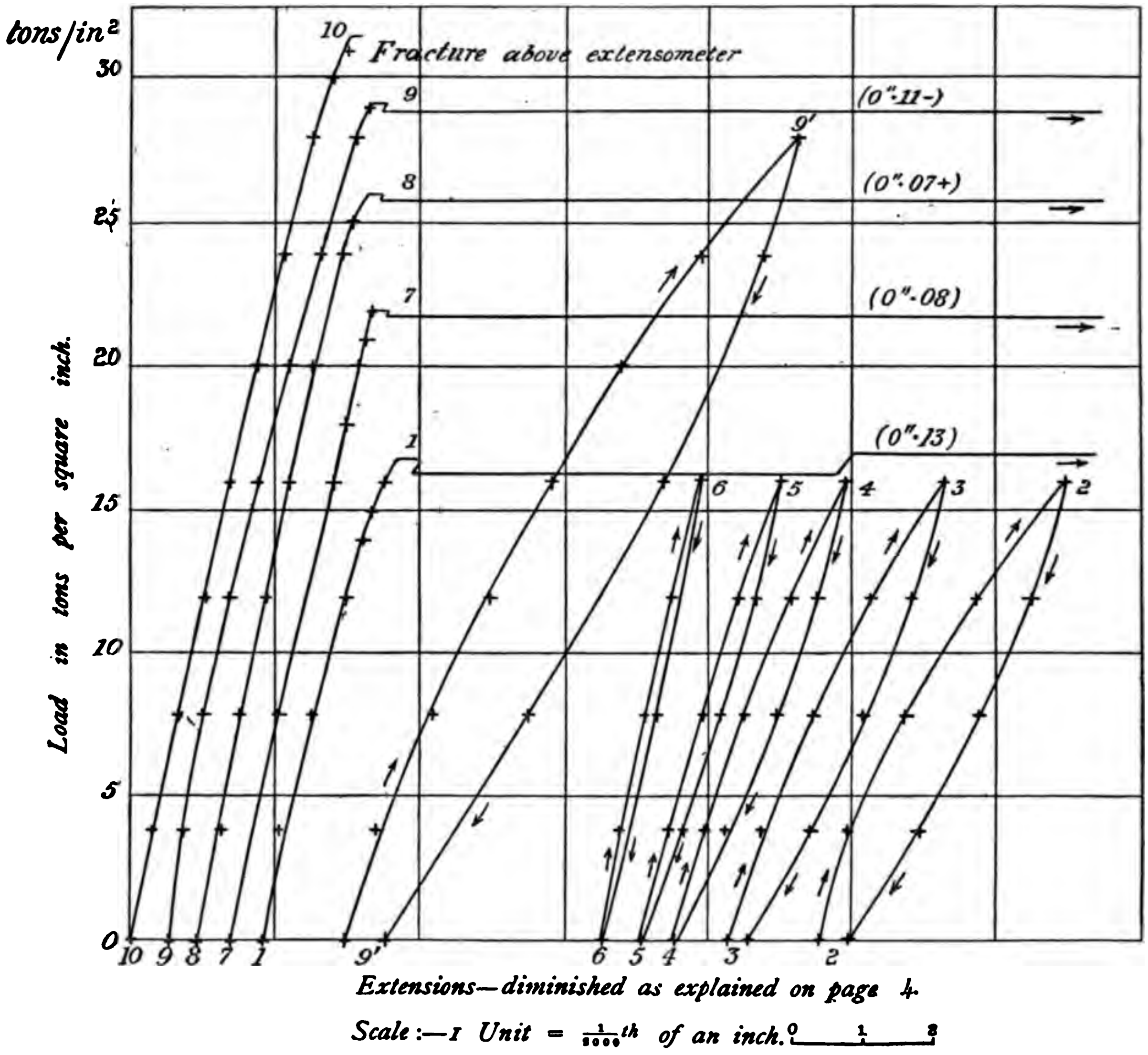
* Although 17 tons were applied before considerable yielding began, it was found that this yielding continued under a stress slightly over 16 tons/sq. inch, so that the yield-point should perhaps be placed at this latter stress.

Diagram No. 4. (Lowmoor Iron as supplied.)



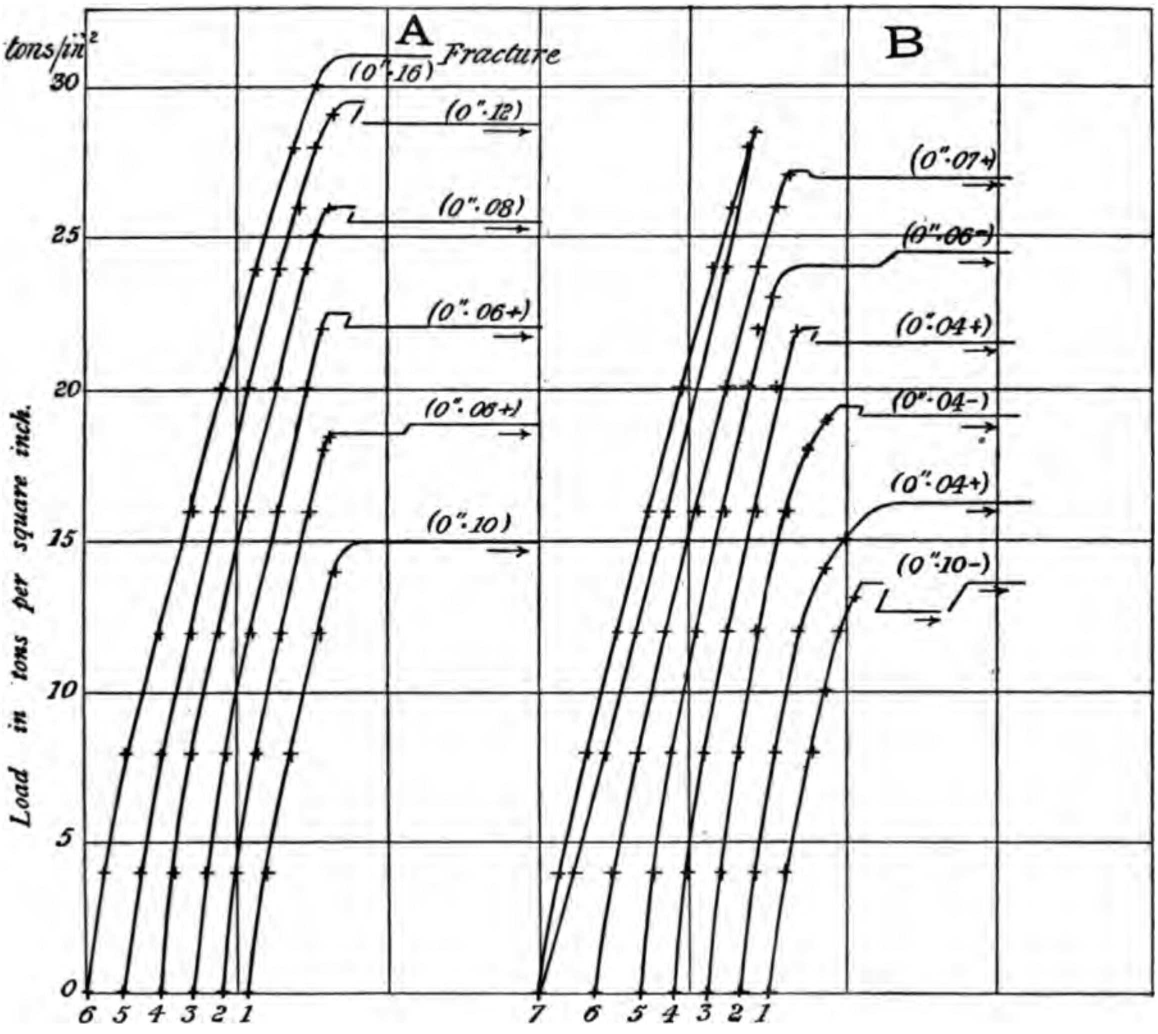
Curve No. 1—Primary test.
„ „ 2—Shortly after No. 1.
„ „ 3—½ hour „ „ 2.
„ „ 4—4 hours „ „ 2.
„ „ 5—21 „ „ „ 2.
„ „ 6—After heating to 230° C.

Curve No. 7—Immediately after No. 6.
„ „ 8—After heating to 150° C.
„ „ 9— „ „ „ 110° C.
„ „ 9'—Shortly after No. 9.
„ „ 10—After heating to 140° C.

The first series of curves (A) in Diagram No. 5 gives the history of another specimen from the same bar of Lowmoor iron, but after the specimen had been heated to 770° C. and allowed to cool down in the air. The primary yield-point is shown to have been lowered from 17 to about 15 tons per sq. inch, and the step by which the yield-point is raised after recovery from overstrain has been slightly diminished. Recovery from the various overstrains was effected by heating to about 150° C.

The series of curves marked B in Diagram 5 shows by comparison with the first series of curves in the diagram the influence of time on the effect of annealing. The specimens from which the two series of curves were obtained had both been heated

Diagram No. 5. (Annealed Lowmoor Iron.)



Extensions—diminished as explained on page 4.

Scale:—1 Unit = $\frac{1}{1000}$th of an inch. 0 1 2

A—Lowmoor Iron annealed for a few minutes at 770° C. and allowed to cool in air.

B— „ „ „ 6 hours at 750° C. and cooled very slowly.

to about the same temperature, but specimen A was simply raised to that temperature (770° C.) and then allowed to cool slowly in the air, while specimen B was kept at the temperature (about 750° C.. and never higher than 780° C.) for six hours, and then very slowly cooled by gradually reducing the gas supply while the specimen was kept in the furnace. The effect of prolonged annealing is shown in the further lowering of the primary yield-point, in the diminution of the step by which the yield-point is raised, and in the reduction of the amount of stretch which occurs at a yield-point.

The Lowmoor iron, whose elastic properties have just been considered, was also examined by means of the microscope. A small piece was cut from each of the three specimens used to obtain the curves of Diagrams 4 and 5; these small pieces of material were polished, etched with dilute nitric acid (2 per cent. strength), and examined under a magnification of 100 diameters. Fig. 1 (Plate 1) shows the microstructure of the material when in the condition as supplied by the makers. The granules were comparatively small, and slag was fairly uniformly distributed in small quantities all over the section. Fig. 2 (Plate 1) shows the structure of the iron which had been annealed at 750° C. for six hours, that is, this figure shows the structure of the specimen whose elastic properties are illustrated at B, Diagram 5. The granules are shown to be much larger than in the material in the condition as supplied; the slag had consequently segregated into larger masses.* The sections in figs. 1 and 2 were transverse to the length of the rod of iron. The photograph reproduced in fig. 1 was taken from a portion of the section where the granules were larger than the average, and where the slag was less thickly distributed. Fig. 2, on the other hand, shows by no means the largest granules found in the material after it had been annealed, but it should be stated that in some portions of the annealed material the granules were still found to be quite small.

The specimen taken from the bar used to obtain curves A of Diagram 5, that is, the specimen of Lowmoor iron which had been annealed by heating to 750° C. for only a few minutes, showed granules larger than in the virgin material, but, taking the average, distinctly smaller than those shown with the material which had been subjected to prolonged annealing. This is, of course, in accordance with ARNOLD'S and STEAD'S results. It should be remembered, however, that the structures of annealed specimens in particular were found to be by no means uniform, so that a complete survey of the sections had to be made in order to get a just comparison of the different granular structures.

To return to the elastic properties of the Lowmoor iron, attention may be called to the fact that the permanent sets which occurred at the various yield-points are marked in both series of curves in Diagram 5, and also at the yield-points in Diagram 4. The total elongation of the specimen of Diagram 4, owing to the position of the fracture, could not be exactly measured, but it was probably about 12 per cent. on a 4-inch length. The breaking load was 31½ tons per sq. inch.

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* This change in structure produced by annealing has been shown by ARNOLD, "On the Influence of Carbon on Iron," 'Proc. Inst. C.E.,' 1895; and by STEAD, "The Crystalline Structure of Iron and Steel," 'Journ. Iron and Steel Inst.,' 1898.

Instead of doing this, the extended series of experiments described below, and illustrated by Diagram 6, were performed on a single specimen, and the results obtained were corroborated by a few simple experiments, using other specimens. It was shown in this manner that a temperature as low as 300° C. could effect tempering or partial annealing, provided the material was sufficiently hardened by tensile overstrain. The harder the material was made, that is, the higher the yield-point was raised by the preliminary stretching, the lower was the temperature which could be shown to have a tempering or annealing action; and the higher the temperature employed, the greater was the tempering or annealing effect produced.

The material employed to obtain Diagram No. 6 (which shows the tempering of steel hardened by stretching) was the semi-mild steel whose analysis is given on a previous page, and whose elastic properties are illustrated by Diagrams 1 and 2. A specimen of this material was turned down (except at the ends) to a diameter of about 0·4 of an inch, and was then annealed by being heated to 820° C., and allowed to cool slowly. It was necessary to anneal the specimen, for otherwise the hardness produced by tensile overstrain would have been superposed on the hardness referred to above as having been produced in the process of manufacture. A 4-inch length was then marked off on the specimen by the aid of the marking instrument, the extensometer was attached, load was applied, and Curve No. 1 of Diagram 6 was plotted from the readings taken. The loading was continued until a yield-point was passed at 28 tons per sq. inch.

Recovery from this first overstrain was effected by heating the specimen to over 200° C., and allowing it to cool slowly. The specimen was then hardened still further by the two successive overstrains illustrated by Curves 2 and 3, Diagram 6.

After recovery from the third overstrain it was known that the material should bear a load about 7 tons higher than the last overstraining load (7 tons being the step between the yield-points shown by Curves 1, 2, and 3); that is, the material should bear a load of about 50 tons per sq. inch before a yield-point was passed. As, however, experience had shown that fracture was liable to occur when the material was stressed to this extent, no further hardening of the material was attempted.*

Curve No. 4, Diagram 6, is given in order to show that after recovery from the third overstrain (by heating to 270° C.) the material could bear a load of at least 48 tons per sq. inch without reaching a yield-point. As indicated above, probably a load just under 50 tons could have been safely applied. Slight imperfection of elasticity is shown by Curve No. 4, but this could not be got rid of by the ordinary means adopted to procure restoration of elasticity after overstrain. With this material there was always a slight departure from HOOKE'S law before a yield-point was

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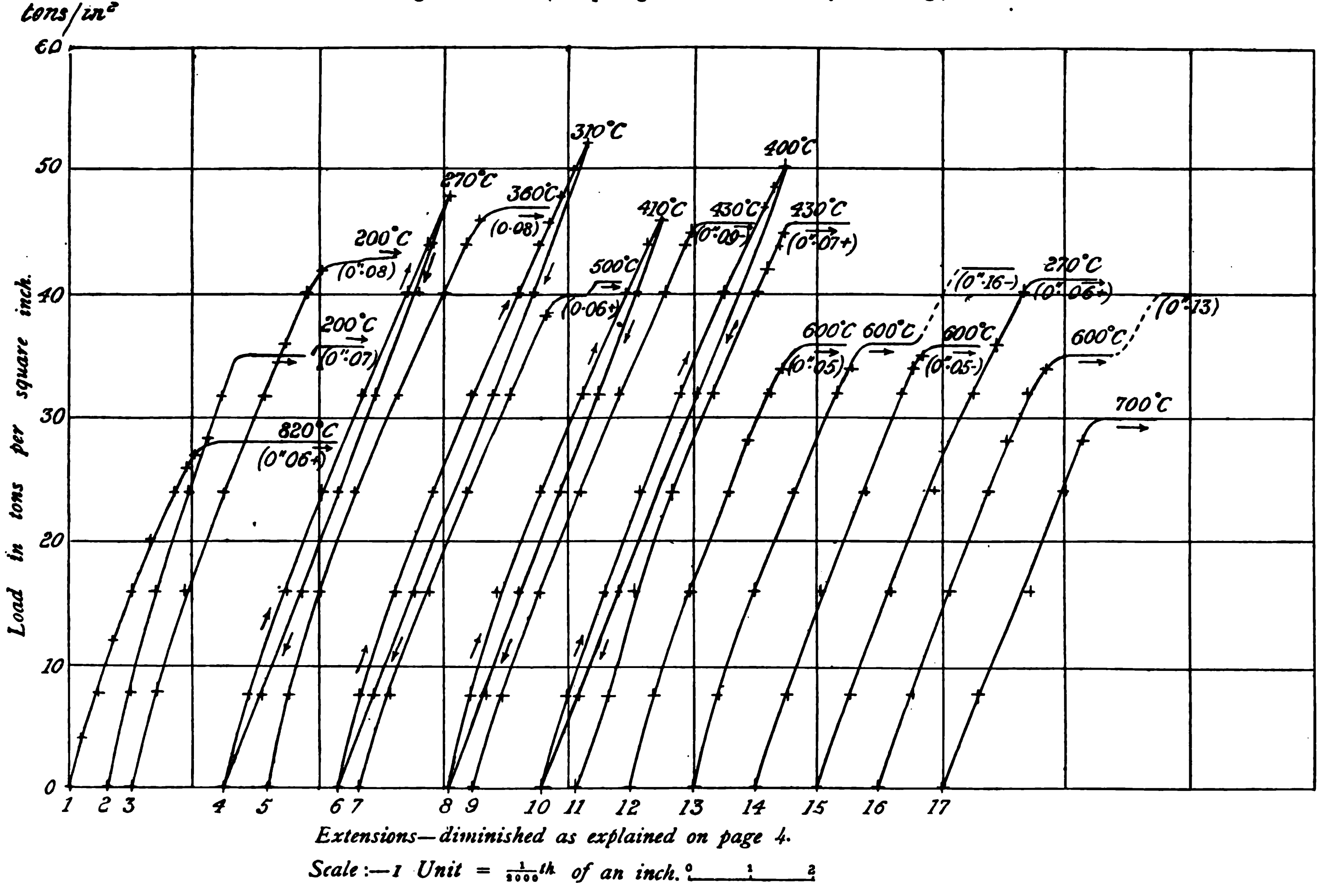
* Another specimen of the same rod, after annealing at 775° C., was found to give yield-points at about 29, 37, 44, and 52 tons per square inch; but after the yielding at the last stress had just spread throughout the 4-inch length under test, a neck formed and fracture occurred.

Diagram No. 6. (Tempering of Steel hardened by stretching.)



have been expected at 59½ tons had only restoration of elasticity been effected by the heating to slightly over 300° C. Another specimen of the same steel also showed that 350° C. was not sufficiently high a temperature to effect any softening of the material when it was in a condition having an elastic range to 52 tons per sq. inch. A temperature of about 300° C. may therefore be taken as the lowest which could be shown with the steel in question to produce tempering from the condition of hardness produced by tensile overstrain.

It is further shown, by a comparison of Curves 9 and 11 and of Curves 12 to 16 in Diagram 6, that a definite annealing temperature corresponds to a fairly definite position of the yield-point. This simple relation, however, is not shown to have held throughout all the experiments of Diagram 6. Curve No. 5 shows that 360° C. has sufficed to lower the yield-point to 47 tons per sq. inch, while in Curve No. 10 no yield-point is shown up to the stress of 51 tons, although the material had previously been heated to 400° C.

Further experiments were made to test how far a definite annealing temperature corresponded to a definite position of the yield-point. A specimen of the same steel as employed in the experiments described above was primarily annealed by being heated to 780° C., and slowly cooled, and was then overstrained by simply loading to the primary yield-point, which occurred at 29 tons per sq. inch. The specimen was then heated to 375° C., and after cooling loaded to 35 tons per sq. inch without a yield-point being passed. The yield-point, raised in consequence of recovery from overstrain, would have been expected at about 36 tons per sq. inch. The specimen was then heated successively to 400, 450, 500, and 550° C., and still the yield-point was found to be above 35 tons of stress. A temperature of 580° C. was next tried, and on testing the specimen after cooling, a yield-point was obtained at 33 tons per sq. inch. This result with a specimen which had been but slightly hardened by tensile overstrain, is practically in agreement with the results shown in Diagram 6, where after severe overstraining, and some tempering from the condition of hardness, a temperature of 600° C. is shown to have brought the yield-point to about 35 tons per sq. inch.

Another annealed specimen of the same steel rod was largely overstrained by carrying the primary loading far beyond the yield-point, which occurred at slightly over 28 tons per sq. inch. The load was steadily increased until a stress of 38 tons per sq. inch of original section was attained. This corresponded to a stress of about 40 tons on the actual reduced section of the bar. The extension produced by this loading was 0·22 of an inch on 4 inches of length. Recovery of elasticity was then effected by heating the specimen to 300° C. When cooled and tested, the specimen was found to bear a load of 44 tons per sq. inch without yielding. The specimen was then heated to 400° C., and after cooling, a load of 44 tons per sq. inch was again applied without a yield-point being passed. A temperature of 500° C. was next tried, and it was observed on testing the specimen that large yielding began at the stress of 42 tons per sq. inch. A stress of 40½ tons was, however, found sufficient to cause the yielding, once it had started, to spread throughout the length of the specimen. This result obtained with a specimen which had been largely overstrained by a single loading is, like the result of the experiment last described, in practical agreement with Diagram 6. Curve No. 7 of that diagram shows that, with a specimen which had been largely overstrained by the passage of several yield-points, a temperature of 500° C. brought the yield-point to a stress of 40 or 41 tons per sq. inch.

The Tempering of Lowmoor Iron Hardened by Stretching.

The tempering of Lowmoor iron which has been hardened by stretching now remains to be considered. A specimen of the soft iron whose properties were described in a preceding section of this paper, was annealed by heating to 750° C. with slow cooling, and was then subjected to the treatment illustrated by Diagram 6. The procedure adopted will be readily understood by reference to the diagram: it was practically identical with that adopted to obtain Diagram 6.

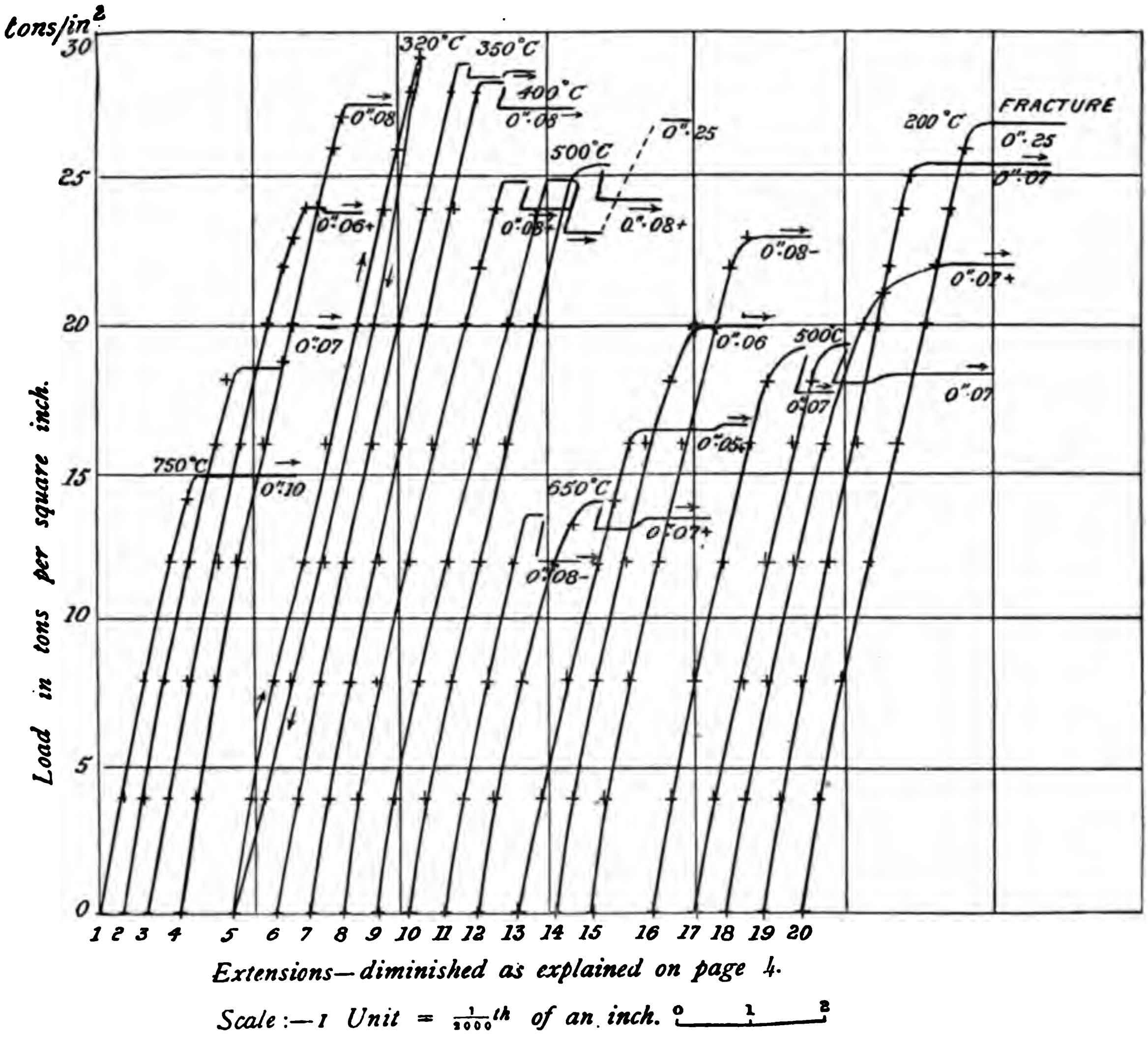
The specimen was first hardened by overstrain in the manner illustrated by Curves 1, 2, 3, and 4, Diagram 7. A temperature of 320° C. was then shown (Curve No. 5) to have had no tempering action on the hardened material, while the amount by which the yield-point was lowered in consequence of heating the specimen to 350 to 400 and to 500° C. is shown by Curves 6, 7 and 8 respectively.

Perhaps reference should again be made here to the method adopted to detect yield-points. The load was very slowly applied as its critical value was approached; the extensometer reading was continually observed in order to detect "creeping" within the 4-inch length under observation, and the lever of the testing machine was watched for any indication of yielding starting outside the measured length of the specimen. When "creeping" had been detected, the load was immediately reduced so as to keep the material yielding slowly. A considerable lowering of this load at the yield-point could often be effected, as is shown by the dip in various curves of the present Diagram No. 7, without causing the extension of the yield-point to stop; but since the rate of extension fluctuated somewhat as the stretching action travelled along, the load had sometimes to be slightly increased again in order to get the yielding to spread at a reasonable rate throughout the whole length of the specimen. The exact position of a yield-point is thus a little indefinite. By slightly reducing the load as the yielding at a yield-point was occurring, the danger of fracture supervening at the higher yield-points was somewhat reduced.

After curve No. 8 of Diagram 7 had been obtained, the specimen was again heated to 500° C., and Curve No. 9 shows that the yield-point had been brought to very much the same position as before. The loading in this test was carried far beyond the yield-point, and then the specimen was once more raised to 500° C. Curve No. 10

shows that the material now gave a yield-point at a stress only slightly higher than before.

Diagram No. 7. (Tempering of Lowmoor Iron hardened by stretching.)



A temperature of 650° C. was next tried, and it was found that this temperature lowered the yield-point by a remarkably large amount, Curves Nos. 11 and 12 showing that the material yielded at a stress distinctly less than that required to produce yielding in the very first test of this specimen, which it may be recalled had been annealed by heating to 750° C. Curves 11 and 12 thus show that the overstraining and tempering which the specimen received as the result of the operations illustrated by Curves 1 to 10 had brought the material into a condition more sensitive to the influence of temperature.

The material was now hardened again by successive overstrain in the manner illustrated by Curves 13, 14 and 15, Diagram 7, recovery from overstrain being effected by warming to between 100 and 200° C., and then the specimen was heated to 500° C. to see if the yield-point would thus be brought to approximately the same position as in Curves 8, 9, and 10, which were obtained after the specimen was previously heated to this temperature. Curves Nos. 16 and 17 show that the yield-point occurred at a much lower stress than in the Curves 8, 9, and 10, so that the material was in an entirely changed condition as regards the effects of temperature.

A further hardening of the material was next attempted, but fracture inadvertently occurred, as illustrated by Curve No. 20. A temperature of 200° C. had been employed to effect recovery from the overstrain illustrated by Curve 19, but Curve No. 20 seems to indicate that this temperature had been sufficient to produce annealing action, because large yielding and then fracture occurred at a load distinctly lower than would naturally have been expected. This test was perhaps not quite conclusive as showing tempering by 200° C., for the specimen after so many overstrains was not very uniform in section, and the fracture occurred at what was known to be a rather thick part of the bar. It is thus just possible that the fracture was due to the bar not having been thoroughly overstrained by the preceding loading, although in that case yielding might have been expected to have started below 26 tons per square inch instead of at 27 tons per square inch. On measuring the fractured specimen it was found that the length, which had been originally 4 inches long, was now 5·80 inches.

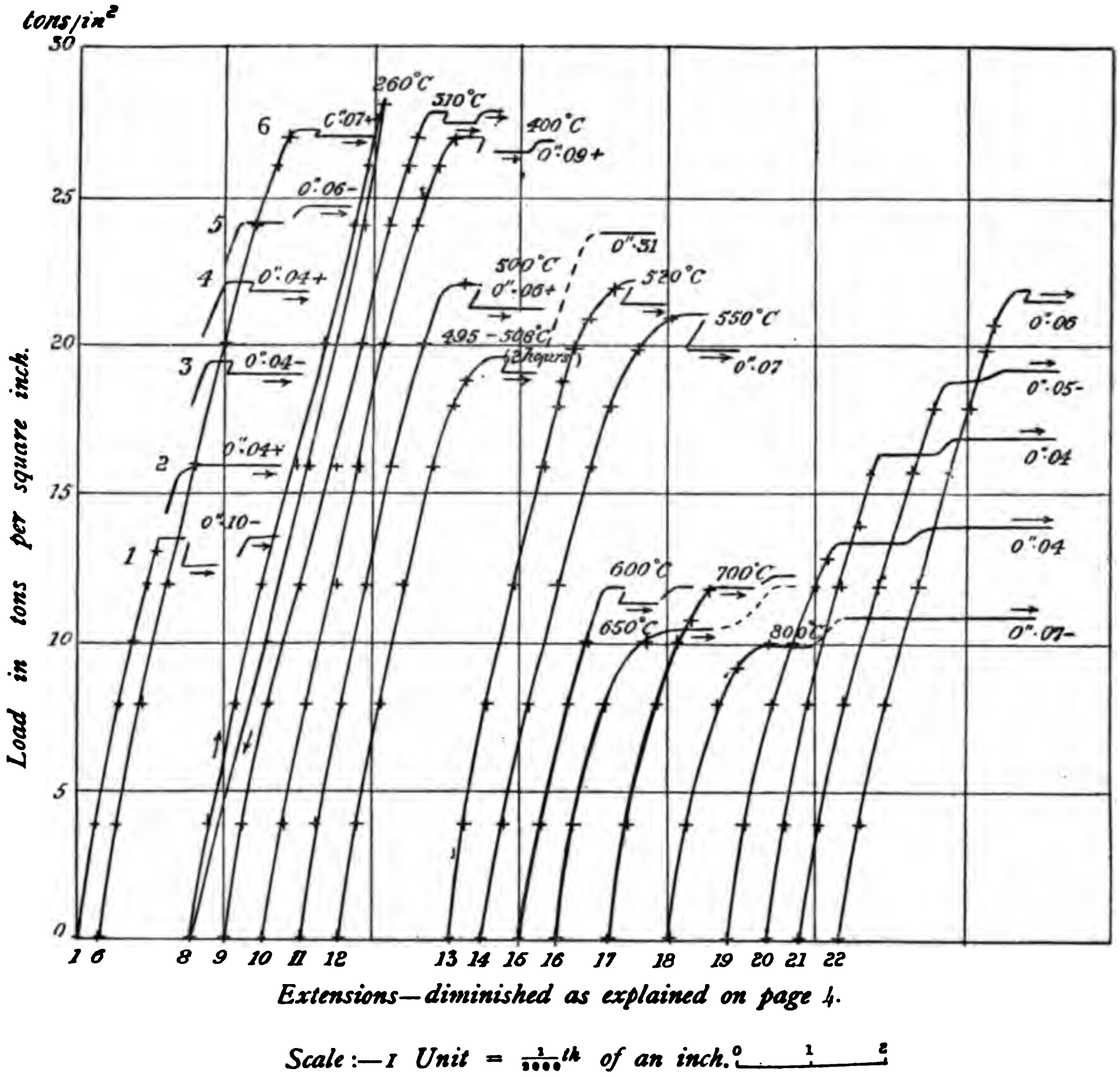
Had fracture not inadvertently occurred, as explained above, there seems to be no reason why this specimen should not have been overstrained and tempered or annealed an indefinite number of times, and so the material have become drawn out into the form of a wire.

The history of another specimen of the same Lowmoor iron is given in Diagram No. 8, and the results recorded in that diagram serve generally to corroborate those illustrated by Diagram 7. The specimen employed has already been referred to in the preliminary examination of the Lowmoor iron described above. It was the specimen which was subjected to prolonged annealing at 750° C., and then hardened in the manner illustrated by the curves marked B in Diagram No. 5. In Diagram No. 8 there are illustrated the first and last overstrains in this hardening of the specimen, and also the position of the four intermediate yield-points.

Curve 8, Diagram 8, shows that 260° C. was too low a temperature to produce annealing, while Curve 9 shows that 310° C. sufficed to lower the yield-point by almost 2 tons per square inch. It is just possible that in the preliminary hardening of this specimen a seventh yield-point at a stress of about 29 tons per square inch might have been safely passed, and so the material after restoration of elasticity brought into a condition capable of bearing a load of 31 tons. Had this been safely accomplished, then no doubt a temperature lower than 310° C. would have sufficed to produce a tempering of the material.

A comparison of Curves 11 and 12, Diagram 8, indicates that time has some influence on the tempering of Lowmoor iron hardened by overstrain. Curve No. 11 shows that 500° C. had lowered the yield-point to 21 or $21\frac{1}{2}$ tons per square inch, while Curve 12 shows that, by keeping the specimen for two hours at about 500° C., the yield-point was lowered to 19 tons per square inch. Time has no doubt a slight effect on the similar tempering of steel; but in both cases temperature is distinctly the main determining cause.

Diagram No. 8. (Tempering of Lowmoor Iron hardened by stretching.)



Curve No. 15, Diagram 8, which was obtained after heating to 600° C, shows a very large drop in the yield-point similar to the drop obtained at 650° C. with the specimen whose history is given in Diagram 7. The yield-point shown by Curve 15 is lower than that shown by Curve 1 of Diagram 8, that is, it is lower than the yield-point given with material which had been annealed for a long time at 750° C., but which had never been subjected to any tensile overstrain.

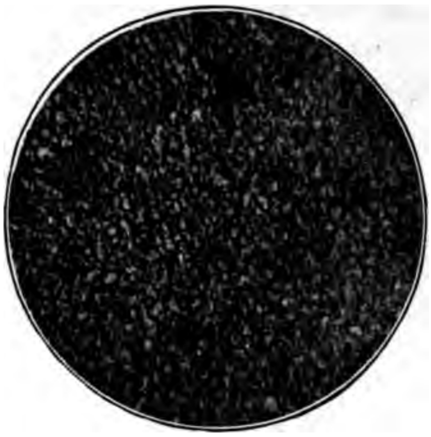


Fig. A—½" Steel as supplied. × 150 D.
(Etched with HNO_3.)

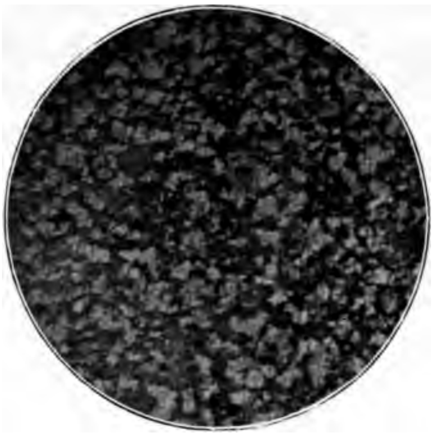


Fig. B—½" Steel annealed for a few minutes
at 750° C. × 150 D.
(Etched with HNO_3.)

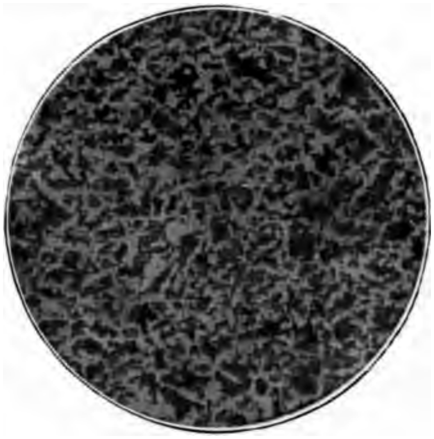


Fig. C—1" Steel as supplied. × 150 D.
(Stained with Cocoa.)

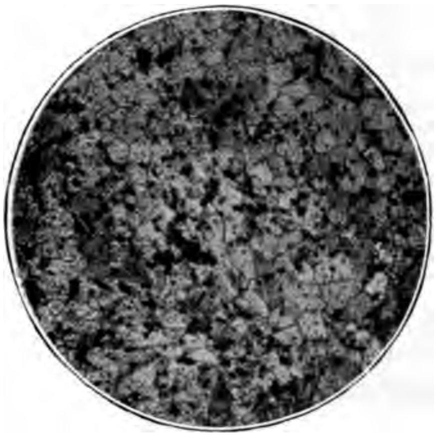


Fig. 1—Lowmoor Iron as supplied. × 100 D.
(Etched with HNO_3.)

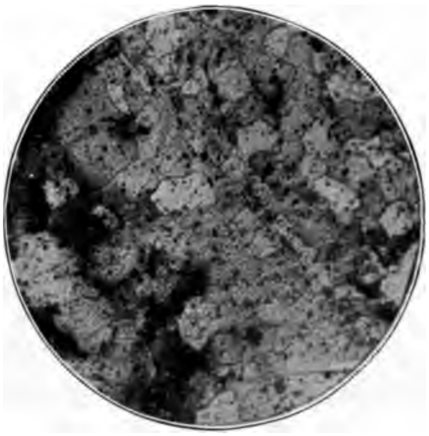


Fig. 2—Lowmoor Iron annealed for 6 hours
at 750° C. × 100 D.
(Etched with HNO_3.)

The specimen whose history is given by Diagram 8 was ultimately heated to 800° C., and Curve 18 was obtained. The successive overstrains illustrated by Curves 19, 20, 21, and 22 were then applied, and the material was left in a condition of hardness, the tempering from which could be made the subject of further investigation.

In conclusion, it may be stated that many experiments other than those described above were performed during the course of this research. The preliminary experiments, which showed qualitatively the tempering of iron hardened by tensile overstrain, were performed in the Engineering Laboratory of the University of Glasgow, Professor Barr having kindly placed his laboratory and all necessary apparatus at the author's disposal. All the experiments described in this paper were carried out in the Engineering Laboratory of the University of Cambridge, and the author desires to express his indebtedness to Professor Ewing for suggestions and advice given from time to time as the work was in progress.

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31.1.1902

II. *The Measurement of Magnetic Hysteresis.*

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Communicated by Professor J. J. THOMSON, *F.R.S.*

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Introduction.

§ 1. Two ideal physical processes have been devised as the foundations of two methods of deducing mathematical expressions for the energy dissipated in magnetic material through magnetic hysteresis; these processes are due to Professor E. WARBURG and to the late Dr. J. HOPKINSON.

In WARBURG'S theory[*] the specimen, in the form of a slender wire, is placed in a magnetic field due to a pair of permanent magnets so arranged as to produce magnetic force parallel to the length of the specimen. The mechanical work spent in moving these magnets through such a cycle of changes of position, that the iron is subjected to a cycle of magnetic changes, is clearly equal to the energy dissipated on account of magnetic hysteresis in the specimen. In terms of the magnetic quantities the energy dissipated per cub. centim. per cycle is $-\int \mathrm{I}d\mathrm{H}$ or $\int \mathrm{H}d\mathrm{I}$ ergs, where H is the magnetic force and I the intensity of magnetisation. Professor J. A. EWING[†] has applied the principle involved in WARBURG'S theory to the design of a simple instrument by which the hysteresis of any specimen of sheet iron (for the range of induction $\mathrm{B} = 4000$ to $\mathrm{B} = -4000$ C.G.S. units approximately) is determined by comparison with two standard specimens supplied with the instrument, and previously tested for hysteresis by the ballistic method. The principle has also been employed by W. S. FRANKLIN,[‡] by H. S. WEBB,[§] and by G. L. W. GILL[||] to obtain absolute determinations of hysteresis.

The theory of the late Dr. JOHN HOPKINSON[¶] proceeds in a different manner. The specimen now takes the form of a fine wire bent into a large circular ring, the ends of the wire being welded together; the length of the wire is l centim., and its cross-section A sq. centim. Let this ring be uniformly overwound with insulated wire at the rate of N turns per centim. so that the total number of turns is $\mathrm{N}l$, and let the wire be without resistance. Then if C be the current at any time, the magnetic force acting on the iron is $\mathrm{H} = 4\pi\mathrm{NC}$. If B be the magnetic induction in the iron, the number of linkages of lines of induction with the electric circuit at any instant is $\mathrm{A}l\mathrm{NB}$, and hence, when B changes, there is by FARADAY'S law a voltage $\mathrm{A}l\mathrm{N}d\mathrm{B}/dt$ between the ends of the coil. We have supposed here that the wire is closely wound upon the iron. The power spent in driving the current against this voltage is $\mathrm{A}l\mathrm{NC}d\mathrm{B}/dt$ ergs per second.

Using the relation $\mathrm{H} = 4\pi\mathrm{NC}$, and noticing that $\mathrm{A}l$ is the volume (v) of the iron, the expression becomes $v/4\pi \,.\, \mathrm{H}d\mathrm{B}/dt$.

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[*] 'Wied. Ann.,' vol. 13 (1881), p. 141.

[†] 'Magnetic Induction in Iron and other Metals,' 3rd ed., revised, § 199.

[‡] W. S. FRANKLIN, 'Physical Review,' vol. 2, p. 466.

[§] H. S. WEBB, 'Physical Review,' vol. 8, p. 310.

[||] G. L. W. GILL, 'Science Abstracts,' vol. 1 (1898), p. 413.

[¶] 'Phil. Trans.,' vol. 176 (1885), p. 466.

The work spent per unit volume during any finite change is thus

$$\frac{1}{4\pi}\int H\frac{dB}{dt}\,dt = \frac{1}{4\pi}\int H\,dB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the expression found by Hopkinson.

When the change is cyclic, so that B and H have the same values at the end as at the beginning of the cycle, we can throw the expression into a different form. For since $B = H + 4\pi I$ we have $dB = dH + 4\pi dI$. But when H goes (1) through the cycle $+ H_0, - H_0, + H_0$ or (2) goes from $+ H_0$ to $- H_0$, then $\int H\,dH = 0$, and thus we recover Warburg's expression $\int H\,dI$.

In the present paper we are only concerned with the work spent in causing a complete cycle of magnetic changes. We shall always use W to denote the energy dissipated by hysteresis per cub. centim. per cycle of magnetic changes, and we shall express W in ergs per cub. centim. per cycle. We have thus

$$W = \frac{1}{4\pi}\int H\,dB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

To obtain the value of W by means of this expression, it has been usual to construct a cyclic B–H curve, best by the method described by Ewing,* and to find its area. This process is easy enough, but since it involves the observations necessary to find at least ten or twelve points on the B—H curve, and the subsequent estimation of the area of the curve after it has been plotted, quite an hour is required for each determination of W.

§ 2. We have given the sketch contained in § 1 for the purpose of contrasting the physical ideas involved in the two mathematical methods by which the formulæ $W = -\int I\,dH$ and $W = 1/4\pi . \int H\,dB$ have been obtained, and of showing the manner in which the subject presented itself to one of us in 1895.

The remarks of § 1 refer only to *mathematical* processes and not to the experimental methods of studying the effects of hysteresis as exhibited in the relation of I or B to H. To the *experimental* knowledge of the subject the first contributions were made by the independent and nearly simultaneous papers of Warburg and Ewing. In addition to the theory noticed in § 1, Warburg gave magnetometric observations of cyclic I—H curves, but his observations were few. Ewing made a much more systematic attack on the subject, using the ballistic as well as the magnetometric method, and determining the values of $-\int I\,dH$ for a graded series of I—H curves for the same specimen of iron. An account of the subsequent development of the subject, in which Ewing has had a great share, will be found in his book on 'Magnetic Induction in Iron and other Metals.' We owe much to Professor Ewing. It was his hysteresis tester which formed the initial incentive to the research described in the present paper, and, further, much of the general knowledge of magnetism needed in carrying out that research has been gained from his writings and from conversations with him.

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* Ewing, 'Magnetic Induction in Iron and other Metals,' 3rd ed., revised, § 192.

§ 3. It occurred to one of us some years ago that just as EWING had, in effect, applied WARBURG'S theory to produce a practical hysteresis tester, so it might be possible to apply HOPKINSON'S theory to the design of a method which should give absolute determinations of the energy dissipated through hysteresis as quickly and as accurately as changes in magnetic induction are found by the aid of a ballistic galvanometer. It was evident that if this could be attained it would be possible to investigate the effects of various physical conditions—stress, temperature, the passage of an electric current, &c.—upon the hysteresis, with a comparatively very small expenditure of time. A preliminary account of the theory of the method was published in 1895,* and since then much time has been spent in working out some details which make the method practical; as only a few weeks in each year have been available for the work, progress has been slow.

In essentials the method is of course well known. In one of its forms it is in constant use among electrical engineers in testing by means of a watt-meter the energy dissipated in a transformer when its primary coil is traversed by an alternating current. In this case there is a *steady* deflexion of the watt-meter, and thus the watt-meter method is convenient in commercial work.

From the scientific standpoint, the watt-meter method has the disadvantage that it is not possible to find the limits between which the magnetic induction alternates without the use of revolving contact-makers, or oscillographs, or other appliances. The "effective" voltage is indeed easily measured, but unless the wave-form of the curve of voltage is known, the limits of the induction cannot be found.

With transformers of commercial dimensions the "effective" voltage is considerable, but when the iron is reduced to a single wire only 1 or 2 sq. millims. in section the "effective" voltage for 100 alternations per second does not exceed a few tenths of a volt, even when the secondary coil contains 1000 turns of wire. We know of no method by which so small an alternating voltage can be measured with any accuracy.

The idea which occurred to one of us in 1895 was to use a single reversal of the current to produce a "throw" of a ballistic electro-dynamometer instead of an alternating current to produce a steady deflexion of a watt-meter. The present paper contains an account of the development of this idea and of its applications to magnetic research.

So far as we know, the ballistic method of measuring hysteresis is novel. In the endeavour to make it a practical method, we have met with many difficulties, and the main part of the work has been devoted to overcoming those difficulties.

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* G. F. C. SEARLE, "A Method of Measuring the Loss of Energy in Hysteresis," 'Proc. Camb. Phil. Soc.,' vol. 9, Part I., November 11, 1895.

Of the many advantages of the ballistic method, two may be mentioned. Thus the induction can be measured simultaneously with the hysteresis far more simply than when an alternating current is used. Further, the ballistic method enables measurements to be made so quickly as to render experiments easy which would otherwise be practically impossible on account of the very great time required for the numerous determinations of hysteresis necessary in investigations on the effects of stress or of temperature.

It will appear from the paper that several of the effects of physical changes upon *hysteresis* which we have studied presented themselves to us as we worked out the method. We found afterwards that some of these effects had already been discovered by EWING or others, or might have been deduced from their experiments. These cases we refer to in foot-notes. The effects of the various physical changes upon the *induction* have received so much attention from others that we have not thought it necessary to point out how much of that part of our work has consisted merely in going over old ground.

Approximate Theory of the Method.

§ 4. Let an iron ring of section A sq. centim. and mean circumference l centim. be wound with N turns of primary windings per centim., so that the total number of turns is Nl. The current C, which passes through the primary and magnetises the iron, producing the magnetic force $H = 4\pi NC$, passes also round the fixed coils of a sensitive electro-dynamometer. A secondary coil of n turns is wound over the iron, and is connected in series with the suspended coil of the dynamometer, and with an earth inductor, the total resistance of the secondary circuit being S. In finding the total induction through the secondary circuit, we must remember that the secondary will not generally be closely wound upon the iron. A certain number of lines of induction will in consequence pass through the secondary circuit due to the magnetic force H produced by the primary current. The total number of linkages of lines of induction with the secondary circuit is thus $AnB + MC$ where M is some constant.

Let the primary current C change gradually (1) from $+ C_0$ to $- C_0$ and back again to $+ C_0$, so as to complete a cycle, or (2) from $+ C_0$ to $- C_0$, making a semi-cycle. During the change the voltage $AndB/dt + MdC/dt$ is set up in the secondary circuit. If the "time constant" of the secondary circuit be very small compared with the "time constant" of the primary circuit, the effect of the self-induction of the secondary circuit may be neglected, and the current in the secondary circuit may be taken to be

$$c = \frac{An}{S}\frac{dB}{dt} + \frac{M}{S}\frac{dC}{dt}.$$

Let the couple experienced by the suspended coil, when the currents in the fixed and suspended coils are C and c respectively, be qCc dyne-centims. Then since $H = 4\pi NC$, when the magnetic force due to the secondary current is negligible, we have for the couple at any instant

$$\text{Couple} = qCc = q\frac{An}{4\pi NS}H\frac{dB}{dt} + q\frac{M}{S}C\frac{dC}{dt}.$$

If the time of vibration of the moving coil be so great compared with the time occupied by the cycle or semi-cycle, that the cycle or semi-cycle is completed before the coil has sensibly moved from its equilibrium position, the angular momentum acquired by the coil is

$$K\omega = \int qCc\,dt = \frac{qAn}{4\pi NS}\int H\frac{dB}{dt}dt + \frac{qM}{S}\int C\frac{dC}{dt}dt \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where K is the moment of inertia of the coil, and ω is the angular velocity imparted to the coil by the electro-magnetic impulse. Now when C goes through either a cycle or a semi-cycle $\int C\,dC$ vanishes, and thus

$$\frac{1}{4\pi}\int H\,dB = \frac{NS}{An}\int Cc\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Let θ be the greatest angular displacement or "throw" of the coil, and f the restoring couple exerted by the suspension per radian of displacement. Then by the principle of the conservation of energy, we may equate the initial kinetic energy to the potential energy at end of swing and thus obtain $\frac{1}{2}K\omega^2 = \frac{1}{2}f\theta^2$,

or

$$K^{\frac{1}{2}}\omega = f^{\frac{1}{2}}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The three constants q, K, and f are eliminated, and the "constant" of the dynamometer is determined in the following manner. Let a constant current C′ flow in the primary circuit through the fixed coil, and while this current is passing, let the earth inductor be inverted so as to produce a known change, P, in the number of linkages of lines of induction with the secondary circuit. The time-integral of the current thereby produced is P/S, and hence the initial angular momentum of the suspended coil is

$$K\omega' = qC'P/S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If ϕ be the throw produced, we have by (5) $\omega/\omega' = \theta/\phi$, and hence by (6) and (3),

$$\int Cc\,dt = \frac{C'P}{S\phi}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Thus by (4)

$$\frac{1}{4\pi}\int H\,dB = \frac{NC'P}{An\phi}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Since it is only the ratio of the angles θ and ϕ which appears in the formula for $1/4\pi \, . \int H dB$, we may take for the ratio θ/ϕ the ratio of the two "throws" of the spot of light along the scale, provided that the "throws" are not so large that $\tan 2\theta$ differs appreciably from 2θ.

The effect of damping has so far been neglected. If λ be the logarithmic decrement, we must write (5)

$$K'\omega = f'\theta(1 + \tfrac{1}{2}\lambda),$$

where θ is now the *observed* "throw." The "throw" ϕ must be treated in the same way, and hence the ratio θ/ϕ is unaffected and the expression (8) remains true.

§ 5. If, instead of making the primary current go through a complete cycle, we make it go through a semi-cycle from $+ C_0$ to $- C_0$ or *vice versâ*, we can very conveniently combine the measurement of the hysteresis by the dynamometer with the measurement of the magnetic induction by a ballistic galvanometer. For if a ballistic galvanometer be connected in series with a second secondary coil wound upon the specimen, the throw of the galvanometer gives the value of $B_1 - B_2$, where B_1 and B_2 are the algebraical values of B corresponding to $+ C_0$ and $- C_0$. In the ideal case $B_2 = - B_1$, and then $B_1 = \frac{1}{2}(B_1 - B_2)$.

Whether this ideal state obtain or not, we shall denote $\frac{1}{2}(B_1 - B_2)$ by B_0, calling it the mean maximum induction. When the specimen takes the form of a ring, the ballistic galvanometer only enables us to find B_0; it gives no information as to B_1 or B_2. This knowledge could, however, be obtained in the case of a long straight specimen by slipping the secondary coil off the specimen.

When, as in our case, there are two observers, the observations for W and B_0 can be made simultaneously.

§ 6. To take the specimen through a complete cycle, we must make the current go through the semi-cycle $+ C_0$ to $- C_0$ and then from $-C_0$ to $+ C_0$. If the iron has reached a cyclic state the throws of the galvanometer will be the same in magnitude, though opposite in direction for the two reversals of the current. But it will often happen that, on account of the previous treatment of the iron, B_1 differs considerably from $- B_2$, and in this case the throws of the dynamometer differ in magnitude for the two reversals, though they are in the same direction.

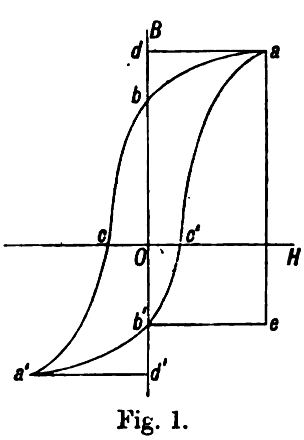

Fig. 1.

A little consideration will show us how to proceed in this case. Let *abca'b'c'a* (fig. 1) be a B—H curve, and let *ad*, *a'd'* be drawn parallel to OH. By reversal of the current, let the iron be caused to go through the changes represented by the curve *abca'*. From *a* to *b* H is positive and dB is negative, while from *b* to *a'* both H and dB are negative. Hence the value of $\int H dB$ for the semi-cycle $+ C_0$ to $- C_0$ is given by (area $bca'd'$) — (area abd). Similarly $\int \mathrm{H}d\mathrm{B}$ for the semi-cycle $-\mathrm{C}_0$ to $+\mathrm{C}_0$ is given by (area *b′c′ad*) — (area *a′b′d′*). The value of $\int \mathrm{H}d\mathrm{B}$ for the whole cycle is thus the area of the B—H curve *abca′b′c′a*.

If θ_1 and θ_2 be the two throws of the dynamometer for the two semi-cycles, we have by (8)

$$\frac{1}{4\pi}\int_{\mathrm{B}_1}^{\mathrm{B}_2} \mathrm{H}d\mathrm{B} = \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}\theta_1, \qquad \frac{1}{4\pi}\int_{\mathrm{B}_2}^{\mathrm{B}_1} \mathrm{H}d\mathrm{B} = \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}\theta_2,$$

and hence, since W is the sum of the two quantities on the left sides of these equations,

$$\mathrm{W} = \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}(\theta_1 + \theta_2) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9).$$

Thus to find W we have simply to add together the two throws for the two semi-cycles, and then multiply the result by the factor $\mathrm{NC'P}/\mathrm{A}n\phi$.

The expression (9) is the fundamental formula of the method employed by us for the measurement of hysteresis. From it we see that all that is needed in addition to the electro-dynamometer for the measurement of W is an ampere-meter by which to determine C′, and an earth-inductor or some other means of producing a known change of induction (P) through the secondary circuit.

Complete Theory of the Method.

§ 7. In the elementary theory of § 4, the resistance, S, of the secondary circuit is supposed to be so large, and the induced current in consequence so feeble, that the magnetic force due to the induced current is negligible in comparison with that due to the primary current. It is thus necessary to proceed to a closer examination of the theory in order to find the correcting term which appears when S is only finite. In this examination we take account also of the energy dissipated by the eddy currents, which circulate in the specimen in consequence of the variations of the applied magnetic force. We use the notation already employed in the elementary theory.

§ 8. In the primary circuit let the magnetising coil and the dynamometer coil be placed next to each other in the circuit, and let E be the voltage at any instant between the ends of the conductor so formed. Now the induction through either circuit depends not only upon the magnetic induction, B, in the iron, but also upon the magnetic force in the space between the coils and the iron, as well as upon the magnetic force near the coils of the dynamometer. In each case the magnetic force depends only upon C and *c*, and not at all upon B, since the iron is formed into a ring. Thus if R be the resistance of the primary circuit between the two points between which the voltage is E, the equations for the primary and secondary currents may be written

$$\mathrm{E} = \mathrm{RC} + \frac{d}{dt}(\mathrm{N}l\mathrm{AB} + \mathrm{L'C} + \mathrm{M}c) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (10),$$

$$0 = \mathrm{S}c + \frac{d}{dt}(n\mathrm{AB} + \mathrm{MC} + \mathrm{L}c) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11),$$

The terms in this formula which involve M are the complete differential of MCc, and thus vanish on integration, since $c = 0$ at both limits. We thus have

$$(\mathrm{W}_1 + \mathrm{X}_1)\mathrm{A}l = \int \mathrm{CN}l\mathrm{A}\frac{d\mathrm{B}}{dt} + \int c\frac{d}{dt}(n\mathrm{AB})\,dt - \mathrm{T}' + \mathrm{T}.$$

If we multiply (11) by NlC/n and integrate we obtain

$$\frac{\mathrm{N}l\mathrm{S}}{n}\int \mathrm{C}c\,dt = -\int \mathrm{C}\left[\mathrm{N}l\mathrm{A}\frac{d\mathrm{B}}{dt} + \frac{\mathrm{N}l}{n}\frac{d}{dt}(\mathrm{MC}) + \frac{\mathrm{LN}l}{n}\frac{dc}{dt}\right]dt.$$

Thus, by addition of the last two equations,

$$(\mathrm{W}_1 + \mathrm{X}_1)\,\mathrm{A}l = -\frac{\mathrm{N}l\mathrm{S}}{n}\int \mathrm{C}c\,dt - \frac{\mathrm{N}l}{n}\int \mathrm{C}\frac{d}{dt}(\mathrm{MC})\,dt + \int c\frac{d}{dt}(n\mathrm{AB})\,dt$$
$$- \frac{\mathrm{LN}l}{n}\int \mathrm{C}\frac{dc}{dt}\,dt - \mathrm{T}' + \mathrm{T}.$$

The first integral in this expression is proportional to the "throw" of the moving coil, for, by (7),

$$-\frac{\mathrm{N}l\mathrm{S}}{n}\int \mathrm{C}c\,dt = \frac{\mathrm{N}l\mathrm{C}'\mathrm{P}}{n\phi}\,\theta_1,$$

if we measure θ_1 in the right direction.

As regards the second integral,

$$\int \mathrm{C}\frac{d}{dt}(\mathrm{MC})\,dt = \Big[\mathrm{MC}^2\Big]_0^\infty - \int \mathrm{MC}\frac{d\mathrm{C}}{dt}\,dt.$$

But since the time of vibration of the suspended coil is comparatively large, the change in C is practically complete before the coil has moved far from its equilibrium position. Hence M may be treated as constant for the whole range of integration, and thus, since $\mathrm{C}^2 = \mathrm{C}_0{}^2$ at both limits, the value of the integral is zero.

As regards the fourth integral,

$$-\int \mathrm{C}\frac{dc}{dt}\,dt = -\Big[\mathrm{C}c\Big]_0^\infty + \int c\frac{d\mathrm{C}}{dt}\,dt = \int c\frac{d\mathrm{C}}{dt}\,dt,$$

since c vanishes at both limits.

Collecting these results, we have

$$(\mathrm{W}_1 + \mathrm{X}_1)\,\mathrm{A}l = \frac{\mathrm{N}l\mathrm{C}'\mathrm{P}}{n\phi}\,\theta_1 + \int c\frac{d}{dt}\left(n\mathrm{AB} + \frac{\mathrm{LN}l}{n}\mathrm{C}\right)dt - \mathrm{T}' + \mathrm{T}.$$

The integral in this expression is the correction which makes its appearance when we take account of the finite conductivity of the secondary circuit. It will suffice to use in it the value of Sc which obtains when Ldc/dt is negligible in comparison with

$d(nAB + MC)/dt$, viz., $Sc = -d(nAB + MC)/dt$.* We may also treat M as constant. When this is done we obtain

$$(W_1 + X_1)Al = \frac{NlC'P}{n\phi}\theta_1 - \frac{1}{S}\int\left(nA\frac{dB}{dt} + \frac{LNl}{n}\frac{dC}{dt}\right)\left(nA\frac{dB}{dt} + M\frac{dC}{dt}\right)dt - T' + T.$$

If we add together the results for a pair of semi-cycles, the quantities T and T′ disappear when a cyclic state has been established, for then the magnetic energy is the same at the end as at the beginning of a cycle. We may then replace $W_1 + X_1$ by $W + X$ and θ_1 by $\theta_1 + \theta_2$, but we must remember that the integral is to be taken completely round a cycle.

The expression can be put into a more convenient form if we notice that by the rules of approximation c is to be taken as zero in the terms under the integral sign. We thus replace $4\pi NC$ by H, and then writing $4\pi N dB/dH . dC/dt$ for dB/dt, we have, on division by al,

$$W + X = \frac{NC'P}{An\phi}(\theta_1 + \theta_2) - \frac{N}{SAn}\int\left(\frac{4\pi n^2 A}{l}\frac{dB}{dH} + L\right)\left(4\pi nAN\frac{dB}{dH} + M\right)\frac{dC}{dt}dC. \quad . \ (12),$$

the final formula.

Now, unless the specimen be so thick, and the variation of H so rapid, that during part of a complete cycle the *average* induction for the whole section increases while the applied magnetic force diminishes, dB/dH is always positive. Hence, since dC/dt always has the same sign as dC, the correcting integral is always positive. Thus the true value of $W + X$ is less than that calculated from the throw of the dynamometer on the assumption that S is infinite.

§ 9. Before we can apply equation (12) in finding W, we must either know that X

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* We can easily obtain with rough approximation the condition which must be satisfied in order that Ldc/dt may be negligible in comparison with $d(nAB + MC)/dt$. From (11) we have

$$\left(S + L\frac{d}{dt}\right)c = -\frac{d}{dt}(nAB + MC),$$

whence

$$-Sc = S\left(S + L\frac{d}{dt}\right)^{-1}\frac{d}{dt}(nAB + MC) = \left(\frac{d}{dt} - \frac{L}{S}\frac{d^2}{dt^2}\ \ldots\right)(nAB + MC).$$

Thus the approximation $-Sc = d(nAB + MC)/dt$ is valid if $L/S . d^2(nAB + MC)/dt^2$ is small in comparison with $d(nAB + MC)/dt$. A sufficient idea of the magnitudes involved is obtained by treating B as proportional to C. In this case $L/S . d^2C/dt^2$ must be small in comparison with dC/dt. But, if E denote the voltage of the battery, the characteristic of C is

$$KdC/dt + RC = E,$$

where R is now the resistance of the whole of the primary circuit, and K depends mainly upon the choking coil (§ 33) in the circuit. Hence, treating K as constant,

$$Kd^2C/dt^2 + RdC/dt = 0.$$

Thus the condition for the validity of the approximation can be expressed in simple form by saying that L/S must be small in comparison with K/R.

is negligible or else be able to determine it. In most of our experiments the specimens have been fine wires, and X has been insignificant in comparison with W, but with rings of solid metal, such as those used by Mr. R. L. WILLS,* with a sectional area of over 1 sq. centim., X becomes of real importance. Now from the behaviour of the eddy currents in a rod of circular section, we may assert that when the "time constant" (inductance/resistance) of the primary circuit is large compared with $\pi\mu a^2/\sigma$, where a is the radius of the largest circle inscribable in the section, and μ and σ are the permeability and specific resistance of the material, then the eddy current at any point may be calculated on the assumption that the magnetic induction has at any instant the same value at all points of the section (Appendix I.). In this case the eddy current at any given point is proportional to $d\mathrm{B}/dt$, and hence, as the method of "Dimensions" shows, the *space-average* of the rate at which heat is generated per unit volume may be written

$$d\mathrm{X}/dt = \mathrm{QA}\,(d\mathrm{B}/dt)^2/\sigma = 16\pi^2\mathrm{N}^2\mathrm{AQ}/\sigma\,.\,(d\mathrm{B}/d\mathrm{H})^2\,.\,(d\mathrm{C}/dt)^2 \quad . \quad . \quad . \quad (13),$$

where Q is a constant depending upon the *form* of the section. On reference to the meaning of X it will be seen that the total rate of heat production by eddy currents per unit length of the specimen is $\mathrm{QA}^2(d\mathrm{B}/dt)^2/\sigma$. For a circular section $\mathrm{Q} = 1/8\pi = \cdot03979$, and for a square section $\mathrm{Q} = \cdot03512$. (Appendix I.)

We can now write (12) as follows :—

$$\begin{aligned}\mathrm{W} &= \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}(\theta_1 + \theta_2) \\ &\quad - \int \left[\frac{16\pi^2\mathrm{N}^2\mathrm{QA}}{\sigma}\left(\frac{d\mathrm{B}}{d\mathrm{H}}\right)^2 + \frac{\mathrm{N}}{\mathrm{SA}n}\left(\frac{4\pi n^2\mathrm{A}}{l}\frac{d\mathrm{B}}{d\mathrm{H}} + \mathrm{L}\right)\left(4\pi n\mathrm{AN}\frac{d\mathrm{B}}{d\mathrm{H}} + \mathrm{M}\right)\right]\frac{d\mathrm{C}}{dt}\,d\mathrm{C} \\ &= \mathrm{U} - \mathrm{X} - \mathrm{Y} \quad (\S 13) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).\end{aligned}$$

If the specimen be built up of p similar wires so that the total cross-section is A, then X is $1/p$ times the value for a single wire of section A. The p wires must be insulated from each other so that there are no eddy currents from wire to wire.

The quantity $d\mathrm{B}/d\mathrm{H}$, which occurs in the correcting integral, is for a given specimen a nearly definite (double valued) function of H and therefore of C, for given limits $\pm\mathrm{C}_0$, provided that the "time constant" of the primary circuit is large compared with $\pi\mu a^2/\sigma$. (Appendix I.) Quite apart from the effects of eddy currents, dB/dH appears to depend slightly upon the rate of variation of H, but so slightly that we may for our present purpose consider dB/dH as having two definite values for any given value of C when C varies between given limits.

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* Mr. R. L. WILLS, of St. John's College, 1851 Exhibition Scholar, began in 1900, at the Cavendish Laboratory, a series of experiments on the effect of temperature upon the energy dissipated through hysteresis in iron and alloys of iron, in continuation of his work on the "Effects of Temperature on the Magnetic Properties of Iron and Alloys of Iron" ('Phil. Mag.,' July, 1900). At the suggestion of Professor J. J. THOMSON he employed the method described in the present paper, while we gladly furnished him with some of the apparatus employed by us in our own researches. It was plain that in the specimens used by Mr. WILLS the energy dissipated by eddy currents was comparatively much greater than in our own specimens, and so might be far from negligible. We were thus led to extend the theory so as to take account of the eddy current loss, and to devise a method of determining that loss,

Hence for a given specimen, the correction to be subtracted from the value of W calculated from the throw of the dynamometer by the elementary theory will be increased p-fold if, for every value of C which occurs in the cycle, dC/dt be increased p-fold. The part of the correction involving the resistance, S, of the secondary circuit, is inversely proportional to S. It also depends upon L and M and diminishes with those quantities.

When the secondary coil contributes practically the whole of L and M we have approximately

$$L = 4\pi n^2 (G - A)/l, \qquad M = 4\pi N n (G - A) \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where G is the mean area of one turn of the secondary coil.

In this case, unless G is many times greater than A, L and M are negligible in comparison with the quantities added to them in (14), since for iron dB/dH is generally very large.

Referring to the definition of L and M in § 8, we see that as far as these quantities are concerned the correction is made as small as possible by winding the secondary windings as closely as possible upon the iron, and by making the self-induction of the rest of the secondary circuit as small as possible. To secure the latter point the earth inductor employed to standardise the dynamometer should be removed from the secondary circuit, and a non-inductive coil of equal resistance inserted in its place when the dynamometer readings for W are taken.

§ 10. For different specimens of iron the correction will depend upon dB/dH. Indeed since L and M can be made comparatively small, the correction depends mainly upon the *square* of dB/dH.

It is, of course, impossible to assign any definite value to dB/dH for any particular specimen, for the ratio varies very greatly for different parts of the B—H curve. All we can say is that if for any part of the B—H curve dB/dH has a very high value, and if for the corresponding value of C the rate dC/dt is not very small, then this portion of the B—H curve may make a considerable contribution to the correcting integral in (14).

Now the maximum value of dB/dH varies greatly in different kinds of iron, being small in steel and very large in good soft iron. According to EWING'S experiments in the case of considerable magnetic forces, the maximum value of dB/dH for glass-hard steel is about 300, while for soft-iron wire, suitable for the manufacture of transformers, its value rises to 13,000. The maximum value of $(dB/dH)^2$ for the iron is accordingly about 2000 times its value for the steel. It is thus evident that though the method may give fairly accurate results for steel without any special precautions to secure that $d\text{C}/dt$ should never be large, it may give quite inaccurate results for soft iron unless the proper precautions are taken. A considerable portion of the time spent upon this research was occupied in investigating the precautions necessary to secure accuracy.

§ 11. If we consider the effect of changing the area of section of the iron we shall see that the part of the correction due to the conductivity of the secondary circuit increases as the section diminishes. For neglecting L and M, (14) becomes

$$\text{W} = \frac{\text{NC}'\text{P}}{\text{A}n\phi}(\theta_1 + \theta_2) - 16\pi^2\text{N}^2\text{A}\left(\frac{\text{Q}}{\sigma} + \frac{n^2}{\text{S}l}\right)\int\left(\frac{d\text{B}}{d\text{H}}\right)^2\frac{d\text{C}}{dt}\,d\text{C}. \quad . \quad . \quad . \ (16).$$

Now for a given kind of iron, if we wish to obtain the same throw of the dynamometer, when we halve A we must double n. Thus we must regard An as determined by the sensitiveness of the dynamometer. Hence the part of the correction which depends upon S is proportional to n, and therefore inversely proportional to A. This explains the difficulty we met with in the earlier stages of the work. We were able to get accurate results for specimens built up of fine wires, with a total section of 1 sq. centim., though we failed to do so for specimens with a section of 1 or 2 sq. millims.

The part of the correcting term in (16) depending upon S is inversely proportional to l. The circumference of the ring should therefore be large. When, as in most of our experiments, a straight wire of finite length is used, the secondary coil should be wound upon a long bobbin and not be heaped up at one part of the wire.

§ 12. *Effect due to Air alone.*—If there be no iron in the system B = H everywhere, and W = 0 ; if, further, there be no metal, *e.g.*, brass, inside either the primary or the secondary coil X = 0 also. In this case (14) reduces to

$$\frac{\text{C}'\text{P}}{\phi}(\theta_1 + \theta_2) = \frac{1}{\text{S}}\left(\frac{4\pi n^2\text{A}}{l} + \text{L}\right)(4\pi n\text{AN} + \text{M})\int\frac{d\text{C}}{dt}\,d\text{C} \quad . \quad . \quad . \quad . \ (17).$$

Since dt is necessarily positive, $d\text{C}/dt$ has the same sign as $d\text{C}$, and hence the integral cannot vanish. There will, therefore, always be a small throw of the dynamometer even when there is no metal in the coils. But since $d\text{B}/d\text{H}$ is always positive and is generally large, it is easily seen that unless L and M (§ 8) are very large in comparison with the quantities to which they are added in (17), the throw, when the iron is out, is small compared with the part of the throw which is due to the correcting integral in (14), when the iron is in. This deduction we verified by experiment, for in many trials we never found any case in which we could detect any throw when there was no metal in the magnetising coil.

Determination of the Correction for the Finite Conductivity of the Secondary Circuit, and of the Energy dissipated by Eddy Currents.

§ 13. It is convenient to have a single symbol for the quantity $\text{NC}'\text{P}(\theta_1 + \theta_2)/\text{A}n\phi$, and we shall use U for this purpose. Thus U is the value of the hysteresis loss

Let the poles of a battery of voltage E be joined to the points F, G (fig. 2), and let the resistance between F and G through the battery be T. Let F and G be also connected through the primary coil, the fixed coils of the dynamometer, and a choking coil, the resistance of this part being R. Let the steady current, when F and G are not in contact, be C_0. Now let F and G be put into contact. The current in the primary coil then has the characteristic

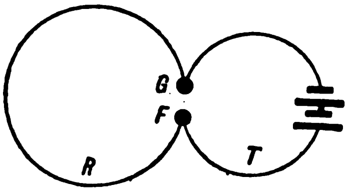

Fig. 2.

$$K\,dC/dt + RC = 0 \quad \ldots\ldots\ldots\ldots \quad (21),$$

where K depends upon C, since it involves the value of dB/dH for the core of the choking coil (§ 33).

After the current has sunk from C_0 to some very small value, let F and G be separated again, but before the separation let E be reversed. The characteristic is now

$$K\,dC/dt + (R + T)\,C = -E = -(R + T)\,C_0 \quad \ldots\ldots \quad (22).$$

The current now gradually attains the value $-C_0$. The whole process is practically complete in a fraction of the time of vibration of the dynamometer coil.

The key described in § 32 carries out this process exactly, provided that the resistances D are made zero. We find that the time for which no E.M.F. acts on the circuit through the primary coil is about $\frac{1}{4}$ second, while a rough calculation shows that with our apparatus the current sinks very nearly to zero in that time.

Suppose now that E, R and T are all increased p-fold. Then C_0 remains unchanged, and for any given value of C, during the whole time of variation, dC/dt is also increased p-fold. Thus, if the ratio R/T be kept constant as well as the maximum current C_0, dC/dt for any value of C is proportional to E. Hence

$$Z = X + Y = zE \quad \ldots\ldots\ldots\ldots \quad (23),$$

where z is a quantity independent of E. A method of finding Z similar to that employed for Y is now available. For if U, Z correspond to E, and U′, Z′ to E′, then $U' - U = Z' - Z$, since W remains unaltered, *unless the hysteresis depends upon the speed at which the magnetic changes occur.* But $Z/E = Z'/E'$, and hence

$$Z = (U' - U)\,E/(E' - E) \quad \ldots\ldots\ldots\ldots \quad (24).$$

Thus, by observing the values of U found in two experiments with two values for E, the values of Z and Z′ can be determined.

Practical examples of these two methods are given in §§ 36, 41, 42.

The secondary current is given by

$$Sc = An\,dB/dt + (G - A)\,n\,dH/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

Substituting for C from (27) we find

$$4\pi N'S \int Ccdt = An \int (H + h)\,dB + (G - A)\,n \int (H + h)\,dH.$$

Remembering that $\int HdH = 0$ for a cycle or a semi-cycle, we have, on using (7),

$$\frac{C'P}{\phi}\,\theta = S \int Ccdt = \frac{An}{4\pi N'} \left\{ \int HdB + \int hdB + \frac{G - A}{A} \int hdH \right\} \quad . \quad . \quad . \quad (29).$$

Hence the throw of the dynamometer is larger than that corresponding to $\int HdB$, and thus if we calculate W from the throw of the dynamometer we must subtract a correction depending upon the integrals $\int hdB$ and $\int hdH$. The correction can be found only when h is known both as a function of B and also as a function of H.

§ 18. Lord RAYLEIGH* was the first to show how to correct an I—H curve for the demagnetising force. He supposes that the specimen is an ellipsoid, and that the applied magnetic force is constant throughout its volume; the magnetic quantities H, B, I, and h are, in this case, constant throughout the specimen, and thus we can write $h = p\mathrm{I}$, where p is a constant factor (usually denoted by N). This relation between H and I allows the I—H curve for an infinitely long ellipsoid to be deduced from the curve for a short ellipsoid by "shearing" through a distance everywhere proportional to I.

Some investigators, supposing that the demagnetising force, h, at the centre of a long cylinder is the same as that for the ellipsoid inscribable in the cylinder, have applied Lord RAYLEIGH'S construction to the case of long cylinders. Others, such as Dr. H. DU BOIS,† though avoiding this error, have assumed that h can be expressed in the form $h = p\mathrm{I}$, where p is a function of the ratio of the length to the diameter, but is independent of I, the intensity of magnetisation at the centre of the cylinder. But it is easy to see that, quite apart from the influence of hysteresis, p cannot be constant, since the permeability of iron is not independent of the magnetic force. For, on account of the demagnetising action of the ends of the rod, the magnetic force near the ends differs from that near the centre of the rod, and thus the rod has, in effect, different values of μ in different parts. If μ were constant for each part, p would still be constant, but in the actual case, when the applied magnetic force is changed, the value of μ for each part of the rod changes, the change being greater for some parts of the rod than for others. The magnetism which appears at any part of the rod is thus not proportional to the value of I at the centre of the rod, and accordingly p is not constant. The curves obtained by Mr. C. G. LAMB* show to how great an extent the distribution of magnetism depends upon the applied magnetic force.

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* "On the Energy of Magnetised Iron," 'Phil. Mag.,' 1886, vol. 22, p. 175, or 'Scientific Papers,' vol. 2, art. 139.

† 'The Magnetic Circuit in Theory and Practice,' p. 41. Dr. DU BOIS, however, describes (p. 123) the experiments of LEHMANN upon the magnetisation of a toroid with a radial slit, and points out that LEHMANN'S results show that the demagnetising factor increases, but only gradually, as I increases up to about half its maximum value; beyond this point the increase is more rapid. The experiments were not arranged so as to show the effects of hysteresis.

But, further, if the rod be put through cycles of magnetic changes, as the ends of the rod are approached the range of the magnetic force changes, on account of the demagnetising action of the ends. Hence, since the behaviour of iron in regard to hysteresis depends upon the range of the magnetic force, the demagnetising force, h, at the centre does not depend simply upon the intensity of magnetisation, I, at the centre of the rod, but depends also upon the manner in which that value of I has been reached. Thus h will exhibit hysteresis with respect to I and hence also to B and H, each quantity referring to the centre of the rod.

§ 19. We give in Appendix II. a simple experimental method of determining h both as a function of B and as a function of $4\pi N'C$, the magnetic force due to the solenoid. Since $\int h dh = 0$ we may replace H by $4\pi N'C$ in the second correcting integral in (29). When h has been found in terms of B and H by this method, the values of $\int h dB$ and $\int h dH$ can be found by measurement of the areas of the h—B and h—H curves. Now in (29) the integral $\int h dH$ has the factor (G—A)/A, and hence disappears when G = A, *i.e.*, when the secondary coil is wound infinitely closely upon the iron. In most cases it will be convenient that the secondary coil should be wound upon a tube large enough to pass easily over the rod, and G will then be considerably larger than A, though G/A need not exceed 10 or 20 except for very thin rods. Now for a given step dB the corresponding step dH is always by comparison small, since dH/dB is very small—perhaps 1/10000—in the steep parts of the cyclic B—H curve, and never rises above 1/100 unless the iron is well "saturated." Hence the term (G—A)/A . $\int h dH$ will generally be negligible in comparison with $\int h dB$.

For a bundle of ten iron wires 47 centims. long and ·0412 sq. centim. in total area of section, placed in a solenoid 47 centims. long, and furnished with a secondary for which G = ·785 sq. centim, so that (G—A)/A = 18, we found (§ 78)

$$\int H dB = 89200\,, \quad \int h dB = 949\,, \quad \frac{G-A}{A}\int h dH = 1{\cdot}57 \times 18 = 28{\cdot}2\,.$$

In this case the second integral introduces a correction of about 1 per cent., and the third one is negligible. The limits of B were ± 9450, and those of H ± 10·65. The value of h at these limits was ± ·106.

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* "On the Distribution of Magnetic Induction in a Long Iron Bar," 'Proc. Physical Society,' vol. 16, p. 509, or 'Phil. Mag.,' September, 1899.

For a single wire 47 centims. long and ·00412 sq. centim. in section, placed in a solenoid 60 centims. long, we had (G—A)/A = 37, and found (§ 79),

$$\int \mathrm{H}d\mathrm{B} = 102100\,, \quad \int hd\mathrm{B} = 72{\cdot}3\,, \quad \frac{\mathrm{G}-\mathrm{A}}{\mathrm{A}}\int hd\mathrm{H} = {\cdot}118 \times 37 = 4{\cdot}36\,.$$

The limits of B were ± 9850, of H ± 10·67, and of h ± ·0084.

§ 20. These corrections are also applicable to the area of a cyclic B—H curve drawn by the aid of a ballistic galvanometer. With respect to the maximum induction B_0, when the current C_0 is reversed, the throw of the galvanometer measures $\mathrm{AB}_0 + (\mathrm{G}-\mathrm{A})\,\mathrm{H}_0$, where it will be sufficient to take H_0 as $4\pi\mathrm{N}'\mathrm{C}_0$, since $(\mathrm{G}-\mathrm{A})\,\mathrm{H}_0$ is small compared with AB_0. But in finding the maximum magnetic force H_0, the correction h_0 may not be negligible, and must be subtracted from the quantity $4\pi\mathrm{N}'\mathrm{C}_0$. The value of h_0 corresponding to H_0 is easily found by the method described in Appendix II.

DESCRIPTION OF THE APPARATUS.

§ 21. We now pass on to describe the instruments employed in the experiments. It will perhaps conduce to clearness if we leave till last the description of the special appliances which were designed in order to secure accuracy.

The Ampere-Meter.

§ 22. In all except the preliminary part of the work a Weston direct reading ampere-meter was used for the measurement of the primary current. The instrument only read up to 1·5 amperes, but when the largest number of windings on the solenoid (§ 24) was employed this current gave a magnetic force of about 108 C.G.S. units. The instrument reads in one direction only, and thus it is necessary to join one of its terminals directly to one pole of the battery, so that the current always flows through it in the same direction.

The Earth Inductor.

§ 23. The earth inductor, which served throughout our experiments to produce a standard change of induction, was a simple coil of 100 turns, having a mean radius of about 18·2 centims.; its resistance is 1·345 ohms. Using a Clark cell as a standard of E.M.F., the change in the number of linkages of lines of induction with the circuit when the coil is turned over through 180° about a horizontal axis was found to be $8{\cdot}72 \times 10^4$ C.G.S. units.

We prepared a resistance coil of copper wire with the same resistance as the earth inductor. We could thus substitute this coil for the earth inductor when we desired, for the reason given in § 9, to get rid of the self-induction of the earth inductor and at the same time to keep the resistance of the circuit unchanged. Since both coils were of copper, their resistances were equal when their temperatures were equal.

The Magnetising Solenoid.

§ 24. In our experiments the specimens of iron have taken the form of straight wires, and in consequence the magnetising coil has been a straight solenoid. We used a straight wire instead of a ring in order to be able to apply tension or torsion to the specimen. The solenoid is formed of several independent layers of wire wound upon an ebonite tube 47 centims. in length. It is essential that the tube should be of non-conducting material, for otherwise the currents induced in it would cause the magnetic force to differ considerably from that calculated from the formula $H = 4\pi NC$; this effect would cause a considerable error in the measurement of the hysteresis. By combining the independent layers in different ways we could vary the magnetic force due to unit C.G.S. current by steps of about 50 from 50·55 to 718·6 C.G.S. units. The magnetic forces due to unit current in each of the four coils are as follows :—

AA, 50·55, BB, 50·55, XY, 212·2, MN, 405·1.

The resistances of the coils are ·54, ·54, 2·00, and 4·85 ohms respectively.

The Secondary Coils.

§ 25. The secondary coils were formed of fine insulated copper wire wound on narrow tubes of ebonite or glass, through which the specimen passed. Insulating material was used for the bobbins to avoid the induction of currents in them. In our earliest experiments it was found that metal tubes caused very large errors by the action of the currents induced in them. Several coils were used with the dynamometer, the number of windings varying from 300 to 1285; the coil of 1285 turns had a bobbin 11·5 centims. long, thus conforming to the recommendation of § 11. The mean area of this coil was ·785 sq. centim.

The Ballistic Galvanometer.

§ 26. The ballistic galvanometer was made by one of us, with some assistance from Mr. W. G. PYE. The magnet system consists of two vertical magnetised wires, each about 11·3 centims. long and 1 millim. in diameter. These are fixed parallel to each other at a distance of 1 centim., the north pole of one magnet being opposite the south pole of the other. When parallelism is secured, the magnetic system is necessarily astatic, however much the magnets may vary in strength. The system is suspended from a torsion head by a single phosphor-bronze wire $\frac{1}{1000}$ inch in diameter and 12 centims. in length. The torsion of this bronze wire supplies the restoring couple. There are four coils, each containing about 250 turns of No. 18 B.W.G. copper wire, the total resistance of the coils in series being 2·14 ohms. The coils are arranged so that the upper end of the magnet system is at the centre of the upper pair of coils and the lower end is at the centre of the lower pair of coils. The motion of the magnets is observed by the aid of a lamp and scale.

The galvanometer is very sensitive, and thus measurements of B for quite thin iron wires can be made with a comparatively small number of turns of secondary winding; this is often a point of some convenience.

We have found the instrument very efficient. The time of vibration, 12 seconds, is long enough to enable the ballistic throws to be read with ease. Since the needle is practically astatic, the zero depends only on the action of the bronze wire and not upon the earth's magnetic field. The result is that the zero is remarkably constant, often not changing by more than one-tenth millim. during several hours. The only disadvantage is that there is so little damping that, to bring the needle to rest in any reasonable time, it is necessary to use a coil of wire placed near the galvanometer in conjunction with a Leclanché cell and a tapping key. A little practice enables the observer to bring the spot quickly to rest.

The restoring couple varying as the angle of deflexion instead of as its sine, the time integral of a transient current is proportional to the angle of throw instead of the sine of half that angle. We verified by experiment that this law is accurately obeyed.

With rise of temperature the magnetic moment of a magnet diminishes, and we consequently found that with the same resistance in circuit the throw due to a given change of induction was rather less on a hot than on a cool day.

The logarithmic decrement depends upon the resistance in circuit with the galvanometer. But in every case the resistance of that circuit was kept constant during a set of observations, and thus all error due to this cause was avoided.

When we desired to draw a cyclic B—H curve for a specimen of iron, we practically followed the method described by Professor J. A. EWING*.

The Electro-dynamometer.

§ 27. The electro-dynamometer employed in the later experiments has a pair of fixed coils, each formed of 250 turns of No. 20 B.W.G. cotton-covered wire wound on ebonite bobbins. The mean radius of the coils is about 5 centims., and their resistance in series about 4 ohms. In order to avoid induced currents there were no large pieces of metal near the coils. In the central space hangs the suspended coil. This consists of 190 turns of No. 40 B.W.G. silk-covered wire. The mean radius of the coil is 1·7 centims. and its resistance about 28 ohms. The coil is attached along a diameter to a stiff brass wire, whose upper end carries a mirror. The mirror is placed so far above the centre of the coils that the beam of light from the lamp passes above the outside of the fixed coils. The moving coil is suspended by a phosphor-bronze wire $\frac{1}{1000}$ inch in diameter, and about 4 centims. long. The other connexion inductive coil D of 100 ohms being put in parallel with the primary circuit, PC, as in fig. 3. When the self-induction of the primary circuit is large the current does not stop very rapidly when contact is broken, for the primary current can still flow on through D, and thus the decay of the current is much less rapid than when it has to overcome the high resistance of the spark, as in § 28. When contact is re-established the current rises in exactly the same manner as with the simple rocking key. The practical objection to the method is that it does not allow of the current being directly measured by the convenient Weston ampere-meter, for the current must always flow in the same direction through this instrument. A Kelvin graded galvanometer or a shunted mirror galvanometer, MG, was accordingly used with this key.

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* "Magnetic Induction in Iron and other Metals," 3rd Ed. Revd., § 192.

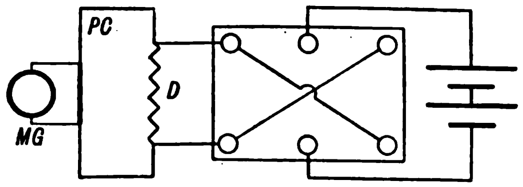

Fig. 3.

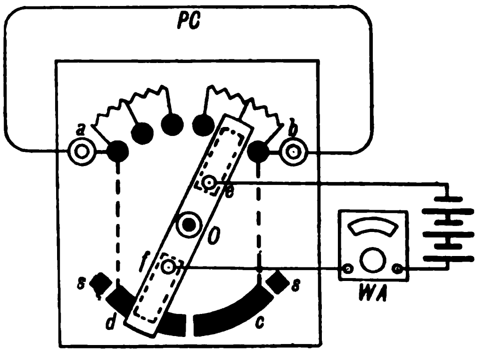

Fig. 4.

§ 31. The fourth reversing key used is shown in fig. 4. The battery is connected through the Weston ampere-meter WA with the terminals *e*, *f*, on an ebonite arm working about a pivot at O. The terminals *e* and *f* are connected to two brass springs, one at each end of the arm. One spring slides over a series of studs, while the other slides over two brass sectors *c*, *d*. Half the studs are connected to the terminal *a* by resistance coils in the manner shown in fig. 4, the remaining studs being connected to the terminal *b*. The sectors *d*, *c* are connected to the terminals *a*, *b*; the primary circuit, PC, joins the key at *a* and *b*. The resistance coils which connect the studs have resistances of 40 and 200 ohms. The two stops *s*, *s* serve to limit the motion of the arm, so that in its extreme positions the spring connected with *e* presses upon the stud nearest to either *a* or *b*. The width of the springs is sufficient to ensure that before the spring leaves one stud it is in contact with the next one.

When the arm is in either of its extreme positions the total resistance of the circuit is R + T, where T is the resistance of the battery and ampere-meter and R is the resistance of the rest of the circuit. When the spring connected with *e* comes on to the next stud the resistance is R + T + 40, and one more step makes the resistance R + T + 240. When the spring comes to the next stud the E.M.F. acting on the primary circuit is reversed.

Since the spring connected with *e* does not leave any stud before it touches the next one, as the arm is moved from one side to the other, the primary circuit is never broken except possibly in the central position of the arm. The circuit will be broken for an instant in the central position unless the spring connected with *e* touches the two central studs at the same instant that the spring connected with *f* touches both the sectors *c*, *d*. It is more or less a matter of chance whether this break of the circuit occurs, but if it does occur it is only after the current has been reduced to a small value by the introduction of the large resistance of 240 ohms. Except for this uncertainty we may say that the current is reversed in several steps which (except possibly those occurring in the uncertain part of the motion) are not sudden because of the great self-induction of the choking coil (§ 33). Though (with a possible exception) there is no sudden change in the current, the rate of variation of the current is no doubt much greater at some stages of the change than at others.

The key generally worked well; it was, however, subject to slight uncertainties.

§ 32. A fifth key was designed in 1900. Our aim was to ensure that the primary circuit should never be broken, and also that the resistances introduced into the circuit should be as small as possible, so that the rate of change of the current should, at every part of its variation, be as small as possible.

The battery is connected through the Weston ampere-meter, WA (fig. 5), with the terminals *g*, *h* on an ebonite arm working about a pivot at O. The terminals *g*, *h* are connected to two brass springs, *k*, *l*, one at each end of the arm. These springs slide over a series of sectors *a*, *b*, *c*, *d*, *e*, cut out of a brass ring. The sectors *a* and *b* and

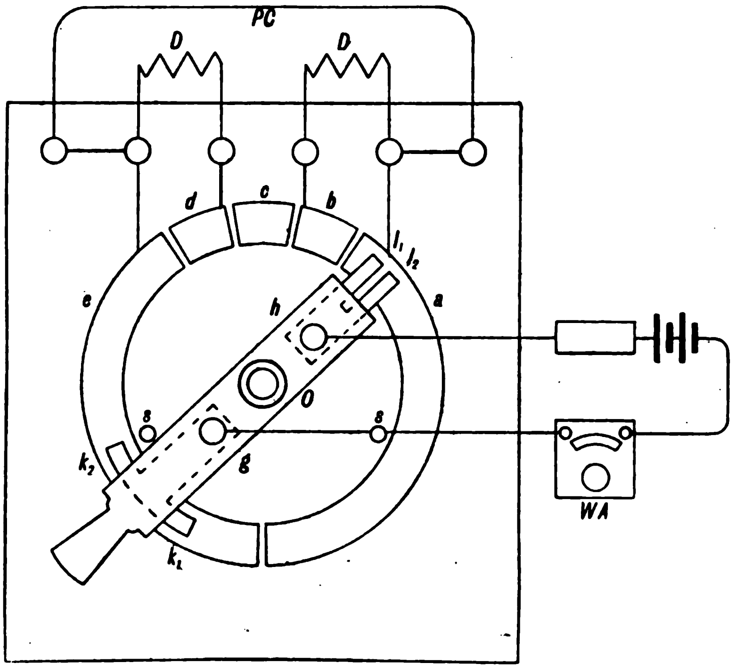

Fig. 5.

the sectors d and e are connected by the resistance coils D, D, while the sector c is insulated. The spaces between the sectors, about 1·5 millims. wide, are filled with ebonite so that the springs pass smoothly over them. The primary circuit PC joins the sectors a, e. The two stops s, s* arrest the motion of the arm when l has got well on to either a or e, so as to be clear of b or d. To avoid any possible break in passing from one sector to the next, the spring l is made in two portions, l_1, l_2, each of which presses upon the sectors. The spring k is arranged so as to touch the sectors only at its ends, k_1, k_2. The lengths of the sectors and of the springs are arranged so that the key performs the reversal in the following manner:—At the start l is on a, and k is on e; both parts of l reach b before k reaches a. While l is on b the primary current falls from $E/(R + T)$ towards the limit $E/(R + T + D)$, where E is the voltage of the battery, T the resistance of the battery and ampere-meter, and R the resistance of the rest of the circuit. Before l leaves b, k_1 reaches a, and then there is no E.M.F. acting round the primary circuit except what arises from the very small internal resistance of the key, and then also the resistance offered to any current flowing in the primary circuit is reduced from $R + T + D$ to R. When l is on c, one pole of the battery is insulated, and there is no E.M.F. at all acting on the primary circuit, whose resistance remains R. When l reaches d, and k_2 is still on e, a very small E.M F., due to the internal resistance of the key, acts in the reversed direction on the primary circuit. The resistance offered to a current in the primary circuit is still R. While l is on d and before it reaches e, k_2 leaves e; the resistance of the circuit is now raised to $R + T + D$, and the voltage E is introduced. When l reaches e the resistance is diminished to $R + T$, the voltage remaining E.

The resistances D need not be large; they are only used at all in order to save the battery and the ampere-meter. If the resistances D were made very small, and if T were small also, a large current would flow through the ampere-meter when l is on b, k_1 on a, and k_2 on e, causing damage to the meter. The resistances D need only be great enough to prevent the ampere-meter from being damaged by too strong a current.

We have only had opportunity to use this key in a few experiments, but as far as we have tested it we are satisfied with its action. The motion of the spot of light on the scale of the dynamometer is now quite free from the sudden jumps which it exhibited when the key of § 31 was used.

The Choking Coil.

§ 33. The "choking coil," which was inserted in the primary circuit to introduce great self-induction, has a core built up of 149 armature rings of an average thickness ·0714 centim., the inner and outer radii being 8·5 and 11·7 centims. respectively. The mean circumference is thus 63·7 centims. and the cross-section 34·0 sq. centims. If the core had been solid and not laminated, the eddy currents induced in it would have rendered the coil ineffective in "choking" any sudden variation of the current due to a sudden change of E.M.F. or of resistance. The iron, which was supplied by Messrs. CROMPTON, was tested for magnetic quality with the following results :—

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* By an oversight the wire from g to WA has been drawn through s.

H_0	·182	·371	·570	·854	1·09	1·53	2·10	3·37	4·98	6·90	8·58	11·15	16·20
B_0	35·4	103	190	390	660	1510	3090	5260	7300	8560	9470	10200	11500
μ	194	278	334	457	606	987	1470	1560	1470	1240	1100	914	710

In magnetic quality, the iron is very nearly the same as the soft thin sheet iron (Ring V.) tested by Professor EWING and Miss KLAASSEN.*

The core was wound with three independent layers of cotton-covered copper wire as follows :—

Layer.	B.W.G.	Turns.	Ohms.	Magnetic force per unit C.G.S. current.
1	No. 15.	225	·475	44·5
2	No. 15.	205	·50	40·4
3	No. 18.	265	1·44	52·3

Stout wire was used in order to avoid any considerable increase in the resistance of the primary circuit. This resistance must be kept low if the choking action is to be efficient.

§ 34. This choking coil is perhaps unnecessarily large. It might be better to use a core smaller both in diameter and cross-section and to form each of the coils on it with wire of the same gauge as that used for a corresponding coil on the solenoid. By this expedient, if we always employ corresponding coils on the choking coil and the solenoid, the magnetic force in the choking coil has the same value as in the solenoid. Thus, if the iron used in the choking coil is of approximately the same quality as the specimen under test, the coil will most effectively choke the primary current approximately at the time when the choking is most needed, viz., when the value of dB/dH for the specimen has its greatest value, as was explained in § 10. If the iron plates used in the core of the choking coil are of good quality it is very unlikely that any specimen will be so "soft" in the magnetic sense as to have a value of dB/dH so many times greater than the value of dB/dH for the core as to lead to inaccurate measurements. If the specimen is of "hard" iron or steel the value of dB/dH for the specimen is by comparison so small that it is of little consequence if the most efficient choking action takes place when dB/dH for the specimen is not at its maximum value. It would be necessary to make the core smaller than in the coil used by us, so that in spite of the smaller section of the wire the resistance might still be small.

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* "Magnetic Qualities of Iron," 'Phil. Trans.,' A, vol. 184 (1893), p. 1003.

Arrangement of the Apparatus.

§ 35. After this description of the apparatus we proceed to explain, by the help of fig. 6, how it was arranged. The battery SC, consisting of one or more small storage cells, was connected through the adjustable resistance R_1 and the Weston ampere-meter WA to the special reversing key K_1. An adjustable length of German-silver wire included in the circuit enabled us to keep the current to a definite value for any length of time. On leaving the key K_1, the current passes round the fixed coils of the electro-dynamometer ED, round the magnetising solenoid MS and the choking coil CC, and finally round a compensating coil F. By adjusting the position of F, the small effect upon the ballistic galvanometer of the current passing through ED and MS was completely annulled.

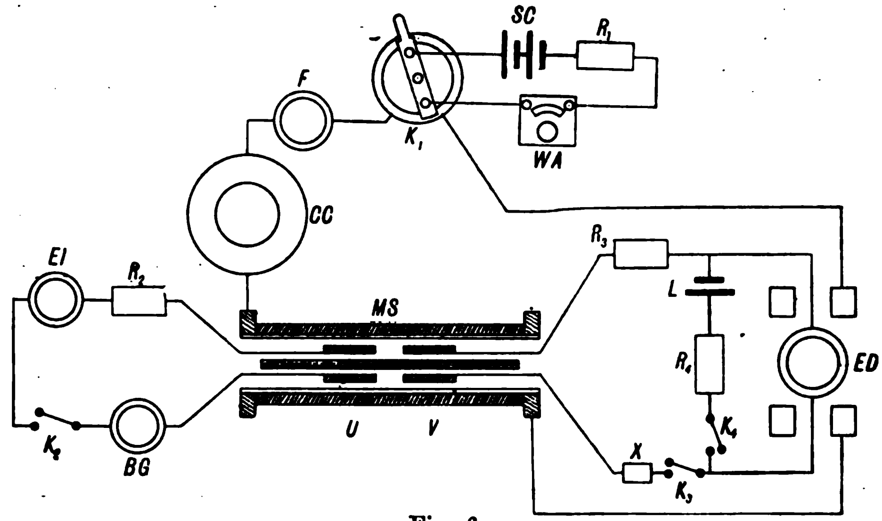

Fig. 6.

The circuit of the ballistic galvanometer BG contains a resistance box R_2 and the earth-inductor EI, as well as the secondary coil U and the key K_2.

The circuit of the suspended coil contains the resistance box R_3, the secondary coil V, the key K_3, and the resistance coil X formed of copper wire adjusted to have the same resistance as the earth-inductor EI, as described in § 23.

To bring the suspended coil to rest a current of the order of 1/200,000 ampere was sent through the suspended coil by depressing the tapping key K_4, a Leclanché cell L providing the current. In the actual experiments a system of shunts was used instead of the single high resistance R_4, which, for the sake of simplicity, is represented in the diagram; the effect is, however, the same in both cases. When we desired to damp the more violent motions of the coil, R_4 was diminished so that a current of 1/20,000 ampere could be sent through the suspended coil, the key K_3 being opened to prevent any of this stronger current from passing round the secondary coil and affecting the magnetisation of the specimen. The final damping was always done with K_3 closed and R_4 at its large value. We verified by experiment that currents many times larger than those actually employed for this purpose had no appreciable effect upon the hysteresis of the specimen.

When the dynamometer is to be standardised the resistance coil X is taken out of the circuit and the earth-inductor is put in its place.

Practical Example of the Method.

§ 36. To illustrate the working of the method, we now give a set of observations made to determine the hysteresis loss, W, as well as the mean maximum induction B_0, for a definite range of magnetic force $\pm H_0$, for a specimen of soft iron :—

Area of cross-section (circular)	$A = \cdot178$ sq. centim.
Change of induction due to earth inductor	$P = 87200$.

Magnetic force per unit C.G.S. current .	$4\pi N = 212\cdot2$.
Maximum current	$C_0 = \cdot0706$ C.G.S.
Maximum magnetic force	$H_0 = 15$.

Number of turns on secondary coil . . .	$m = 25$.
Throw of galvanometer due to earth inductor	$\delta = 12\cdot73$ centims.
Mean throw on reversing primary current	$\beta = 15\cdot63$ centims.

Mean maximum induction $B_0 = P\beta/2mA\delta = 770\beta = 12030$.

Current when earth inductor is turned .	$C' = \cdot0527$ C.G.S.
Throw of dynamometer due to earth inductor	$\phi = 4\cdot68$ centims.
Number of turns on secondary coil . . .	$n = 300$.
Resistance of secondary circuit	$S = 43$ ohms.
Length of secondary coil	$l = 6\cdot6$ centims.
Mean area of one turn of secondary coil .	$G = \cdot65$ sq. centim. $= 3\cdot66A$.
Throw, H from +15 to −15 (E = 8 volts)	$\theta_1 = 23\cdot85$ centims.
Throw, H from −15 to +15 (E = 8 volts)	$\theta_2 = 23\cdot30$ centims.

Hence $U = \dfrac{C'PN}{An\phi}(\theta_1 + \theta_2) = 311\cdot8\,(\theta_1 + \theta_2) = 14700$ ergs per cub. centim. per cycle.

In order to find Z, the observations for θ_1 and θ_2 were repeated with other voltages (§ 15), with the result

$$E = 8 \qquad E' = 12 \qquad E'' = 16 \text{ volts.}$$
$$U = 14700 \qquad U' = 15130 \qquad U'' = 15540$$
From U and U', $Z = (U' - U)E/(E' - E) = 860$

From U and U'', $Z = (U'' - U)E/(E'' - E) = 840$

Mean value of $Z = 850$ ergs per cub. centim. per cycle.

Hence $W = U - Z = 14700 - 850 = 13850$ ergs per cub. centim. per cycle.

In this case L and M are negligible, and thus, as in § 15, $X/Y = QSl/n^2\sigma$. Taking σ as 10^{-5} ohm per centimetre cube, and putting $Q = 1/8\pi$, since the section is circular, we have $X/Y = 12{\cdot}5$. But $Z = X + Y = 850$, and hence

$$X = 787, \quad Y = 63 \text{ ergs per cub. centim. per cycle.}$$

The value of Y is so small in comparison with U that it could not be determined satisfactorily by varying S (§ 14). A series of experiments was made in which S was varied, but the small irregularities rendered the observations useless for the determination of so small a quantity as Y is in this case. [See § 41 (*d*).]

§ 37. A word should perhaps be said as to the accuracy aimed at in our experiments. The throws were all recorded to $\frac{1}{10}$ millim., the throws themselves varying from 25 centims. to 1 centim. The difference between $\frac{1}{2}\theta$ and $\frac{1}{4}\tan 2\theta$, seldom amounting to more than 1 per cent., and usually much less, has generally been neglected.

The calculations were mostly effected by a 10-inch slide-rule, and the numbers recorded in the tables are the numbers read from the rule.

Tests of the Accuracy of the Method.

§ 38. We may now pass on from the theory of the method, and the description of the apparatus used for applying it in actual measurements of hysteresis, to an account of the tests which we have made in order to determine if, in our experiments, the theoretical conditions are so nearly satisfied that the dynamometer yields accurate measurements of hysteresis.

§ 39. *Test by Comparison with the Cyclic B—H Curve.*—We were without the guidance of the completed theory till 1900, and thus till that date we did not know what measurements were required for the determination of the correction arising from the eddy currents and the finite conductivity of the secondary circuit. Under these conditions the only way of testing the accuracy of the dynamometer method was to make a cyclic B—H curve by the use of the ballistic galvanometer, to calculate W from the area of the curve, and to compare this value with the value of U (§ 13) found by the dynamometer for the same range of magnetic force. We made this test on many corresponding to H_1 may differ somewhat from the value B_1 found by the galvanometer.

The experiments of §§ 45–49 below show that, for a given value of H_0, W may vary considerably from causes depending upon the magnetic history of the iron even when its temperature and strain remain constant. Thus, unless it is possible to give an accurate account of the history of the iron and of the manner in which the magnetic force changes from H_0 to $-H_0$, it is impossible to assign any definite meaning to W, and hence, in the absence of such an account, it is useless to attempt a very close comparison between the value of U found by the dynamometer and the value of W found from the cyclic B—H curve. But the agreement between the two values as recorded in the table will perhaps suffice to give general confidence in the method.

§ 41. *Test by Variation of the Resistance of the Secondary Circuit.*—The process of comparing the value of U found by the dynamometer with the value of W found from the area of the cyclic B—H curve is laborious, and after all does not furnish an absolute test of the equality of U and W for that particular law of variation of the magnetic force which is obeyed when the measurement is made, as was explained in § 40. When, as was the case in most of our experiments, the specimen is a fine wire not exceeding 2 millims. in diameter there is a second method, described in § 14—it may be termed, in contrast with the first, a self-contained method—which is easy of application, enabling us to test the accuracy of the dynamometer measurements by the dynamometer itself. If, in any case, we can find the effect of Y, we have a superior limit to the effect of X, for by (14) $X/Y < QSl/n^2\sigma$. We now give the results of some applications of this second method.

It will generally suffice to take two values of S, one double the other. Since, when S is varied, ϕ is inversely proportional to S, it follows that $U\ [= NC'P(\theta_1 + \theta_2)/An\phi]$ is proportional to $S(\theta_1 + \theta_2)$, provided that ϕ and θ are corrected for damping (§ 14). Hence, in testing the accuracy of the dynamometer measurements, it is sufficient to compare the values of $S(\theta_1 + \theta_2)$ for the two values of S. If S be doubled the consequent diminution in $S(\theta_1 + \theta_2)$ is, by § 14, equal to the amount which must be subtracted from the product for the larger resistance in order to obtain the value of the product corresponding to $S = \infty$.

(*a*.) We will first refer to the results obtained by one of us in November, 1895, for an iron wire ·0201 sq. centim. in section, with $H_0 = 8{\cdot}34$ and $B_0 = 10170$, the value of U being 17780 when $S = 59{\cdot}7$ ohms, while W was found from a cyclic B—H curve to be 7409. A simple mercury rocking key was used, and the primary circuit had only the self-induction of the dynamometer and the solenoid. As S rose from 59·7 to 584 ohms, U fell from 17780 to 14550, being closely represented by $U = 14070 + 221900/S$. From this we find by (20) that when $S = 59{\cdot}7$ ohms, $Y = 3720$. In these experiments $n = 1285$, $l = 11{\cdot}5$ centims., $G = {\cdot}785$ sq. centim., and hence if L and M (§ 11) be neglected, $X/Y = 1{\cdot}66$ when $S = 59{\cdot}7$ ohms. Thus $Z = X + Y = (1{\cdot}66 + 1)3720 = 9890$, when S is 59·7 ohms. Subtracting this from the corre-

It was only in the later experiments that we were guided by the complete theory to test the accuracy of the measurements by varying S, but from the tests just described we may conclude that the value of U, obtained from the dynamometer throw, was in all our experiments very nearly equal to W.

Energy dissipated by Eddy Currents.

§ 42. In our experiments the ratio X/Y was small, so that, Y being small in comparison with U, X was small also, and thus U and W were nearly equal.

We now consider the case in which the section of the specimen is large, so that S has to be made large and n small, in order to reduce the sensitiveness of the apparatus. Under these conditions X/Y is large, while Y is now so small in comparison with U that it eludes observation, and thus cannot be determined by varying S, the resistance of the secondary circuit. These conditions have prevailed in the experiments made by Mr. R. L. WILLS with some of our apparatus; in these experiments the eddy current loss was so large that systematic measurements were made to determine X for every value of H_0 employed.

Mr. WILLS used the key described in § 32, and found that for a given total resistance of the primary circuit the "throw" of the dynamometer is practically independent of the resistances denoted by D in the description of the key. This key divides the primary circuit into two portions; the resistance of the portion which includes the battery is denoted by T, and the resistance of the other portion by R. When the voltage E was to be changed in order to make the observations suggested in § 15, Mr. WILLS changed the total resistance, R + T, of the primary circuit, mainly by changing R. The resistance T was small compared with R, and was used as a means of obtaining an exact adjustment of the current to definite values. No attempt was made to make T bear any fixed ratio to R.

The reversal of the current does not take place quite in the manner described in § 15. In addition to the effect of the resistances D, there is the further point of difference that although R + T is adjusted to be accurately proportional to E, the voltage driving the current C_0, yet R is not accurately proportional to E because T does not bear any fixed ratio to R. The value of dC/dt and, consequently, the power absorbed by the eddy currents while the primary current is sinking to zero is thus not quite proportional to E for a given value of C.

During the subsequent rise of the current the whole resistance R + T comes into play, and, the resistances D being comparatively small, the value of dC/dt for a given value of C is nearly proportional to E.

Now the increment of current dC contributes to X a quantity proportional to $(dB/dH)^2 . dC/dt$. Thus although, for a given value of C, dC/dt is much greater during the fall than during the rise of the current, because dB/dH for the iron of the choking coil is much smaller during the first than during the second stage, yet, The agreement between $2U - U'$ and $3U' - 2U''$ is satisfactory; it may be taken as evidence that the theory sketched in § 15 is practically applicable to Mr. WILLS's experiments. The ratio W/Y ranged from 1040 to 66,000, and thus no appreciable error was introduced by neglecting the correction due to the conductivity of the secondary circuit.

In the second example the specimen was a ring similar to the last, the numbers now being $a = 1·985$, $b = ·735$, $l = 20$ centims., $A = 1·46$ sq. centims., $b/a = ·37$, $Q = ·0236$, $n = 50$. The material was an alloy of iron and aluminium, and the experiments were made when the specimen was at 645° C. We do not know σ; if it was 10^{-5}, then X/Y varied from 430 to 1380. The results are shown in the following table, where it will be seen that there is again good agreement between $2U - U'$ and $3U' - 2U''$. The agreement is the more significant because the eddy current loss, X, now forms a very considerable part of the whole energy dissipated in each cycle.

H_0.	B_0.	S. ohms.	U. $E = 8$.	U′. $E' = 16$.	U″. $E'' = 24$.	$2U - U'$.	$3U' - 2U''$.	W. (mean).	X. $E = 8$.
·20	1747	23	77·2	80·3	83·4	74·1	74·1	74·1	3·1
·34	3357	23	223·8	251·5	276	196·1	202·5	199·3	24·5
·48	4406	23	332·0	390·8	448	273·2	276·4	274·8	57·2
·68	5507	73	461	570	669	352	372	362	99
1·02	6661	73	615	773	926	457	467	462	153
1·36	7238	73	714	917	1107	511	537	524	190
2·04	8282	73	858	1096	1335	620	618	619	239
2·72	8654	73	952	1214	1472	690	698	694	258
3·40	8968	73	1007	1264	1528	750	736	743	264

Complete Cycles and Semi-cycles.

§ 43. The theoretical investigation of § 8 shows that the throw due to a complete cycle should be equal to the sum of the throws due to a pair of semi-cycles. We have not spent any considerable time in testing this point, for the arrangements used by us have not been well adapted for that purpose. In order that U shall be sensibly equal to W, it is necessary, when the specimen is a thin wire of soft iron, that the self-induction of the choking coil should be very large, and in this case, after the current has been reversed by the key, it does not at once rise to its full strength. Thus, if the key be worked very rapidly so as to make a pair of semi-cycles in quick succession, the current may never rise to its full value in the middle of the cycle. On the other hand, if a definite halt be made after the current has been reversed for the first semi-cycle, the dynamometer coil will have moved considerably from its zero position when it receives the impulse due to the second semi-cycle. It is perhaps for these reasons that we have not found good agreement between the throw due to a complete cycle and the sum of the throws due to a pair sq. centim. An ordinary mercury rocking key was used, and the choking coil was inserted in the circuit.

We now give the results of one of these experiments. The numbers heading the columns indicate the cycle to which the pair of semi-cycles in any column belong.

Cycle.	1.	2.	3.	4.	5.	6.	28.	50.
$\frac{1}{4\pi}\int HdB$ for $+H_0$ to $-H_0$	5440	4820	4650	4570	4480	4470	4260	4190
$\frac{1}{4\pi}\int HdB$ for $-H_0$ to $+H_0$	4540	4380	4290	4230	4210	4160	4030	3990

In this example the inequality is greater and more persistent than in most cases. The inequality is generally much more marked for small than for large magnetic forces. It will be noticed that the hysteresis diminishes with continued reversals.*

Effect of Continued Reversals.

§ 46. In the course of the experiments undertaken in the hope of finding the cause of the inequality of the two throws of the dynamometer, θ_1 and θ_2, for the two semi-cycles belonging to a single cycle, we had occasion to put the specimen through many cycles. We then discovered that the hysteresis diminishes very considerably with continued reversals of the magnetic force. To investigate this matter more completely a systematic set of experiments was made in 1899 with the object of determining how the effect depends upon the limits of the magnetic force. The experiments were carried out in the following manner:—The specimen, demagnetised by annealing or "by reversals," was placed in the solenoid when no current was flowing. The magnetic force was first applied in the positive direction; the magnetic force was then reversed from H_0 to $-H_0$ and the throws of the dynamometer and the galvanometer were read simultaneously. The next reversal $-H_0$ to H_0 was observed in like manner. These two reversals constituted the first cycle. The observations were repeated for other cycles as shown in the tables, where in each case the first column shows the number of the cycle. The two throws of the dynamometer, θ_1 and θ_2, sometimes showed an inequality which was rather persistent, though it was never greater than that recorded in § 45. We have therefore thought it sufficient to give the value of the hysteresis deduced from each complete cycle. The mean maximum magnetic induction, B_0, which diminished in much the same manner as the hysteresis, is also recorded in the tables.

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* The initial asymmetry of the hysteresis loop and its gradual shrinking with continued reversals are well shown in fig. 154 in Professor EWING'S Magnetic Induction . . .', 3rd Edition. Much labour would have been saved had we realised the significance of Professor EWING'S curves.

The experiments of Professor J. A. EWING* on soft iron show that after the few dozens of reversals which are necessary to avoid the effects of the initial diminution of B_0, the values of B_0 and of W remain constant, even after 70,000,000 of cycles of reversal when the temperature of the iron is not allowed to rise above that of the atmosphere. In his experiments the maximum magnetic force H_0 in these cycles was about 4, and the maximum induction, B_0, about 8000. It is now well known that if iron be raised to a temperature only 50° C. above the atmospheric temperature, and be maintained in that state for many hours, a large increase in the hysteresis often results.

§ 47. *Soft Iron demagnetised by Annealing.*—The specimen was a wire of soft iron ·0265 sq. centim. in section. In each case it was annealed before the observations for any given value of H_0 were made; the previous magnetic history of the iron was thus wiped out. The key of § 31 was used.

No.	$H_0 = 2·50$.		$H_0 = 4·98$.		$H_0 = 7·57$.		$H_0 = 10·86$.		$H_0 = 14·70$.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
1	2220	598	8150	5250	10550	9500	11800	12450	12840	14610
2	2040	519	7970	4870	10430	8980	11680	12050	12830	14340
3	2000	488	7950	4730	10340	8830	11660	11960	12760	14230
11	1940	452	7740	4530	10250	8520	11660	11850	12730	14140
21	1890	440	7700	4420	10250	8590	11670	11760	12720	13990
41	1840	433	7650	4360	10220	8450	11660	11760	12730	14070

No.	$H_0 = 20·05$.		$H_0 = 25·15$.		$H_0 = 30·24$.		$H_0 = 40·65$.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
1	13480	17420	13270	20020	14360	21300	14760	22600
2	13430	17300	13260	19850	14300	21260	14740	22600
3	13420	17200	13280	19770	14300	20940	14720	22610
11	13420	17060	13230	19690	14290	20820	14720	22400
21	13430	17100	13280	19720	14290	20900	14730	22560
41	13420	17080	13250	19640	14310	20820	14710	22630

It will be seen that the effect of continued reversals of H_0 in producing a diminution of B_0 and W rapidly decreases as H_0 increases, both B_0 and W becoming sensibly constant when H_0 reaches the value 40·65. The following table illustrates this fact, and shows that the percentage change of W is always greater than the percentage change of B_0. The change recorded is that which occurred in the first 41 cycles:—

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* 'The Electrician,' vol. 34, January 11, 1895.

H_0	2·50	4·98	7·57	10·86	14·70	20·05
Percentage change in B_0 .	17·1	11·0	3·1	2·0	·9	·4
Percentage change in W .	27·5	17·0	11·0	5·5	3·7	1·9

§ 48. *Soft Iron demagnetised "by Reversals."**—The same piece of wire and the same key were used as in § 47. The wire was very completely demagnetised by reversals before the tests for B_0 and W were made; a mirror magnetometer served as indicator. The results are shown in the table below:—

No.	H_0 = 2·50.		H_0 = 7·45.		H_0 = 20·50.	
	B_0.	W.	B_0.	W.	B_0.	W.
1	2068	711	9950	8000	13480	16670
2	1997	662	9980	8000		
3	1984	669	9960	8000		
4	1997	652				
11	1960	638	9940	7910	13510	16590
21	1960	633	9940	7850		
41	1950	613	9890	8000	13510	16580
81	1913	603			13480	16610
101			9900	7820		

For $H_0 = 7{\cdot}45$ and $H_0 = 20{\cdot}50$ the values of B_0 and W are constant within the errors of observation. For $H_0 = 2{\cdot}5$ there is a considerable diminution in both B_0 and W during the first 41 cycles, but much less than in the case of the wire demagnetised by annealing, the changes in the first 41 cycles being now for B_0 5·8 per cent., and for W 13·8 per cent.

The wire had been annealed before the tests for $H = 2{\cdot}5$ in § 47 were made. Without further annealing, the tests of the present section for $H_0 = 7{\cdot}45$, 20·50, and 2·5 were made in the order written. This treatment has had the result of diminishing the initial value of B_0 and raising its value after 41 reversals; the value of W is at the same time considerably increased.

§ 49. *Steel Rod.*—A few experiments were made upon a steel rod ·0925 sq. centim. in section, using the key of § 31. We endeavoured to demagnetise it by reversals before the tests for B_0 and W were made, but we were only partially successful. The dynamometer was not sensitive enough to enable us to go to small magnetising forces. The induction remained practically constant for both values of H_0; in 41 cycles the hysteresis fell by 10 per cent. when $H_0 = 7{\cdot}29$ and by 2·5 per cent. when

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* See EWING, 'Magnetic Induction . . .,' 3rd Edition, fig. 155.

$H = 20 \cdot 54$, the changes in W being roughly the same as those for the iron wire, demagnetised by annealing, under approximately the same magnetic forces.

No.	$H_0 = 7 \cdot 29$.		$H_0 = 20 \cdot 54$.	
	B_0.	W.	B_0.	W.
1	1275	853	11430	32840
2	1255	838	11370	32670
3	1275	838	11370	32530
4			11370	32580
11	1240	787	11310	32110
21	1275	779	11340	32110
41	1275	768	11280	32040

$W - B_0$ Curves for Zero Stress.

§ 50. The energy dissipated through hysteresis can be varied in many ways. We describe, in the sequel, many experiments in which the hysteresis loss for constant values of H_0 was caused to vary by varying the stress applied to the specimen; Mr. WILIS has varied the hysteresis loss for constant values of H_0 by varying the temperature of the specimen. But perhaps the most natural, and certainly the most usual, way of varying W is to vary H_0, while the stress is kept at a constant zero value. The curve representing W as a function of B_0 under these conditions is the curve which is useful to engineers when designing transformers. To distinguish it from the curves, which represent W as a function of B_0, when H_0 is kept constant and B_0 is varied by varying the stress, we call it a $W - B_0$ curve for zero stress. It is the curve for which Mr. C. P. STEINMETZ has proposed the formula $W = \eta B_0^{1 \cdot 6}$.

Effects of Stress.

§ 51. We now pass on to describe a number of experiments in which we studied the effects of tension and torsion upon the mean maximum magnetic induction, and upon the energy dissipated by hysteresis when the magnetic force ranged between definite limits $\pm H_0$. In each case the same series of stresses was applied to the specimen for each value taken for H_0.

Effect of Tension on Soft Iron Wire.

§ 52. The first systematic experiments on the effect of stress upon the energy dissipated in hysteresis were made in 1898 upon a soft iron wire.

The wire was placed horizontally, perpendicular to the magnetic meridian, and the tension was applied by a flexible silk cord which passed over a pulley and supported a weight. The reversing key described in § 30 was used, the current being measured by a shunted d'Arsonval galvanometer. We satisfied ourselves by the comparison of the values of W found (1) from cyclic B—H curves, (2) by the electro-dynamometer, that the method was yielding at least approximately exact values of W. Before the observations corresponding to any given value of H_0 were made, the wire was subjected to several cycles of loading and unloading, the maximum load being 24 kilogrammes. The magnetic observations were taken only as the load was being increased. The section of the wire was ·00708 sq. centim., so that a load of 1 kilogramme gives a tension of $1·39 \times 10^9$ dynes per sq. centim. The results are given in the following table and in fig. 7. (See also § 67.)

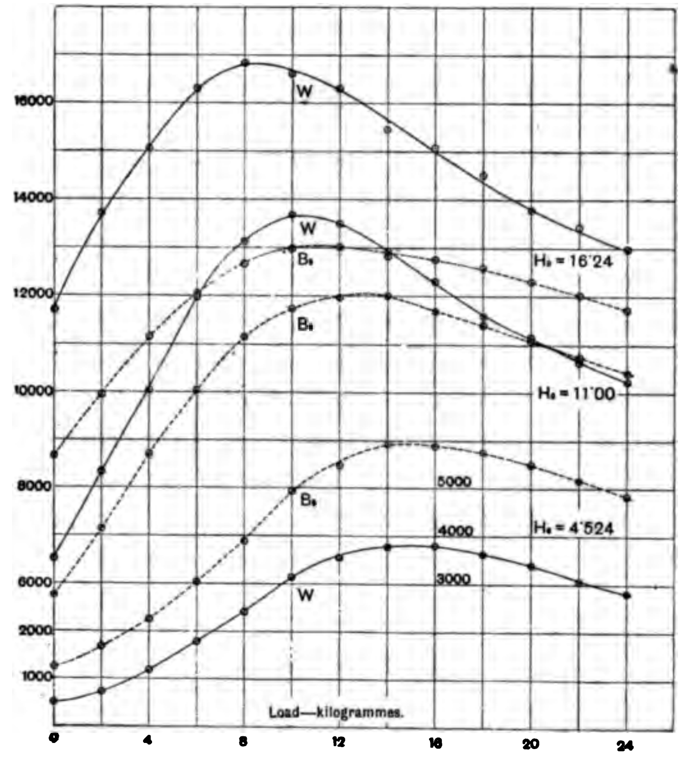


Fig. 7.

Observations on a soft iron wire during both loading and unloading showed that the curves for loading and unloading are not quite identical, though the difference between them is not large.

Load, kilogrammes.	$H_0 = 4·524$.		$H_0 = 11·0$.		$H_0 = 16·24$.	
	B_0.	W.	B_0.	W.	B_0.	W.
0	1233	494	5750	6520	8660	11710
2	1663	726	7150	8320	9940	13700
4	2250	1183	8700	10030	11150	15050
6	3030	1776	10030	11970	12050	16300
8	3880	2402	11150	13120	12650	16830
10	4915	3140	11730	13680	12980	16600
12	5475	3540	11970	13500	13010	16300
14	5920	3770	12000	12920	12820	15550
16	5870	3820	11690	12300	12760	15080
18	5750	3610	11400	11590	12580	14500
20	5490	3400	11070	11150	12300	13780
22	5160	3060	10740	10630	12010	13430
24	4840	2800	10410	10220	11720	12980

In order to save space on the diagram, the zero for the four upper curves differs from that for the two lower ones. The numbers on the diagram will prevent any confusion.

In each case, as the tension increases, both B_0 and W rise to maximum values, the tension corresponding to the maximum values diminishing as H_0 increases. Next to the similarity between the curves for B_0 and W for a given value of H_0 the most striking feature is the great increase in both B_0 and W occasioned by tension when H_0 is small. Thus for $H_0 = 4·524$ a pull of 14 kilogrammes raises B_0 from 1233 to 5920, and W from 494 to 3770. The magnetic force was not carried to values high enough to obtain the Villari reversal of the effect of tension.

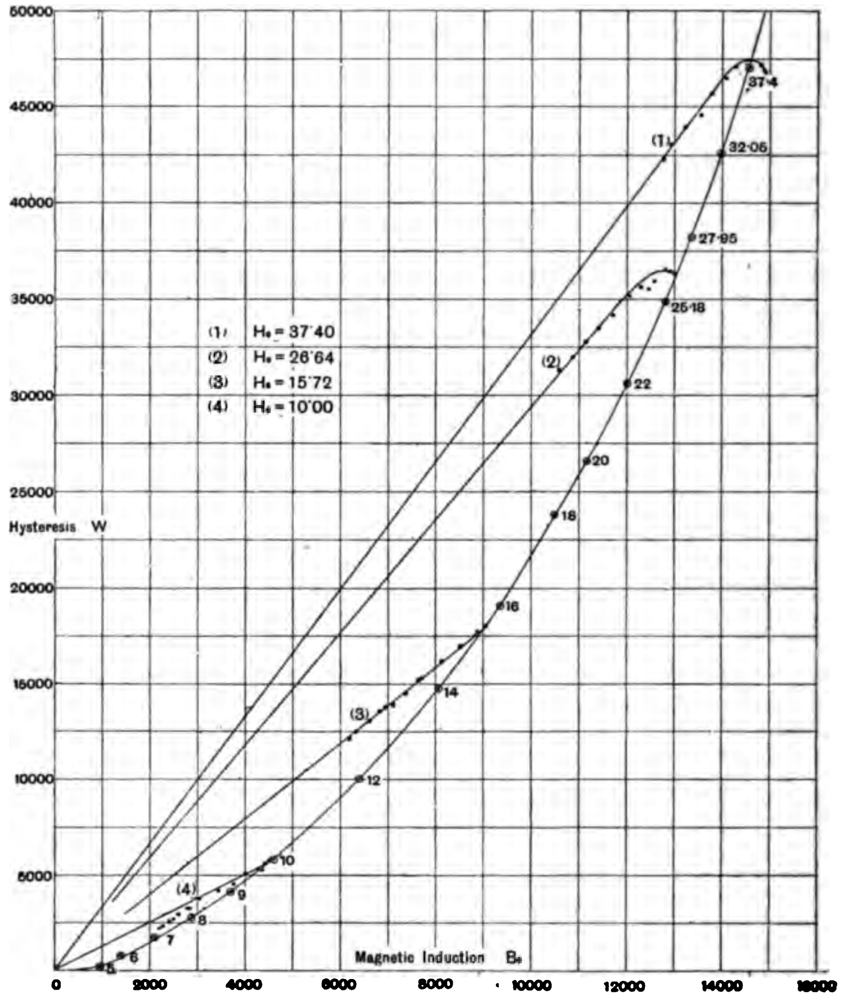
Effect of Torsion within the Elastic Limit.

§ 53. In July and August, 1899, we made a long series of experiments on the effect of torsion upon the magnetic qualities of iron and steel. The arrangements for applying torsion were very simple. A wooden wheel 24·7 centims. in diameter was mounted on a brass tube as an axle, and this tube revolved in a bearing. The wire under test passed through the solenoid and through this tube, and was clamped at one end to the wooden wheel, while the other end was held in a vice. The vice and the bearing of the wheel were mounted on a stiff wooden beam so that the wire was parallel to the beam, which was placed at right angles to the magnetic meridian. The magnetising solenoid was so fixed that the wire passed along its axis.

To apply a torsional couple to the wire, a weight was hung from the edge of the wheel by a flexible string wrapped round the wheel. When we wished to give the wire a definite twist, the wheel was clamped in the desired position.

Experiments on Steel.

§ 54. *$W - B_0$ Curve for Zero Stress.*—The first set of experiments was made upon a steel rod ·09125 sq. centim. in section. To find how B_0 and W depend upon H_0, when there is no torsion, we made a series of observations for B_0 and W, using the key of § 31, and varying H_0 from 37·4 to 5·0; the results are given in the table below. The electro-dynamometer was not sensitive enough to allow us to work with magnetic forces less than 5 units. These observations will serve as an example of the application of the method described in this paper to determine the manner in which W depends upon B_0, when B_0 is made to vary by changing H_0, the stress being constant. The curve representing these observations is shown in fig. 8. The numbers placed along the curve show the values of H_0 for which the values of B_0 and W_0 were found. The curves (1) to (4) are considered in § 66.



Load, grms.	$H_0 = 37 \cdot 4$.		$H_0 = 26 \cdot 64$.		$H_0 = 15 \cdot 72$.		$H_0 = 10 \cdot 0$.		$H_0 = 5 \cdot 0$.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
0	14890	46800	13000	36400	9060	18100	4350	5310	674	190
200	14860	46900	12920	36500	8840	17620	4410	5650	690	196
400	14810	47200	12800	36600	8520	16950	3980	5070	685	194
600	14520	47400	12560	36000	8130	16180	3420	4190	677	206
800	14290	47100	12460	35600	7700	15300	3020	3760	621	165
1000	14060	46500	12040	35200	7360	14560	2840	3320	628	173
1200	13760	45700	11700	34200	7100	13950	2600	2890	602	166
1400	13540	44600	11420	33500	6800	13550	2480	2730	584	151
1600	13200	44000	11130	32800	6600	12980	2390	2540	547	130
1800	12910	43200	10850	32000	6330	12510	2260	2350	547	129
2000	12760	42300	10600	31300	6210	12050	2190	2230	556	140
1600	13210	43900	11100	32700	6520	12660	2330	2310	552	126
1200	13780	45700	11700	34100	6960	13800	2510	2610	585*	144*
800	14290	47000	12300	35700	7630	15230	2870	3230	635†	173†
400	14690	47100	12800	36500	8540	17100	3750	4610	657	171
200	14850	46600	13000	36600	8860	17700	4270	5380	704	206
0	14920	46800	13050	36800	9030	18200	4510	5770	794	265

* 1000 grammes. † 600 grammes.

From $H_0 = 37 \cdot 4$ to $H_0 = 15 \cdot 72$ the curves for unloading agree, within the errors of experiment, with the curves for loading, and only a single line is shown in the diagram. In the cases of $H_0 = 10 \cdot 0$ and $H_0 = 5 \cdot 0$ the curves for unloading differ from those for loading. The curve for $H_0 = 5$ could not be conveniently shown on the diagram. In these two cases there is also an evident want of symmetry, for, if the symmetry were perfect, the lines for loading and unloading would cross each other on the line of no load. It is interesting to notice how closely the curves for W imitate those for B_0. We consider this similarity in § 66 below.

Experiments on Soft Iron.

§ 56. A much more extended series of experiments was made upon soft iron wire, ·0324 sq. centim in cross-section. The wire was "galvanised" when bought, but by heating it to bright redness in a large blowpipe flame all the zinc was burnt off the wire and at the same time the wire was annealed. The wire was twisted by means of the arrangement described in § 53. The reversing key described in § 31 was used in these experiments.

The first step was to test by the method of § 41 whether X and Y could be neglected. The numbers for this specimen, recorded in section (*b*) of § 41, show that X and Y were negligible.

§ 57. *$W - B_0$ Curve for Zero Stress.*—To gain a general idea of the magnetic character of the wire, a set of observations was made to determine B_0 and W for a series of values of H_0, the wire having been annealed and being free from strain.

The magnetic force was put through about 20 cycles of reversal before the tests for B_0 and W were made. The results are given in the following table, and the curve representing them is shown in fig. 10 ; the curves (1) to (5) are considered in § 65.

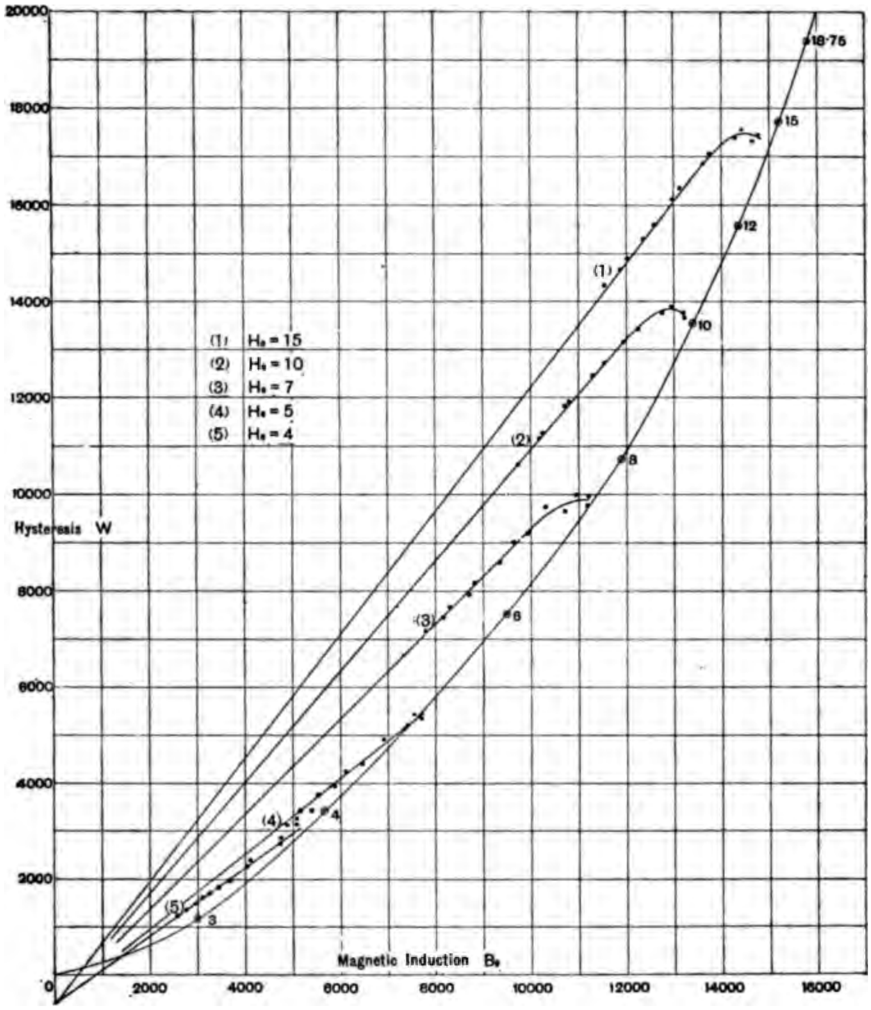


Fig. 10.

H_0.	B_0.	μ.	W.	H_0.	B_0.	μ.	W.
18·75	15800	843	19400	8·0	11880	1485	10720
15·0	15190	1012	17720	6·0	9490	1582	7530
12·0	14350	1196	15580	4·0	5660	1415	3424
10·0	13380	1338	13550	3·0	2990	997	1170

Effect of Torsion within the Elastic Limit.

§ 58. The wire was found capable of sustaining 300 grammes hung from the edge of the wheel without acquiring permanent set. We therefore decided to subject it to cycles of torsion in which the load varied within the limits ± 300 grammes. The changes in the load were made by steps of 50 grammes.

Before the cycle of loading and unloading, during which the magnetic observations were made, the wire was subjected to several cycles of positive and negative torsion, the load having the limits ± 300 grammes. In every case, before the observations for B_0 and W were made for any particular value of the torsion, the magnetic force was put through about 20 cycles of reversal; in this way we avoided the major part of the indeterminateness due to the diminution of B_0 and W which occurs initially with continued reversals of H_0. (§ 46.)

The results of these experiments are given in the table below and are exhibited in fig. 11:—

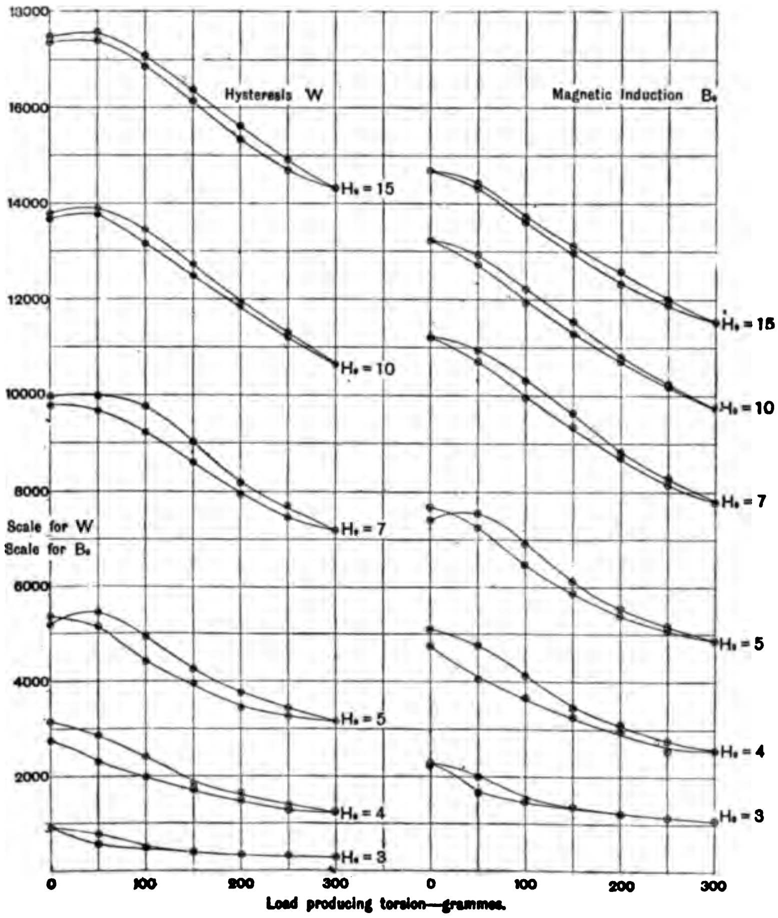


Fig. 11.

Load, grms.	H_0 = 15·0.		H_0 = 10·0.		H_0 = 7·0.		H_0 = 5·0.		H_0 = 4·0.		H_0 = 3·0.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
0	14680	17470	13210	13780	11200	9970	7400	5180	5100	3140	2280	907
50	14420	17560	12920	13900	10940	9990	7540	5440	4750	2860	1980	766
100	13730	17080	12220	13440	10310	9760	6910	4920	4110	2400	1520	484
150	13120	16370	11540	12720	9640	9020	6120	4250	3440	1840	1350	389
200	12570	15610	10800	11930	8820	8180	5550	3760	3100	1630	1220	336
250	12000	14900	10240	11280	8300	7700	5170	3420	2760	1380	1140	302
300	11530	14320	9740	10630	7790	7180	4860	3135	2540	1242	1070	276
250	11880	14680	10190	11190	8150	7460	5070	3260	2550	1265	1120	286
200	12320	15330	10710	11830	8700	7940	5390	3443	2950	1480	1220	329
150	12970	16130	11280	12480	9350	8600	5870	3950	3260	1700	1320	367
100	13610	16850	11930	13150	9960	9210	6460	4405	3660	1980	1460	443
50	14300	17400	12730	13770	10700	9660	7250	5130	4040	2290	1650	563
0	14670	17350	13200	13660	11170	9770	7680	5350	4740	2745	2230	911

In each cycle of torsion both the mean maximum induction B_0 and the hysteresis W are greater when the torsion is increasing than when it is diminishing, except for small parts of those curves in which want of symmetry about the line of zero torsion has caused the crossing-point of the two branches of the curve to lie off that line.

For the smaller fields, both B_0 and W are very sensitive to torsion. Thus, when $H_0 = 3$, a load of 100 grammes hung from the edge of the wheel diminished B_0 by about one-third and W by about one-half of their values for zero load.

The dynamometer was not sensitive enough to allow us to continue the observations for W for fields less than $H_0 = 3$. We could have measured B_0 for much smaller fields, but the observations would not have been of much interest in the absence of the observations for W.

In the case of the smaller magnetic fields we noticed that the hysteresis continued to diminish considerably with continued reversals of the magnetic force even after it had been subjected to many reversals.

The table of § 57, and the top and bottom rows of the table just above, give the values of B_0 and W for zero torsion for various values of H_0. The values of B_0 and W in the first table do not agree very closely with those in the second table, but it must be noticed that the first table was made after the wire was annealed and before it was strained in any way. After the observations of § 57 had been made, experiments were made to find the limits of elasticity of the wire, and in this process the wire was subjected to torsions large enough to give it permanent set; to get rid as far as possible of the effects of this overstraining, the wire was re-annealed before the tests for the second table were made. Close agreement between the two tables is in consequence not to be expected.

These experiments may serve to show the saving of time effected by our method of measuring hysteresis, for the observations for the first four values of the magnetic force, involving fifty-two determinations of hysteresis, were easily taken in one day, though much time was spent in changing the load, in putting the current through 20 reversals before the magnetic tests were made, and in bringing the suspended coil of the dynamometer to rest.

Effect of Torsion beyond the Elastic Limit.

§ 59. So far our experiments were made for torsional strains in which the elastic limit was not overpassed at all, or in any case was not exceeded to such an extent that the wire took a noticeable permanent set. We thought it would be interesting to trace the effect of great torsional overstrain upon the induction and the hysteresis for a constant maximum magnetic force $H_0 = 5$. In 1899 we made experiments in this direction upon two specimens of the same soft iron wire as was used in the experiments of §§ 57, 58. Both specimens were heated in a large blowpipe flame to burn off the zinc coating and to anneal the wire. The arrangement of § 53 was used for applying torsion.

§ 60. *Experiments on Soft Iron Wire* (1); $H_0 = 5$.—In these experiments the wheel was turned, always in one direction, through the desired angle, measured from its position when the wire was unstrained, and was then clamped. A mechanical "counter" served to record the number of revolutions made by the wheel. When the wheel had been clamped in any position, the magnetic force was put through 20 cycles of reversal, and then readings for B_0 and W were made. The wheel was now turned still further and again clamped, and then the magnetic observations were repeated. This process was continued till the wire broke. The fracture occurred when the wheel had been turned through 104 revolutions. Since the length of the wire was 65·5 centims., this angle corresponds to a twist of 1·57 turns per centim. The observations were too numerous to be recorded in a table; we therefore only give a diagram in fig. 12, distinguishing the curves for this specimen by the mark (1).

In these experiments the first result of the torsion was to cause a very rapid diminution of both B_0 and W. The curve for B_0 falls continuously till the wire breaks, but the curve for W shows a well-marked rise and fall, with a maximum at about 7·5 revolutions. The values of B_0 and W were determined immediately after the wire broke, the tests showing that, when the stress was relieved by the fracture, both B_0 and W fell to about half the values they had just before the fracture. Thus B_0 fell from 2950 to 1510, while W fell from 2410 to 1044.

We found that when the wire had been twisted W diminished considerably with continued reversals, even after 20 cycles of magnetisation.

§ 61. *Experiments on Soft Iron Wire* (2); $H_0 = 5·0$.—The second specimen (2) was treated rather differently. The wheel was turned through a definite number of which are placed on the ordinate corresponding to any number, x, of complete revolutions, indicate the values found for B_0 and W when the wire was allowed to free itself from stress after the wheel had been turned through x revolutions.

The curves (2*a*) show that, as in the case of specimen (1), the application of torsion causes a continued decrease of both B_0 and W. The decrease is at first very rapid, B_0 falling from 6520 to 4750 and W from 4380 to 3180 for the first half revolution of the wheel. The curves (2*b*) show that after the first half revolution of the wheel both B_0 and W diminish continually, but at first much less rapidly than is the case for the curves (2*a*). The initial parts of the curves (2*b*) are an exception to the rest, for when the wire was allowed to untwist after the wheel had been turned through the first half revolution both B_0 and W were greater than before the wire was twisted at all, B_0 rising from 6520 to 6720 and W from 4380 to 4590.

Up to about 13 revolutions of the wheel, the relief of torsional stress was followed by an increase of both B_0 and W, but beyond this point the relief caused a decrease of both B_0 and W.

The wire broke at 101 revolutions of the wheel, *i.e.*, at practically the same twist as wire (1); the length was very nearly the same, viz., 65·7 centims. Just before fracture the values of B_0 and W under the torsion due to 100 revolutions were $B_0 = 1050$, $W = 383$, and when the wire had freed itself from stress, $B_0 = 960$, $W = 383$. After fracture the values fell to $B_0 = 787$, $W = 302$, the changes being much smaller than those for specimen (1). It seems probable that the diminution of B_0 and W which occurs on fracture is due to the violent jar which attends the fracture.

The values of both B_0 and W in (2*a*) and (2*b*) are for large twists very much smaller than the corresponding values in (1). It is true that the permeability of (2) before it was strained, viz., $\mu = 1304$ for $H_0 = 5$, is much less than the value $\mu = 1694$ for (1) under the same conditions. But the great difference between (1) and (2) in respect to B_0 and W for large twists can hardly be due to this fact. It is more likely due to the untwisting of the wire at the various stages of the process.

When the wire had been twisted we found just as for (1) that W diminishes considerably even after 20 cycles of magnetisation. We took 20 cycles in each case before making the tests for B_0 and W, but if W is still diminishing with continued reversals, it is hardly possible to assign any very definite value to it.

The wire, initially quite pliable, was very stiff after it had been broken by twisting. Its length originally was 65·7 centims., and this was increased by the torsion to 66·9 centims. The mean diameter diminished from ·0796 to ·0793 inch. Thus the volume of the wire increased by about 1 per cent.

The smallest values reached by B_0 and W, viz., those found after fracture, are very small compared with those found before the wire was strained. Initially $B_0 = 6520$, $W = 4380$, and finally $B_0 = 787$, $W = 302$.

line for B_0 and W in each curve. The numbers marked near each curve will enable the values of B_0 or W to be read off for any point on the curves.

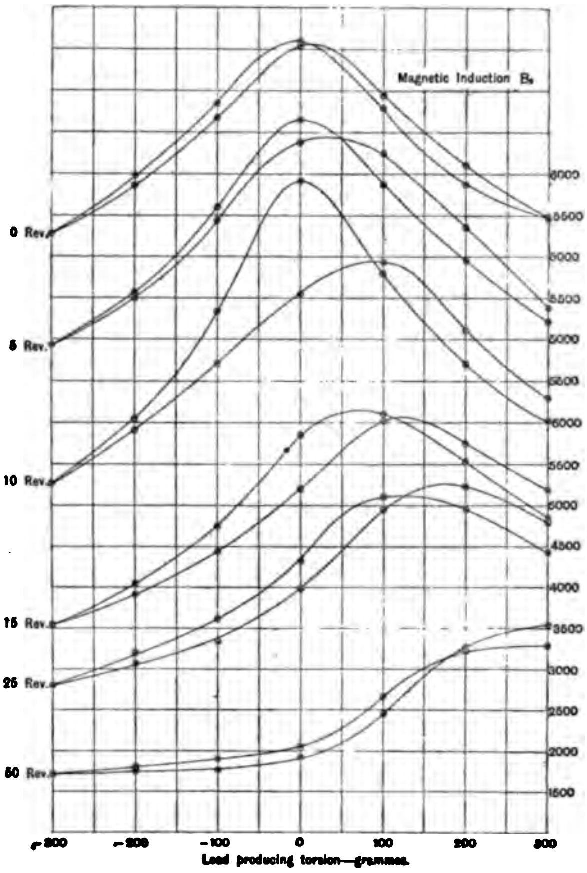


Fig. 13.

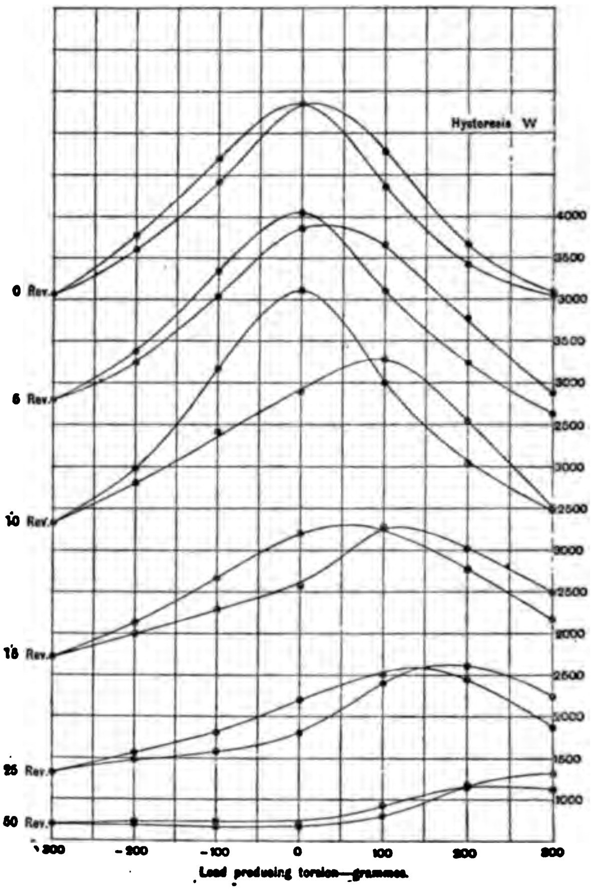


Fig. 14.

Load, grammes.	0 Rev.		5 Rev.		10 Rev.		15 Rev.		25 Rev.		50 Rev.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
+ 300	5460	3040	5210	2630	5020	2480	4850	2170	4430	1870	3290	1130
+ 200	5870	3420	5950	3240	5700	3040	5530	2760	4950	2440	3220	1150
+ 100	6790	4360	6870	4100	6790	4000	6100	3260	5100	2510	2660	930
0	7600	5350	7650	5050	7920	5110	5850	3200	4320	2200	2040	705
− 100	6850	4700	6600	4340	6340	4170	4750	2670	3610	1810	1900	740
− 200	5980	3770	5570	3370	5050	2980	4050	2130	3200	1560	1800	732
− 300	5280	3060	4930	2800	4270	2330	3540	1720	2800	1320	1710	692
− 200	5860	3600	5500	3240	4910	2800	3910	2000	3060	1490	1750	700
− 100	6680	4420	6430	4040	5710	3410	4440	2290	3340	1580	1760	672
0	7540	5360	7380	4860	6550	3890	5200	2570	3990	1800	1920	673
+ 100	6940	4780	7240	4660	6930	4280	6020	3280	4950	2400	2450	802
+ 200	6100	3660	6350	3770	6100	3540	5750	3020	5230	2610	3260	1172
+ 300	5480	3090	5370	2880	5300	2510	5190	2490	4790	2240	3540	1317

negative couples. The process of changing the load from + 300 to − 300 grammes and back again was continued till it appeared that the curves connecting B_0 and W with the load had nearly settled down to a permament form.

In the first experiments, when the permanent set was about 6 revolutions, the first application of a *negative* couple due to 200 grammes caused the wire to yield somewhat, and when the load of −300 grammes was reached, the yielding was great enough to cause the wheel to turn gradually through half a revolution. In the next cycle of loading this effect of a negative couple was small, and it became much smaller for each successive cycle. With the large permanent sets of 20 and 50 revolutions the wire became so hard that the yielding under a negative couple was very small even in the first cycle. The positive couples caused little yielding even in the first cycle with the permanent set of 6 revolutions.

The observations were too numerous to be recorded, except in the form of the curves in figs. 15, 16. In order to show all the curves on a single diagram, we have taken a different position of the zero line for B_0 and W in each curve.

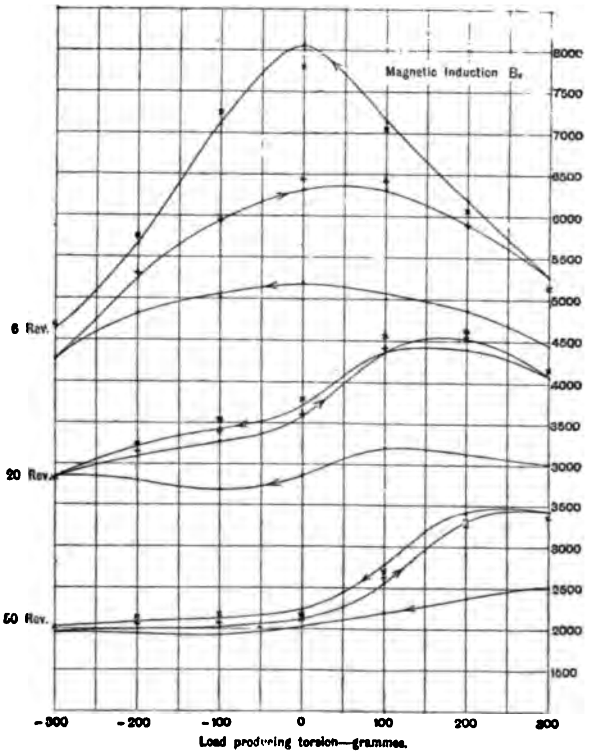


Fig. 15.

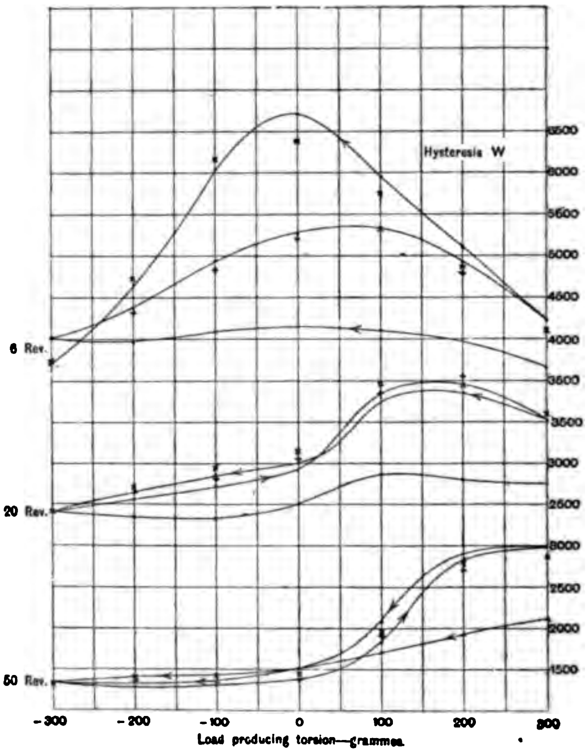


Fig. 16.

In these experiments it was found that after the first application of a negative couple due to 300 grammes the iron had roughly reached a cyclic state, and thus, to avoid confusion, we give only 1½ cycles as continuous curves. For a permanent set of 6 turns 3½ cycles were actually gone through, and in each of the other cases

2½ cycles. In each case the last cycle is indicated by isolated points; when the positive load is increasing a + is used, and when the positive load is decreasing a × is used.

In their general forms the curves due to the last cycle for 20 and 50 turns of permanent set resemble those obtained after several comparatively rapid cycles of torsion for 25 and 50 turns of permanent set in § 62. Any close comparison is impossible on account of the fact that the curves of § 62 do not form closed figures.

In the case of 6 turns of permanent set, the curves for the last cycle differ widely from those for 5 turns of permanent set in the experiments of § 62. The part of the curve which corresponds to a decreasing positive couple now lies always above that part which corresponds to an increasing positive couple instead of crossing it. The two parts of the curve show, however, a tendency to approach each other with repetition of the cycles of torsion.

In this case there is a very great recovery of magnetic quality in the first 1½ cycles of torsion. When the load was reduced to zero for the first time B_0 was 5190, at the second zero load we found $B_0 = 6310$, and at the third zero load $B_0 = 8050$. Since $H_0 = 5$, the permeability rose from 1038 to 1610.

Relation connecting W with B_0 and H_0.

§ 64. A cursory examination of the curves recording the effect of strain upon the hysteresis and the induction, for given values of the magnetic force, is sufficient to make it evident that the changes in B_0, due to strain, are very closely followed by the consequent changes in W. This correspondence is so close that it invites the attempt to express it mathematically. We have here the means of analysing W and determining it as a function of H_0 and B_0; this is impossible with the $W - B_0$ curve for zero stress, since we cannot, without straining the specimen, change B_0 without changing H_0. Our plan has been to plot curves showing W in terms of B_0 for definite values of H_0, the variations of B_0 and W being due to strain.

§ 65. The first example is taken from the experiments on the effect of torsion upon a soft iron wire, the corresponding values of B_0 and W_0 for definite values of H_0 being recorded in § 58. Five curves connecting W and B_0 were plotted for five values of H_0 upon fig. 10, where the $W - B_0$ curve for zero stress is also shown. The points corresponding to increasing torsion lie so closely on the same curves as those corresponding to diminishing torsion, that we have marked both sets in the same manner. For the larger values of H_0 each curve consists of a straight line with a small hook at one end—the end corresponding to zero strain. The straight parts of the curves all pass, on prolongation, through the point −600 on the axis of W, and hence over the greater part of the variation of B_0, caused by torsion under a constant maximum magnetic force H_0, the value of W is given by

$$W = mB_0 - 600,$$

where m is a function of H_0. To determine m in terms of H_0 we plotted m against H_0, and found that m is nearly proportional to $H^{\frac{1}{2}}_0$ as the following table shows :—

H_0	4	5	7	10	15
m	·71	·785	·99	1·15	1·28
$·35H^{\frac{1}{2}}_0$ . . .	·70	·78	·925	1·10	1·35

Hence, with fair accuracy, the formula

$$W = ·35H^{\frac{1}{2}}_0 B_0 - 600$$

represents W as a function of H_0 and B_0, over a large range, provided that the load producing torsion exceeds 100 grammes. The hooked part of each curve corresponds to small torsions due to loads of 100 grammes and under. The value of W for zero strain, recorded in § 57, is given with considerable accuracy by $W = ·326\, H^{\frac{1}{2}}_0 B_0 - 320$ over a considerable region in the neighbourhood of the maximum permeability.

§ 66. The second example is furnished by the experiments on the torsion of a steel rod, described in § 55. The curves connecting W and B_0 are shown in fig. 8 along with the $W - B_0$ curve for zero stress. We again find that for the larger values of H_0 those portion of the curves which correspond to the larger stresses are straight lines radiating from a single point—in this case the origin—and thus obtain $W = mB_0$. The values of m and H_0 are given in the table.

H_0	10	15·72	26·64	37·4
m.	1·25	1·98	2·94	3·32
$·64(H_0 - 6·2)^{\frac{1}{2}}$ . .	1·25	1·97	2·90	3·57

Thus, approximately,

$$W = ·64\,(H_0 - 6·2)^{\frac{1}{2}} B_0 .$$

This expression would naturally fail to represent facts when $H_0 < 6·2$.

The value of W for zero stress, recorded in § 54, is given closely over the whole range by $W = ·57H^{\frac{1}{2}}_0 B_0 - 1800$.

§ 67. We now take the experiments on the effect of tension upon a soft iron wire described in § 52. The curves connecting W with B_0 are shown in fig. 17 along with the $W - B_0$ curve for zero stress. Each of the curves, which shew the effects of stress, is again made up of a straight part and a hook, the straight parts radiating from the point $B_0 = -600$, $W = -950$. For the slope of the lines we have

H_0	4·524	11·00	16·24
m	·741	1·20	1·36
$·353H^{\frac{1}{2}}_0$ . . .	·750	1·17	1·42

so that the straight parts are given by

$$W = ·353H^{\frac{1}{2}}_0(B_0 + 600) - 950 .$$

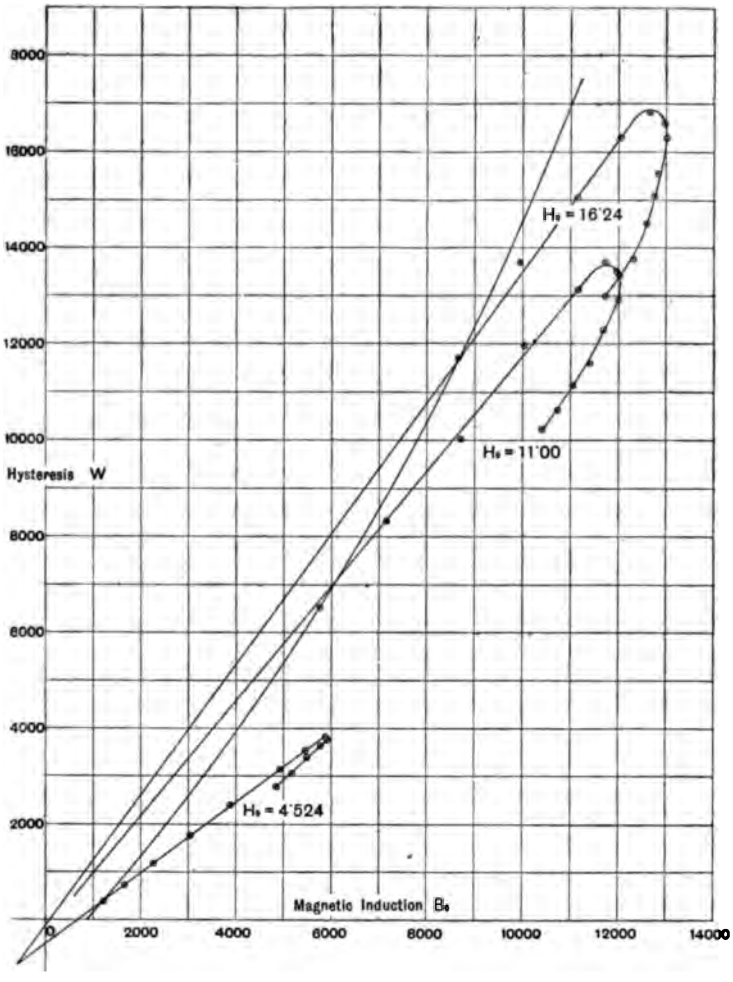


Fig. 17.

The curves just described differ in two important particulars from the curves for torsion in fig. 10. In the case of torsion the straight parts of the curves correspond to large stresses and the hooks to small stresses, but in the case of tension the straight parts correspond to small stresses and the hooks to large stresses; the hook is moreover much more developed in the curves for tension than in the curves for torsion. The second point of difference is that the tension curves lie to the right of the $W - B_0$ curve for zero stress whereas the torsion curves lie to the left.

§ 68. So far the strains have been practically within the elastic limit. We now examine some cases in which this limit was much exceeded.

In the W—B_0 curve, plotted from the experiments described in § 60, the points appear to be irregularly placed till the torsion wheel has made about eight revolutions. For greater strains the points lie well on the straight line, $W = {\cdot}610B_0 + 450$. Since $H_0 = 5$ this may be written

$$W = {\cdot}273H^{\frac{1}{2}}_0B_0 + 450\,.$$

In the experiments of § 61 the points corresponding to both the curves (2*a*) and (2*b*) cluster round a single curve, which when B_0 exceeds 2500 is represented by

$$W = {\cdot}352H^{\frac{1}{2}}_0B_0 - 850 \qquad (H = 5).$$

§ 69. We are not prepared to offer any physical explanation of the formula $W = aH^{\frac{1}{2}}_0B_0 - b$*. But as it has a rational appearance it seemed worth while to test it on some results obtained under zero stress. We plotted W against $H^{\frac{1}{2}}_0B_0$ for several of the tables given by Professor J. A. EWING and Miss KLAASSEN,† as well as for the results for cobalt obtained by Professor J. A. FLEMING.‡ In every case the resulting curve was straight over a considerable range of B_0. For small values of B_0 the value of W given by $W = aH^{\frac{1}{2}}_0B_0 - b$ is too small, while for large values of B_0 it is too large. The value of $aH^{\frac{1}{2}}B_0 - b$ begins to be too large when with increasing B_0 the permeability μ begins to fall rapidly below its maximum value. When, with decreasing B_0, μ falls much below its maximum value, $aH^{\frac{1}{2}}B_0 - b$ becomes too small. It also appears that when B_0 is less than B'_0, the value corresponding to the maximum of μ, μ may, without causing serious error, differ much more from its maximum value than when B_0 exceeds B'_0. The following examples from the experiments of Professor EWING and Miss KLAASSEN will serve as illustrations.

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* [November 4, 1901. Mr. WILLS has obtained from his experiments a series of curves showing how W, for constant values of H_0, depends upon B_0, when B_0 is varied by varying the temperature. These curves exhibit in a striking manner the characteristic features of our own curves. Thus the curve for a given value of H_0 consists of a straight line and a hook, while the straight lines, corresponding to different values of H_0, all radiate from a point on the axis of W. Mr. WILLS, however, finds that the index of H_0 is $\frac{3}{4}$ instead of $\frac{1}{2}$. The relation deduced from his experiments is thus $W = aH^{\frac{3}{4}}_0B_0 - b$.]

† "On the Magnetic Qualities of Iron," 'Phil. Trans.,' A, vol. 184, p. 985.

‡ "The Magnetic Hysteresis of Cobalt," 'Proc. Physical Soc.,' vol. 16, p. 519.

Current (amperes).	Deflexion.	Diminution, per cent.
0	141	0
·143	139	1·42
·212	137	2·84
·350	132	6·39
·518	127	9·93
·664	122	13·5
1·123	109	22·7

In an experiment made on March 7 the wire was subjected to tension due to a load varying from 0 to 20 kilogrammes. The deflexion due to hysteresis was observed (1) when no current flowed through the wire, and (2) when a current of definite strength flowed through the wire. The current was furnished by a single Daniell cell, and, judging from the last experiment, was about 1·3 amperes. In the table the last column shows the percentage diminution of the hysteresis due to the passage of the current. The numbers in this column are rather irregular, but they show quite clearly that the effect of the current diminishes as the tension increases. The initial increase of W and its subsequent decrease, noticed in detail in § 52, are well shown in this experiment.

Tension (kilos.).	Deflexion, current off.	Deflexion, current on.	Diminution, per cent.
0	73	55	24·7
4	90	76	15·6
8	98	82	16·3
12	94	82	12·8
16	84	74	11·9
20	75	67	10·7

§ 71. In concluding this paper we desire to express our thanks to Professor J. J. THOMSON for his encouragement during the progress of the experiments, as well as for the use of the resources of the Cavendish Laboratory. We are also indebted to Mr. JOHN TALBOT, of Trinity College and to Mr. W. G. FRAZER, Fellow of Queens' College, for valuable assistance during the earlier stages of the work, and to Mr. L. N. G. FILON, of King's College, for help in connexion with Appendix I. We have to thank Dr. R. S. CLAY, of St. John's College, for some help in the preliminary experiments. We gladly record our obligation to the writings of Mr. OLIVER HEAVISIDE, for it was by the method of operators, so fruitfully used by him, that we first obtained the complete theory of the method. Our thanks are also due to Mr. W. G. PYE and to Mr. F. LINCOLN, the mechanical assistants at the Cavendish Laboratory, for help and advice on many occasions. Their mechanical skill has been of great service to us.

This assumption is thus equivalent to the assumption that the second, third, . . . terms in (5) are negligible in comparison with the first term. In this case we find for the rate at which heat is generated by eddy currents

$$\pi a^2 \frac{dX}{dt} = \sigma \int_0^a u^2 2\pi r dr = \int_0^a \frac{g^2 r^2}{64\pi^2}\left(\frac{dH}{dt}\right)^2 2\pi r dr = \frac{\sigma g^2 a^4}{128\pi}\left(\frac{dH}{dt}\right)^2,$$

where dX/dt has the meaning assigned to it in § 8. Here, since dH/dt is constant over the section, we may put $\mu = dB/dH$, and thus

$$\frac{dX}{dt} = \frac{A}{8\pi\sigma}\left(\frac{dB}{dt}\right)^2 = \frac{QA}{\sigma}\left(\frac{dB}{dt}\right)^2.$$

Hence, with the notation of (13) § 9, $Q = 1/8\pi = {\cdot}03979$.

The second term of (5) will be negligible in comparison with the first, provided that $\pi\mu a^2/\sigma \, . \, d^2H_a/dt^2$ is negligible in comparison with dH_a/dt, and the third term will be negligible in comparison with the second if $\pi\mu a^2/\sigma \, . \, d^3H_a/dt^3$ is negligible in comparison with d^2H_a/dt^2.

Now, as in § 15, the characteristic of H_a is

$$K dH_a/dt + RH_a = 4\pi NE.$$

Hence, supposing that K may be treated as constant,

$$K d^2H_a/dt^2 + R dH_a/dt = 0.$$

Thus we see that the ratio of each term in (5) to the term before it is small, provided that $\pi\mu a^2/\sigma$ is small compared with the "time constant" K/R. When this condition is satisfied, we may treat dH/dt and also dB/dt as constant over the section of the rod, and may then calculate the eddy current from (1).

On the Heat produced by Eddy Currents in a Rod of Rectangular Section.

§ 73. The section of the rod is supposed so small that the current at any point may be calculated by FARADAY'S law, on the assumption that dB/dt has the same value at all points of the section. We see by the case of the circular rod that this assumption is legitimate, provided that $\pi\mu r^2/\sigma$ is small compared with K/R, r being the radius of the largest circle inscribable in the section.

Let, now, a, b be the sides of the rectangular section, and let the origin be at the centre of the section. Then, since the magnetic force is parallel to the axis of the rod, we have, under the specified conditions

$$\frac{du}{dy} - \frac{dv}{dx} = q \quad . \; . \; . \; (1), \qquad \frac{du}{dx} + \frac{dv}{dy} = 0 \quad . \; . \; . \; (2),$$

$$u = 0 \text{ when } x = \pm \tfrac{1}{2}a \; . \; . \; . \; (3), \quad v = 0 \text{ when } y = \pm \tfrac{1}{2}b \; . \; . \; . \; (4),$$

where u, v are the components of the current, and $q\sigma = dB/dt$.

Now (2) is satisfied if we write

$$u = d\phi/dy, \qquad v = -d\phi/dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

while (1) now becomes

$$d^2\phi/dx^2 + d^2\phi/dy^2 = q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

The solution of (6), appropriate to the problem in hand,* is

$$\phi = -\frac{16q}{\pi^2}\Sigma\Sigma(-1)^{m+n}\frac{\cos(2m+1)\pi x/a \,.\, \cos(2n+1)\pi y/b}{(2m+1)(2n+1)\{(2m+1)^2\pi^2/a^2+(2n+1)^2\pi^2/b^2\}} \,. \quad . \quad (7).$$

This value of ϕ satisfies $\nabla^2\phi = q$, because, m and n both ranging from 0 to ∞,

$$1 = \frac{4}{\pi}\Sigma(-1)^m\frac{\cos(2m+1)\pi x/a}{2m+1} = \frac{4}{\pi}\Sigma(-1)^n\frac{\cos(2n+1)\pi y/b}{2n+1}$$

within the limits $x = \pm\frac{1}{2}a$, $y = \pm\frac{1}{2}b$.

Now by (5) and (7)

$$u = \frac{d\phi}{dy} = \frac{16q}{\pi^2}\Sigma\Sigma(-1)^{m+n}\frac{\cos(2m+1)\pi x/a \,.\, \sin(2n+1)\pi y/b}{(2m+1)(2n+1)\{(2m+1)^2\pi^2/a^2+(2n+1)^2\pi^2/b^2\}} \,.\, \frac{(2n+1)\pi}{b},$$

with a similar expression for v. These expressions satisfy (3) and (4), and thus all the conditions are fulfilled.

The rate at which heat is generated is given by

$$ab\,d\mathrm{X}/dt = \sigma\iint(u^2+v^2)\,dxdy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The necessary integrations are easily effected, for the integral

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\cos(2h+1)\pi x/a \,.\, \cos(2k+1)\pi x/a \,.\, dx$$

is zero unless h and k are equal. When $h = k$, its value is $\frac{1}{2}a$. Similar results hold when two sines are substituted for the two cosines.

We thus obtain

$$\frac{d\mathrm{X}}{dt} = \frac{64q^2\sigma a^2}{\pi^6}\Sigma\Sigma\frac{1}{(2m+1)^2(2n+1)^2\{(2m+1)^2+(2n+1)^2a^2/b^2\}} \,.$$

It is not convenient to calculate $d\mathrm{X}/dt$ from this double series, on account of the slow convergence. We therefore transform it into a single series. Now

$$\cosh\pi z/2 = (1+z^2)(1+z^2/3^2)(1+z^2/5^2) \ . \ . \ .$$

Differentiating the logarithm of both sides, we obtain

$$\frac{\pi}{2}\tanh\frac{\pi z}{2} = 2z\Sigma\frac{1}{(2m+1)^2+z^2} \,. \qquad (m \text{ from } 0 \text{ to } \infty).$$

---

* The method of solution was suggested by notes taken by one of us at a course of lectures on hydrodynamics, given by Mr. R. A. HERMAN, Fellow of Trinity College, in November, 1889.

Expanding in powers of z and comparing the coefficients of z and of z^3, we find

$$\pi^2/8 = \Sigma\,(2m+1)^{-2}, \qquad \pi^4/96 = \Sigma\,(2m+1)^{-4}.$$

Hence
$$\frac{\pi^2}{8z^2} - \frac{\pi}{4z^3}\tanh\frac{\pi z}{2} = \Sigma\frac{1}{(2m+1)^2\{(2m+1)^2+z^2\}}.$$

Now put $z = (2n+1)\,a/b$, and then

$$\frac{dX}{dt} = \frac{64q^2\sigma a^2}{\pi^6}\Sigma\frac{1}{(2n+1)^2}\left\{\frac{\pi^2 b^2}{8(2n+1)^2a^2} - \frac{\pi b^3\tanh(2n+1)\,\pi a/2b}{4(2n+1)^3a^3}\right\}$$

$$= \frac{ab}{\sigma}\left(\frac{dB}{dt}\right)^2\left\{\frac{b}{12a} - \frac{16b^2}{\pi^5a^2}\Sigma\frac{\tanh(2n+1)\,\pi a/2b}{(2n+1)^5}\right\}.$$

Writing this in the form $dX/dt = QA(dB/dt)^2/\sigma$, we have

$$Q = \frac{b}{12a} - \frac{16b^2}{\pi^5a^2}\Sigma\frac{\tanh(2n+1)\,\pi a/2b}{(2n+1)^5} \quad \text{. . . . . (9).}$$

As a/b increases, the expression (9) very rapidly tends to the limit

$$Q = \frac{b}{12a} - \frac{16b^2}{\pi^5a^2}\Sigma\frac{1}{(2n+1)^5} = \frac{b}{12a} - \cdot 05255\,\frac{b^2}{a^2} \quad \text{. . . . . (10)}$$

[since $\Sigma(2n+1)^{-5} = 1\cdot 0045$], the error not exceeding 1 in 4000 when a/b is as small as 2. We give a table of the values of Q for small values of a/b; it will be seen that Q is rather smaller for a square than for a circular rod.

a/b.	1.	1·5.	2.	2·5.	3.	4.	8.	16.	32.	64.
Q	·03512	·03260	·02853	·02492	·02194	·01755	·00960	·00500	·00255	·00129

Mr. L. N. G. FILON has kindly verified our results, employing the mathematical method used by DE ST. VENANT in finding the torsional rigidity of a rectangular prism. (*Cf.* THOMSON and TAIT, 'Nat. Phil.,' Part II., p. 248.)*

APPENDIX II.

On the Demagnetising Force due to Rods of Finite Length.

§ 74. In order to find how the demagnetising force at the centre of a long cylindrical rod depends upon the induction at the centre of the rod, the following experiments were made. The magnetising solenoid was placed at right angles to the leaks out from the wire by the cylindrical surface, thus developing a distribution of magnetism along the wire, h_0 is considerably greater than it would be were the "poles" at the ends of the wire. As B_0 increases, h_0 increases less rapidly, reaches a maximum for a value of B_0 [10300] somewhat greater than B'_0 [8400], the value corresponding to μ' the maximum value of μ, and then diminishes. As B_0 increases still further, h_0 diminishes rapidly till B_0 reaches the stage where μ begins to diminish comparatively slowly. After passing a minimum h_0 again increases, and in this last stage is less than $2I_0A/l^2$.

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* [For an elliptic cylinder, axes $2a$, $2b$, $\phi = \frac{1}{2}q\,(x^2/a^2 + y^2/b^2)\,a^2b^2/(a^2+b^2)$ satisfies (6) as well as the condition that no current crosses the bounding surface.

We hence find $Q = ab/\{4\pi\,(a^2+b^2)\}$.—*December* 26, 1901.]

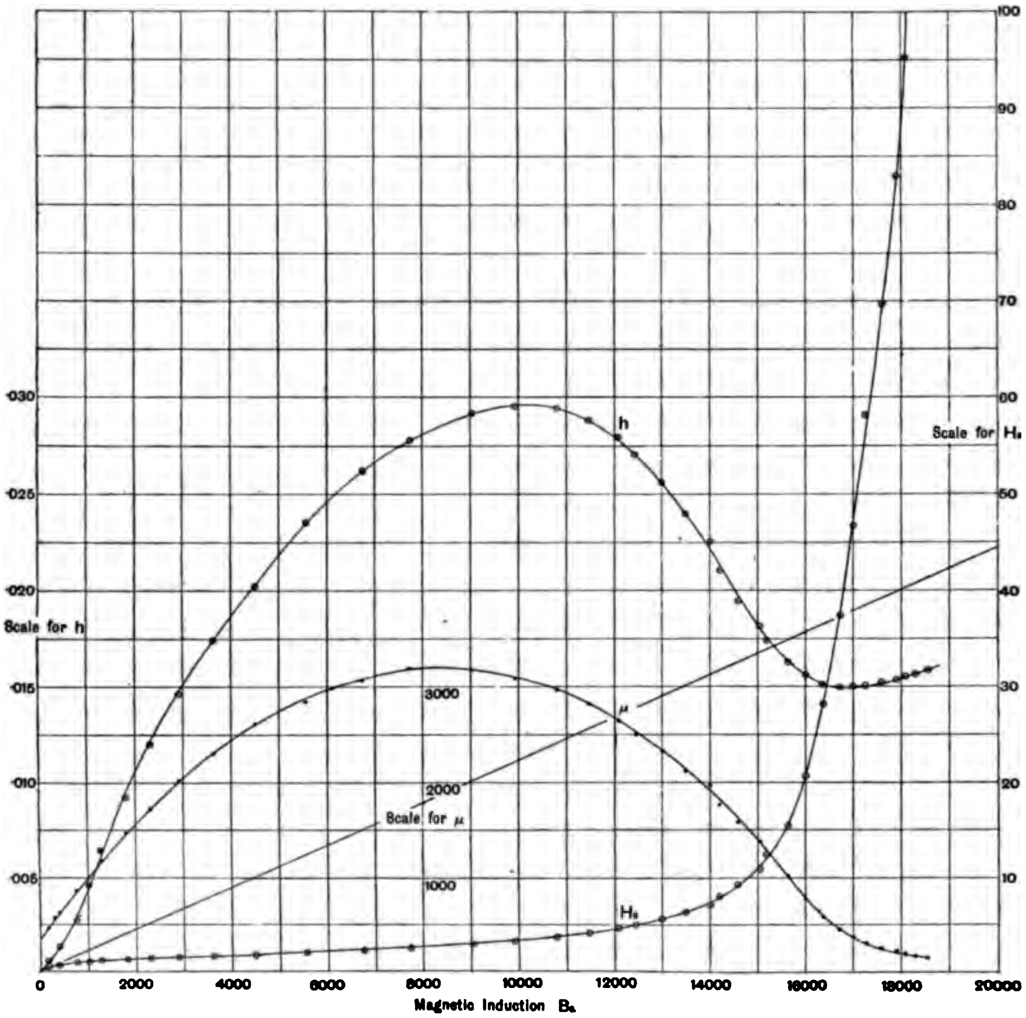

Fig. 18.

§ 76. The experiments of Mr. C. G. LAMB, "On the Distribution of Magnetic Induction in a Long Iron Bar,"* serve to explain the rise of h_0 to a maximum, and its subsequent fall. For he found that as B_0 increases from zero, the percentage of the induction at the centre of the rod which leaks out between the centre and the end of the rod, increases until B_0 reaches B'_0 approximately. Thus as B_0 increases we may regard the "poles" as moving inwards towards the centre. There is thus a double reason why h_0 should increase with B_0 during this stage. When B_0 increases beyond B'_0, he found that the "poles" move out again towards the ends of the rod, and we must suppose that this motion more than compensates for the increasing strength of the poles with increasing B_0.*

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* 'Proc. Phys. Soc.,' vol. 16, p. 509; or 'Phil. Mag.,' Sept., 1899.

In our experiments, for the largest values of B_0, h_0 was *less* than $2I_0A/l^2$. This effect can only arise from the development of a subsidiary pole between the centre and either end of the wire, with sign opposite to that of the pole at the end. This result is easily seen to be impossible with a bar of uniform material as long as the induction is a definite single-valued function of the magnetic force, which increases as the magnetic force increases. It must therefore be due to the effects of hysteresis, which becomes an increasingly important factor as the point of maximum permeability is reached and passed. A different result might perhaps have been obtained if the magnetic force had been gradually increased from zero. But in our experiments, as in those of LAMB, the magnetic force was put through several cycles between the limits $\pm H_0$ before the observations were made.

§ 77. When iron is tested by the ballistic method for hysteresis, the specimen is subjected to an applied magnetic force which is reversed between the limits $\pm H_0$ until the iron has reached a cyclic state. This attained, the points on the cyclic B--H diagram are found from the throws of the galvanometer which occur when the magnetic force is suddenly changed, by means of a special key, from H_0 to a series of values between H_0 and $-H_0$.

We made hysteresis tests by this method upon two specimens, determining also the demagnetising force, h, at the centre of the wire, for each value of H which was employed in constructing the B—H curve. Thus, the magnetic force was changed from H_0 to H, and the throw of the galvanometer, giving $B_0 - B$, was observed, and then, without altering the magnetic force from its value H, the deflexion of the magnetometer was noted. We reckon h_0 positive when its direction is opposite to that of the induction at the centre of the specimen.

§ 78. The first specimen was a bundle of ten iron wires with a total area of section of ·0412 sq. centim. The length of the wire was equal to the length of the solenoid,

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* Dr. L. HOLBORN, in a paper "On the Distribution of Induced Magnetism in Cylinders" ('Sitzungsberichte der Akademie der Wissenschaften zu Berlin,' 17th February, 1898), has obtained a result similar to that found by Mr. LAMB. He used two secondary coils, one a uniformly wound solenoid closely fitting the rod, the other a coil wound about the centre of the rod. The rod and the secondary coils were placed inside a long magnetising solenoid. By comparing the changes of induction through these two secondaries due to a reversal of the primary current, he found the distance λ between the "centres of gravity" of the free magnetism on the two halves of the rod. He found that the "centres of gravity" move towards the centre as B_0 increases, until B_0 reaches B'_0. A further increase of B_0 caused the "centres of gravity" to move out again from the centre. Dr. HOLBORN made similar experiments on ellipsoids, and found that for them λ was remarkably constant.

47 centims., and thus since the internal radius of the windings was about 2 centims. the applied magnetic force was far from uniform near the ends of the wires. The results of this experiment are shown in figs. 19, 20. As we pointed out in § 18, h exhibits hysteresis with respect to both B and H. In fig. 19 the straight line through the origin represents $2\text{BA}/4\pi l^2$, the demagnetising force corresponding to uniform magnetisation, H in these experiments being negligible in comparison with B. From the areas of the curves we found

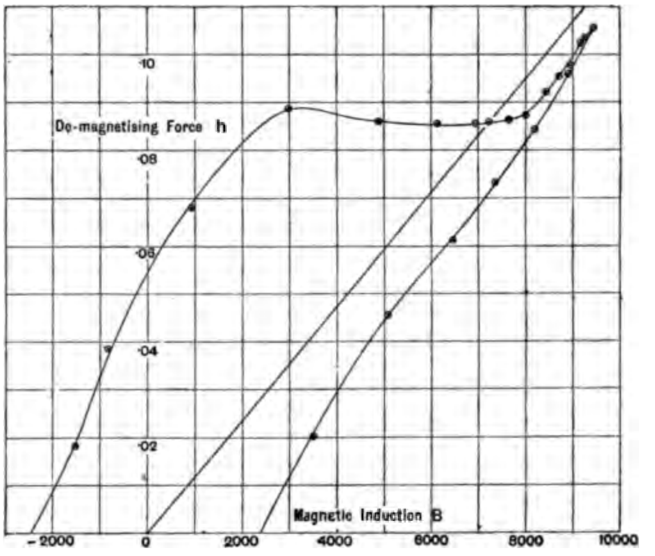

Fig. 19.

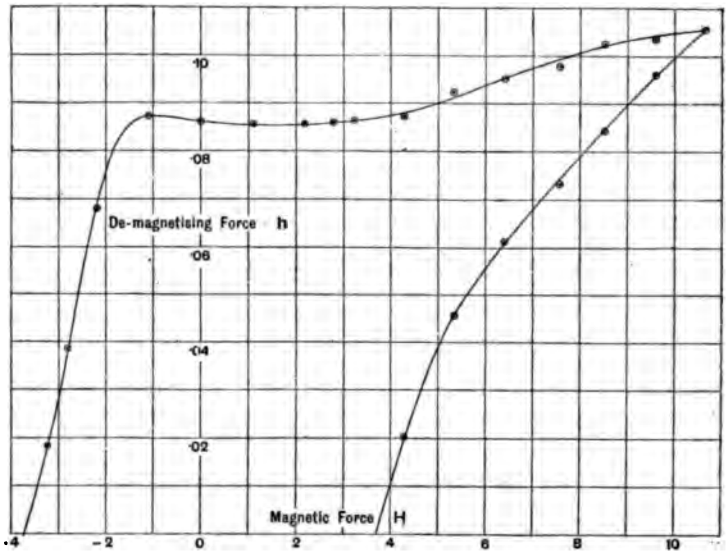

Fig. 20.

$$\int \text{H}d\text{B} = 89200, \qquad \int hd\text{B} = 949, \qquad \int hd\text{H} = 1{\cdot}57.$$

§ 79. The second specimen was a single iron wire 47 centims. in length, and ·00412 sq. centim. in section. The magnetising solenoid was that described in § 75, and thus the applied magnetic force was practically uniform over the whole length of the wire. We examined the polarity of the eastern end of the wire at every stage by means of a small compass. We show the results of the experiment in figs. 21, 22, and also in the following table, where the last column indicates the polarity of the eastern end. It will be seen that the polarity of the end of the wire agrees in every case with that corresponding to the direction of the induction at the centre of the wire. In fig. 21 the straight line through the origin represents $2BA/4\pi l^2$, the demagnetising force corresponding to uniform magnetisation, H in these experiments being negligible in comparison with B.

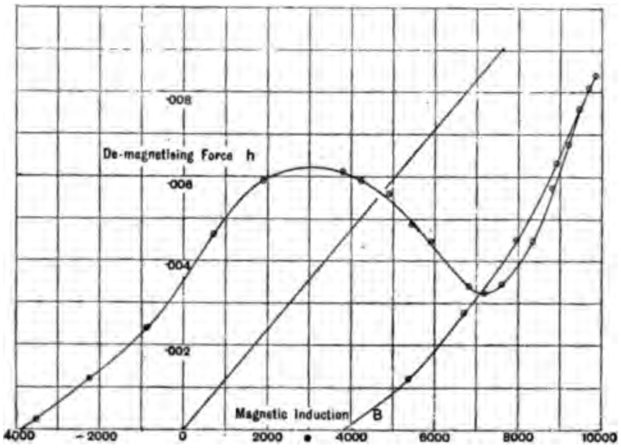

Fig. 21.

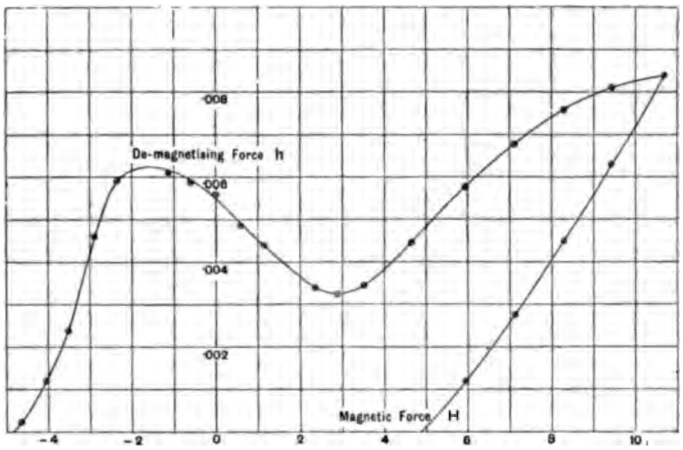

Fig. 22.

H.	B.	*h*.	Pole.	H.	B.	*h*.	Pole.
10·67	9850	·00839	N.	- 0·59	4250	·00587	N.
9·44	9675	·00810	N.	- 1·13	3800	·00610	N.
8·30	9465	·00757	N.	- 2·36	1900	·00590	N.
7·12	9204	·00676	N.	- 2·88	730	·00460	N.
5·95	8800	·00575	N.	- 3·52	- 850	·00238	S.
4·65	8320	·00446	N.	- 4·05	-2200	·00120	S.
3·52	7590	·00344	N.	- 4·65	-3480	·00023	S.
2·88	7190	·00323	N.	- 5·95	-5370	-·00119	S.
2·36	6820	·00338	N.	- 7·12	-6680	-·00276	S.
1·13	5900	·00440	N.	- 8·30	-7950	-·00450	S.
0·59	5450	·00486	N.	- 9·44	-8900	-·00630	S.
0	4900	·00559	N.	-10·67	-9850	-·00839	S.

As B diminishes from 9850, *h* diminishes to a minimum at $B = 7200$, and then increases to a maximum at $B = 3000$; after this it passes to its greatest negative value corresponding to $B = -9850$ without passing through a maximum or minimum. As B increases again to 9850, *h* goes through a similar set of changes, its values from $B = 7000$ to 9850 differing but little from those which it had when B was diminishing. From $B = 9850$ to 4700, *h* is less than $2BA/4\pi l^2$, and from $B = 0$ to $B = -3800$, *h* has the opposite sign to B. Both these cases require that there should be "poles" on the wire between the centre and the ends, with signs opposite to those of the poles at the ends of the wire. Our arrangement of apparatus was not well adapted for detecting these subsidiary poles, since the magnetic force due to a uniformly distributed pole would be at right angles to the wire, and therefore parallel to the magnetic meridian, thus producing no deflexion of the search compass. Still we were able to verify the conclusion when B was -3480, for though the corresponding S-magnetism appeared at the eastern portion of the wire, being most concentrated at a point about 7 centims. from the end, yet we found a weak N-pole at 9 centims. from the same end.

From the areas of the curves we found

$$\int H dB = 102100, \qquad \int h dB = 72{\cdot}3, \qquad \int h dH = {\cdot}118.$$

These experiments add emphasis to Mr. LAMB's remark that "the magnetometric method [of determining I—H curves], although extremely useful for comparative work, must be used with much caution in determinations of an absolute character."

[*December* 23, 1901.—C. BENEDICKS ('Annalen der Physik,' 1901, vol. 6, p. 726) compared the B—H curve for a cylinder of steel with the curve for an ellipsoid formed out of the same piece of metal, and deduces that, as B increases, *h* rises to a maximum and then rapidly decreases, as in our fig. 18. From EWING's experiments ('Phil. Trans.,' 1885, pp. 532, 535) on (1) a ring, (2) straight pieces of iron wire, he deduces the same result. Neither experimenter, however, reached the minimum of *h*.]

INDEX SLIP.

STEELE, B. D.—The Measurement of Ionic Velocities in Aqueous Solution, and the Existence of Complex Ions.
Phil. Trans., A, vol. 198, 1902, pp. 105–145.

Ionic Velocities, Measurement of; Existence of Complex Ions.
STEELE, B. D. Phil. Trans., A, vol. 198, 1902, pp. 105–145.

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change is different in different cases; for lithium chloride, and the chlorides and sulphates of the alkaline earth metals, p diminishes with increasing dilution, whilst for silver nitrate, the opposite is the case. The measurements of the same salt which have been made by different investigators, although they show considerable differences amongst themselves, nevertheless all point to the same general conclusion; namely, that it is only for a very limited number of salts that the transport number is independent of concentration. These differences are probably due in part to the difficulties of the method, and the very great influence of a small experimental error in the determination of small changes of concentration, and partly also to the fact that, following HITTORF, many workers have employed membranes of one kind or another to separate different portions of the solution, and thus prevent mixing by convection currents, whilst others again have altogether avoided their use. BEIN ('Zeit. Phys. Chem.,' 28, 439, 1898) has shown that the use of certain membranes affects in a very remarkable manner the value of the transport number, and it is probable that a considerable number of the discrepancies are to be traced to this cause. Another serious difficulty consists in the fact that an experiment must only be carried on for such a time that no change in concentration can take place in the middle part of the solution, and for many salts the formation of hydrogen and hydroxyl ions at the electrodes further diminishes this time on account of the great velocity of these ions. The danger from hydrogen ions has been minimised by the use of a cadmium anode, and quite recently NOYES ('Zeit. Phys. Chem.,' 36, 63, 1901) has completely overcome this difficulty by the device of adding small quantities of acid and alkali to the solution in the neighbourhood of the anode and cathode respectively.

The measurements of NOYES are probably the most accurate that have been made, and his results again confirm the statement made above.

The importance of and the need for a measurement of the resistance of aqueous solutions of electrolytes was repeatedly referred to by HITTORF; but it was not until the development by KOHLRAUSCH of his well-known method—in which, by the use of an alternating current, the polarisation effect is neutralised—that this problem could be successfully attacked.

From the measurement of the conductivities of salt solutions, KOHLRAUSCH ('Wied. Ann.,' 6, 1, 26, 213) deduced the law of "the independent wandering of the ions," which states that the molecular conductivity of an aqueous solution of an electrolyte is the sum of two constants, of which one depends only on the nature of the cation, and the other only on the nature of the anion; and he further assumed that these constants are proportional to the velocities of the ions, $\mu \propto U + V$. The values of U and V are obtained from the molecular conductivity, which is the sum, and the transport number, which is the ratio, of the two velocities, or $U \propto (1 - p)\mu$ and $V \propto p\mu$.

It is found further, that with increasing dilution the molecular conductivity increases, until it finally reaches a constant value at very great dilutions. This is the potential fall depends on the concentration; any changes in which, that may take place during electrolysis will condition corresponding changes in potential fall. The question of concentration changes and the movements of these during electrolysis is discussed by KOHLRAUSCH ('Wied. Ann.,' vol. 62, 209, 1897), in whose calculations it is assumed that the electrodes are far removed from the part of the system under consideration; the effects of ordinary diffusion are also not dealt with.

If we consider the case of electrolysis in a long narrow column of liquid, and neglect the movements that take place in directions at right angles to the length of the tube, dealing only with those in the direction of the axis, KOHLRAUSCH'S general equation takes the form

$$\frac{\partial c}{\partial t} = -i\frac{\partial}{\partial x}\left(\frac{uc}{\mu}\right),$$

where c is the ionic concentration of one species of ion (cation), that of the anion being necessarily the same;
i is the current density;
x is the length of the tube;
u is the velocity of the cation whose concentration is c;
μ is the molecular conductivity of the solution. If v is the velocity of the anion and

$$\mu = (u + v)\,c;$$

then

$$\frac{uc}{\mu} = \frac{u}{u + v} = (1 - p) = p',$$

where p' is the cation transport number, and the equation takes the form

$$\frac{\partial c}{\partial t} = -i\frac{\partial p'}{\partial x} = -i\frac{\partial p'}{\partial c}\cdot\frac{\partial c}{\partial x}.$$

From this it follows that a change in concentration can be brought about only in the case that there exists an initial concentration change, dc/dx, together with a change in transport number with change in c; when the solution is originally homogeneous no variation in c is caused even if p' varies with c. If, however, the solution is not originally of the same concentration throughout, a portion, s, of the solution being, perhaps, more dilute than the remainder, then if $\partial p'/\partial c = 0$, or if there is no variation in transport number, s remains stationary, "as many ions leave the section as enter it."

If $\partial p'/\partial c$ is positive, that is, if the cation transport number increases with increasing concentration, then s moves in the direction of the current.

If $\partial p'/\partial c$ is negative, the movement is in the opposite direction.

A sharp margin between a concentrated and a dilute solution of the same salt is

If, on the other hand, the following ion has the greater velocity, the velocities of the 2 ions are given by

$$\frac{dx_1}{dt} = \frac{iu^2}{u_1(u_1 + v_1)c}$$

and

$$\frac{dx_2}{dt} = \frac{iu_1}{(u_1 + v_1)c},$$

where u and v are the velocities of the preceding and $u_1 . v_1$ those of the following ions.

The 2 ions no longer move with the same velocities, but a mixing takes place, with the result that no stable boundary is to be expected.

Whetham (*loc. cit.*), avoiding altogether the use of gelatine, measured the velocity of the boundary between two electrolytes having a common ion, and by the device of selecting pairs of solutions which possessed the same, or nearly the same, specific resistance, obtained an approximately uniform potential fall for the whole column. He was thus able to convert the observed velocities into those which would be occasioned by a fall in potential of 1 volt per centimetre.

Although, as previously mentioned, the conditions for stability of the boundary are pointed out by Whetham, in his experiments the values obtained are the means of two sets of measurements in which the boundary moves alternately in opposite directions, and generally with slightly different velocities.

Most of his figures show a very good comparison with those calculated by Kohlrausch, and the measurements as a whole form the first direct confirmation of the theory. Proof of the fact that the velocity is proportional to the potential fall is also given in this paper.

In a second paper, Whetham (‘Phil. Trans.,’ A, vol. 186 (1895), p. 507) measured the velocity of a number of ions in gelatine solution: in some of these experiments the position of the boundary was indicated by means of chemical indicators. The results show a very good agreement with Kohlrausch's figures.

Masson (*loc. cit.*), employing a gelatine solution of the salt, compares directly the velocity of the anion and the cation margins, which he shows to be dependent only on the nature of the ions, provided certain conditions are fulfilled; the potential fall although unknown is the same for both boundaries, since between these the concentration, and so also the resistance, is the same at all points. His experiments afford a striking confirmation of the Kohlrausch theory, since he shows that it is possible to calculate the current by measuring the velocity of the two margins.

The general theory of electrolysis is briefly summed up by the equation,

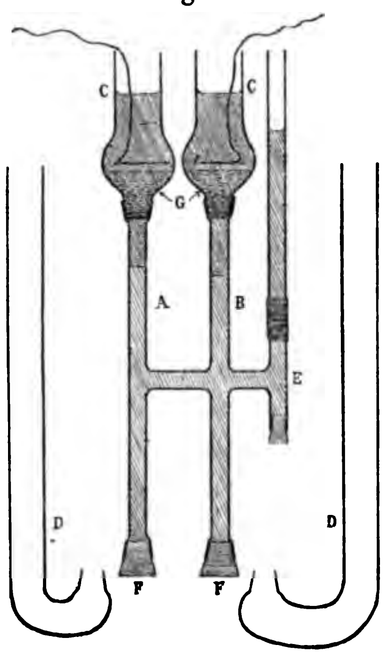
$$C = A\frac{n}{\eta}(U + V) = A\frac{n}{\eta}(u + v)\pi x,$$

since the observed velocities $U = \pi xu$, and $V = \pi xv$.

the ratio $\frac{V}{U}$ or $\frac{V}{U+V}$; as will be seen, it also measures, in the case of salts of the type of sodium chloride, the actual velocity $U = xu$ for any concentration.

The essential feature of the method consists in the imprisonment of the liquid to be measured between two partitions of jelly* containing the indication ions in solutions, thus preventing displacement of the liquid during the course of an experiment.

Fig. 1.



A large number of different forms of apparatus have been tried, and it has been found that for the measurements of the simplest type of salts, the apparatus shown in fig. 1 is most convenient.†

In the figure, A and B are two carefully selected glass tubes having an even bore, and being both of the same area of cross-section; they are joined by a short piece of tubing of larger diameter. The tube B is also provided with a tube E, by means of which the liquid during measurement is exposed to the atmospheric pressure.

Each tube is fitted at either end with the vessel C, C and D, D which are carefully ground in. For an experiment, two of the vessels (C, C if the indicator solutions are lighter, and D, D if heavier than the measured solution) are taken and filled to a depth of 2·5 centims. with a gelatine solution of the indicator to be employed.

The open ends of the tube having been first closed with rubber stoppers F, the apparatus is filled with the solution and the cells C C placed in position, care being taken not to enclose any air bubbles. The apparatus is then completely immersed in a water-bath, which is provided with parallel walls of good plate-glass, in order that the observations may not be affected by uneven refraction at the surface.

The electrodes are then placed in position, and the current is started by pouring into the cells solutions of the same indicator as that contained in the jelly.

Since the boundaries, at first coincident with those between gelatine and aqueous solutions, advance shortly after the current is started into the tubes A and B, the presence of the jelly can have no influence on their velocities, which are conditioned sionally be produced even when conditions 1 and 3 are not fulfilled, but, as is seen from the equations of WEBER, already given, its velocity is no longer that of the ion in the intermediate solution. Condition No. 4 is sufficiently obvious, nevertheless it is necessary to point out that it is not sufficient that the difference in density of equimolecular solutions should be known; for, since concentration changes are brought about during electrolysis in the sense that the indicator solution is always (provided condition 1 is fulfilled) of less concentration than the measured solution, it may, and sometimes does, happen that an indicator solution when placed beneath the solution to be measured, and which in equimolecular solutions is more dense than the latter, becomes, through these changes, lighter than and accordingly rises through the overlying solution, with the result that no boundary can be obtained; this behaviour is shown with 2 N potassium chloride solution followed by 2 N copper chloride. According to the relation $c/c' = p/p'$, the concentration of the copper solution becomes reduced to 1·2 N, the density of which = 1·0711, whilst that of 2·0 N potassium chloride = 1·0886. Hence for this reason copper chloride, although its density, as calculated from VALSON'S moduli (see NERNST'S 'Theoretical Chemistry,' p. 333), is much greater than that of potassium chloride, cannot be used as an indicator for the latter from underneath.

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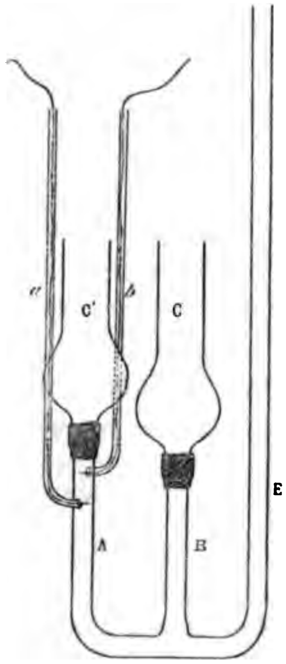
* Attempts have been made to use a porcelain membrane instead of gelatine, but it was always found that the liquid was forced through the membrane in the direction of the anode, probably on account of electric endosmose.

† A modification of this form of apparatus may be made by connecting the tube with the electrode cells by means of rubber tubing, as shown in fig. 3.

In addition to the experimental evidence of these concentration changes that has been given by MASSON, further qualitative evidence is found in the fact that in an experiment in which lithium or magnesium follows potassium in a solution of potassium chromate, so that there is formed electrolytically a solution of lithium or magnesium chromate overlying one of potassium chromate, the indicator solution is seen to be distinctly lighter in colour than the measured. Since the colour depends only on the anion, this shows the concentration of the latter is less in the indicator solution.

Fig. 2.



More direct and conclusive evidence was obtained in the following manner:—

An apparatus, shown in fig. 2, was employed. The tubes A and B, instead of having the anode and cathode cells ground in, are provided with pieces of thick, strong, india-rubber tubing, which are firmly bound on with copper wire, into which the cells C and C′ can be fitted.

It is found in practice that no displacement of the latter takes place during the course of an experiment, and, in order to reduce the danger of melting the jelly by the current, the necks of the vessels C and C′ are made a little larger than the tubes A and B; the tube E serves the double purpose of a support or handle, by means of which the apparatus may be held, and also to allow for any slight contraction or expansion that may take place through unequal heating.

be seen, and a superior limit above which the boundary is rendered useless for observation by the "washing" and "mixing" effect of too great a current density and consequent heating. For a few pairs of solution it has not been found possible to obtain a refraction margin under any conditions that have been tried.

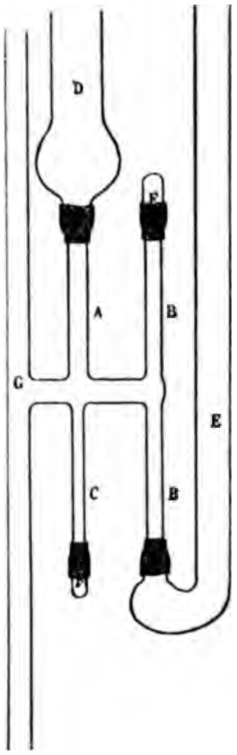
The potential fall in all cases is calculated on the assumption that the solution between the two boundaries is homogeneous, being of the same concentration and specific resistance throughout. Since OHM's Law holds good for electrolytes, E = CR; and from the specific resistance r the resistance of 1 centim. of the liquid column is obtained by dividing by the area of the tube, and hence the potential fall $\frac{d\pi}{dx} = \frac{\mathrm{C}r}{\mathrm{A}}$, where C is the current, r the specific resistance, and A the area of cross-section of the column of liquid.

A striking example of the influence of potential fall on the condition of the margin is seen in the case of a normal solution of barium chloride, using the apparatus shown in fig. 1. With magnesium chloride and sodium acetate as indicators the anion boundary is that between barium acetate and barium chloride, and will be represented by $\mathrm{Ba}\,\frac{\mathrm{Ac}}{\mathrm{Cl}}$, the cation boundary that between magnesium chloride and barium chloride $= \frac{\mathrm{Mg}}{\mathrm{Ba}}\,\mathrm{Cl}$. Starting the experiment with a potential fall of 1 volt per centim., a fair but not very good anion boundary is produced, but there is no sign of a boundary at the other end; on increasing the voltage to 1·20, the anion margin becomes very sharp and easy to read. At the cation end the gas flame, when viewed through the telescope of the cathetometer, is seen to be slightly distorted, but no boundary has yet appeared. With a voltage of 1·5 the anion boundary shows signs of "washing," whilst that at the cation end is still too indistinct for use. At 2·0 volts it has become good and distinct, whilst from the anion margin little whirlpools are seen to rise, and it has become undulating and sharp as though it were cut with a knife. With further increased potential fall the cation boundary remains good, until about 3·5 volts, when it, in its turn, begins to show signs of "washing" and consequent mixing.

For the determination of the ratio $\frac{\mathrm{V}}{\mathrm{U}}$ or $\frac{\mathrm{V}}{\mathrm{U}+\mathrm{V}}$ for salts which, like barium chloride, require a different potential fall at the two boundaries, the form of apparatus shown in fig. 3 has been found suitable.

Here the tubes to be used for the measurement are four in number, and are indicated by the letters A, B, and C. The sectional areas of these tubes are different, and each is carefully calibrated, and its area of cross-section determined. Since the potential fall—$\frac{d\pi}{dx} = \frac{\mathrm{C}r}{\mathrm{A}}$, it follows that, by selecting tubes of different sizes, any required ratio between the potential falls at the two margins may be easily obtained. It is found in practice that three sizes of tubes are sufficient for all the cases that are likely to arise, the ratio of the areas of the tubes being approximately 1, : 1·5, : 2·25. In this apparatus, as in that shown, fig. 2, the electrode cells, instead of being ground into the tubes, are connected by means of thick rubber tubing; the tube G serves as a handle by which the apparatus may be held in a clamp, and also, like E in figs. 1 and 2, to allow for expansion of the liquid.

Fig. 3.



The apparatus shown in the figure permits of the use of the following combinations: The tubes A and B may be used in three ways: with two cells similar to D both indicators may be placed on top; with one vessel, D, and one E, as shown in the figure, one indicator may be used from above and one from underneath; and with two vessels such as E both indicators from underneath. The same combination may be employed with the tubes B and C, whereas with A and C the apparatus can be used only in the second of these three manners; by the use of a second similar apparatus in which the tube C is replaced by B, and *vice versâ*. All possible combinations of these three tubes, two at a time, may be obtained.

For the calculation of $\frac{V}{U}$ from the observed velocity, since the latter is proportional to the potential fall, and this inversely as the sectional area of the tubes, it is only necessary to multiply the velocity of the boundary in one tube by the ratio of the two areas. Thus, using the tubes A and C, the sectional areas of which are α and γ, if U′ and V′ are the actually observed velocities of the two margins, the ratio of the ionic velocities is given by

$$\frac{U}{V} = \frac{\alpha}{\gamma}\frac{V'}{U'}.$$

In Table I. are given the potential falls which have been found to give a measureable boundary in the case of thirty-eight different pairs of solutions. In addition to those tabulated, the following have also been examined, but no good margins could be obtained:

$$H\frac{C_2H_3O_2}{NO_3},\ H\frac{C_2O_4}{NO_3},\ H\frac{C_6H_5SO_3}{NO_3}\left(0{\cdot}5\,N\frac{Cd}{Cu}SO_4\right),\ \frac{Mg}{Li}Cl,\ \frac{Li}{Mg}Cl,\ \frac{Cu}{Li}Cl,\ \frac{Li}{Cu}Cl,$$

and cadmium chloride as indicator following 2N solutions of the chlorides of potassium, lithium, magnesium, and calcium.

In the first column of the table is given the system forming the boundary, thus: $K\frac{A}{C}$ represents the boundary between an acetate and a chloride whose common cation is potassium, and in all cases the indicator ion is placed over the measured ion.

Table I.

Margin.			N = 0·5.		N = 1·0.		N = 2·0.	
	A.	B.	A.	B.	A.	B.	A.	B.
$K\,\frac{Ac}{Cl}$	—	—	0·82	—	—	—	0·84	—
$Na\,\frac{Ac}{Cl}$	—	—	0·735	-	1·13	—	0·92	—
$Li\,\frac{Ac}{Cl}$	—	—	1·57–1·98	—	1·06	—	1·94	1·1–2·3
$Ba\,\frac{Ac}{Cl}$	—	—	1·06	—	1·25	1·1–1·6	1·02	—
$Sr\,\frac{Ac}{Cl}$	—	—	1·40	—	1·02	—	1·32	1·0–1·6
$Ca\,\frac{Ac}{Cl}$	—	—	1·2	0·916–1·61	1·18	—	1·2	0·86–1·61
$Mg\,\frac{Ac}{Cl}$	—	—	1·7	1·55–1·90	1·53	—	1·25–1·41	—
	N = 0·2.							
$Mg\,\frac{Ac}{SO_4}$	2·38	—	2·05	—	2·1	—	2·26	1·1–
$Cu\,\frac{Ac}{SO_4}$	—	—	1·6	1·14–2·7	1·6	—	1·6	—
$K\,\frac{Ac}{CrO_4}$	—	—	1·46–1·89	—	—	—	0·935–1·47	—
$K\,\frac{Ac}{Fe'''Ox}$	—	—	2·6–4·25	—	—	—	—	—
$K\,\frac{Br}{OH}$	—	—	0·81–1·23	0·81	—	—	—	—
$Na\,\frac{Br}{OH}$	1·40	—	1·02–	1·02–2·10	—	—	—	—
	N = 0·1.							
$K\,\frac{Ac}{Br}$	1·213	—	0·981	—	1·192	—	0·841	—
$Na\,\frac{Ac}{Br}$	1·07	—	0·906	—	1·04	—	—	—
$\frac{Li}{K}\,Cl$	—	—	0·82	—	—	—	0·84	—
$\frac{Li}{K}\,OH$	—	—	0·81–1·23	0·81–	—	—	—	—
$\frac{Li}{K}\,CrO_4$	—	—	1·39–2·74	—	—	—	0·89–2·2	–2 3

TABLE I.—*continued.*

Margin.			N = 0·5.		N = 1·0.		N = 2·0.	
	A.	B.	A.	B.	A.	B.	A.	B.
$\frac{\mathrm{Li}}{\mathrm{K}}$ Br	1·21	—	0·981	—	1·192	—	0·85	—
$\frac{\mathrm{Li}}{\mathrm{K}}$ Fe'''Ox		—	2·43	2·15–3·28	—	—	—	—
$\frac{\mathrm{Li}}{\mathrm{Na}}$ Cl	—	—	0·735	—	1·13	—	0·923	—
$\frac{\mathrm{Li}}{\mathrm{Na}}$ Br	—	—	0·906	—	—	—	—	—
	N = 0·2.							
$\frac{\mathrm{Ag}}{\mathrm{H}}$ NO_3	0·48–0·79	—	—	—	—	—	—	—
$\frac{\mathrm{K}}{\mathrm{H}}$ NO_3	0·70–1·20	—	0·3–0·6	0·238–0·85	—	—	—	—
$\frac{\mathrm{Cd}}{\mathrm{Li}}$ Cl	—	—	2·40	—	1·55	—	—	—
$\frac{\mathrm{Mg}}{\mathrm{Ba}}$ Cl	—	—	2·56	—	2·00	1·60–3·50	2·56	-3·57
$\frac{\mathrm{Mg}}{\mathrm{Sr}}$ Cl	—	—	1·40	—	1·02	—	—	—
$\frac{\mathrm{Mg}}{\mathrm{K}}$ CrO_4	—	—	2·50–	2·0–	—	—	—	—
$\frac{\mathrm{Mg}}{\mathrm{Ag}}$ NO_3	—	—	2·02*	—	—	—	—	—
$\frac{\mathrm{Cu}}{\mathrm{Sr}}$ Cl	—	—	—	—	—	—	2·51	—
$\frac{\mathrm{Cu}}{\mathrm{Ca}}$ Cl	—	—	—	—	—	—	2·29	—
$\frac{\mathrm{Cu}}{\mathrm{Mg}}$ Cl	—	—	—	—	—	—	2·05	—
$\frac{\mathrm{Cu}}{\mathrm{Ag}}$ NO_3	—	—	2·79–3·32*	2·22	—	—	—	—
$\frac{\mathrm{Cd}}{\mathrm{Ca}}$ Cl	—	—	1·13–1·50	—	1·18	—	—	—
$\frac{\mathrm{Cd}}{\mathrm{Mg}}$ Cl	—	—	1·50	—	1·53	—	—	—
$\frac{\mathrm{Cd}}{\mathrm{Mg}}$ SO_4	2·38	—	2·05	—	2·10	—	2·26	1 1–
$\frac{\mathrm{Cd}}{\mathrm{Cu}}$ SO_4	—	—	—	—	3·66	—	3·06	2·54
$\frac{\mathrm{Al}}{\mathrm{Cu}}$ SO_4	—	—	3·04	—	3·36	—	—	—

* Boundary good but jelly melted.

The concentration of the intermediate solution is given at the head of the table, a normal solution, here and throughout the paper, being one which contains 1 gramme equivalent of salt in a litre of solution. In the columns A are given for each concentration of the different boundaries the potential fall which has been found to produce a good and measurable refraction margin. In the columns B are given the limits, where these have been determined, within which such a margin may be expected. Thus, for example, for 0·5 N Mg $\frac{Ac}{Cl}$ the inferior limit is 1·5 volts, the superior 1·9 volts per centim. A very faint and indistinct margin may still be found when the voltage is reduced to 1·39, but its velocity is not constant. Above 1·90 volts the margin becomes "washed."

Although an examination of the table fails to show any regularities of a striking character, it is seen that, generally speaking, the slower the ions forming the boundary the higher the potential fall required. On the other hand, contrary to expectation, it is found that the margin between the same pair of ions varies in the voltage required with the nature of the common ion. This is most clearly shown in the case of the $\frac{Ac}{Cl}$ margin; here, with Na as cation, 0·73 volt is more than sufficient to ensure stability, whereas, with Ba, Ca, or Mg as cation, a higher potential fall than this is required before the minimum is reached; the same behaviour is shown by the boundary $\frac{Li}{K}$, when Cl is the common anion, 0·82 volt is more than sufficient; with $Fe'''\overline{Ox}$,* on the other hand, the lowest voltage that will give a stable margin is 2·5.

Greater regularities and a possible explanation of the difficulty of obtaining a margin in certain cases are found by considering, instead of the potential fall in the measured solution, the change of potential slope on passing from indicator to solution, or the difference between the potential fall in the two parts of the system. To obtain this a knowledge of the resistance in the indicator is necessary, and this can be calculated very approximately from the transport numbers of the two ions, which condition the concentration change, and from **Kohlrausch**'s conductivity tables.

Table II. contains the differences in potential for a few boundaries for which the data exist for the required calculation. The numbers under "Potential Fall (A)" give the voltage used for the production of the particular boundary, and in the few cases where the minimum fall is known, this is given in the fourth column under B. In the last column $E_i - E_s$ are given the differences in potential fall between indicator and solution for the voltage under A and B.

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* This symbol has been used as an abbreviation to indicate the complex anion of the ferric oxalates.

TABLE II.—Potential Fall.

Margin.	N.	A.	B.	$E_i - E_s$.
$\frac{K}{H} NO_3$	0·574	—	0·20	1·18
$\frac{Li}{K} Cl$	0·5	0·82	—	1·16
$\frac{Li}{K} Br$	0·5	0·98	—	1·10
$\frac{Li}{Na} Cl$	0·5	0·73	—	0·45
$\frac{\frac{1}{2}Mg}{Li} Cl$	2·0	2·00	—	0·01
$\frac{\frac{1}{2}Mg}{\frac{1}{2}Ba} Cl$	1·0	—	1·60	0·65
$\frac{1}{2}\frac{Mg}{Sr} Cl$	1·0	—	1·02	0·28
$\frac{1}{2}\frac{Cd}{Ca} Cl$	0·5	1·15	—	1·69
—	2·0	0·65	—	1·92
$\frac{1}{2}\frac{Cd}{Mg} Cl$	1·0	1·53	—	2·82
—	2·0	1·25	—	2·67
$\frac{\frac{1}{2}Cd}{Li} Cl$	0·5	1·06	—	1·04
$\frac{1}{2}\left(\frac{Cd}{Mg} SO_4\right)$	0·5	2·04	—	0·47
		1·00	—	0·23
$\frac{1}{2}\left(\frac{Cd}{Cu} SO_4\right)$	0·5	6·95	—	0·30
$K \frac{Ac}{Cl}$	0·5	0·82	—	0·98
$Mg \frac{Ac}{Cl}$	0·5	—	1·39	1·13
$Ca \frac{Ac}{Cl}$	0·5	—	0·916	0·98
$K \frac{Br}{OH}$	0·5	0·81	—	1·32

From this table it is seen that a very different electrical tension across the boundary is required for some pairs of solutions than for others. On the other hand, for the same pair of ions, a different value for the potential fall in the intermediate solution corresponds to approximately the same change in potential slope. This is shown by the $\frac{Ac}{Cl}$ margin with K, Mg, and Ca as common cation. Looked at from this standpoint, it seems that the change in potential fall required is some function of the velocities of the ions, and it is most probable that a connection will be found to exist between this and the Nernst theory of liquid cells. To establish the relation, however, it is necessary that a very much larger number of experiments should be carried out. In the table only five of the experiments correspond to the minimum potential fall. This follows from the fact that the investigation had for its first aim the production of a margin, and not the determination of the limits of voltage within which this may be produced.

Two of the boundaries previously referred to for which no satisfactory conditions could be found are the $\frac{\mathrm{Li}}{\mathrm{Mg}}\,\mathrm{Cl}$ and $\frac{\mathrm{Mg}}{\mathrm{Li}}\,\mathrm{Cl}$. From Table II. it is seen that for this pair of solutions no change in potential slope is brought about. The densities and refractive indices of the two solutions, however, differ very considerably. Perhaps it is on this account that it is possible to obtain an indistinct and undulating refraction margin whether the Li follows Mg or Mg Li; but in neither case is the velocity constant. This is undoubtedly due to the fact that, since there exists no difference in potential fall at the two sides of the margin, there is no controlling force which shall prevent the faster ion diffusing into the slower, or *vice versâ*. Diffusion therefore takes place, the resistances become slightly altered, and so also the potential fall and velocities. A similar case is that of the 0·5 N $\frac{\mathrm{Cd}}{\mathrm{Cu}}\,\mathrm{SO_4}$ margin. Here, with a low voltage, a mixed colour boundary, which cannot be located nearer than perhaps 1–2 millims., results; but when the voltage is increased, a sharp colour boundary ultimately appears at about 6·5 volts, and from the table it is seen that even with 7 volts the change in potential slope only amounts to 0·30 volt. That no refraction margin can be detected with this pair of solutions at this concentration (such a margin is obtained in more concentrated solutions) is explained by the circumstance that the refractive indices of the two solutions lie very close together. NERNST, in describing the experiment previously quoted, refers to the necessity for a high voltage that will allow the experiment to be completed in a comparatively short time, as the margin would otherwise be lost by diffusion. This is another case in which there is no change in potential slope on passing from one solution to the other, the velocities of the $\mathrm{NO_3}$ and $\mathrm{MnO_4}$ ions being practically identical.

The case of cadmium chloride as indicator following 2 N solutions of the chlorides furnishes examples of another difficulty. Here the boundaries obtained are the most distinctly to be seen of any that have been investigated, but the motion is always irregular and much too slow. These irregularities are probably due to a mixing by convection currents set up as previously described by the greater heating in the indicator solution; they may, however, be in some way connected with the nature of the anions in the cadmium chloride solution, which according to HITTORF, are not simple Cl ions, but something more complex.

For the systems $\frac{\mathrm{Li}}{\mathrm{Cu}}\,\mathrm{Cl}$ and $\frac{\mathrm{Cu}}{\mathrm{Li}}\,\mathrm{Cl}$ also an undulating refraction margin results in both cases; but in the former case the copper lags behind the margin, and the resulting lithium chloride solution shows throughout the entire length of the column a uniform blue colour. In the latter case, when Cu is the following ion, it encroaches on the intermediate solution and advances ahead of the refraction margin from 0·5 to 1·0 centim., the colour gradually fading away.

The only case in which no boundary whatever has been obtained occurred in the attempt to measure the velocity of the NO_3 ion in nitric acid.* Three indicators (the acetic, oxalic, and phenylsulphonic ions) were tried in different experiments, always without success. It may be mentioned that with such excellent conductors as the acids the use of a high voltage is practically impossible on account of the great heat development. An apparatus was, however, constructed by means of which a potential fall of 2·0 to 3·4 volts was obtained at the anion boundary; under these conditions the whole column of liquid was seen to be mixed by rapid convection currents.

It has been occasionally noticed that with the same boundary the inferior limit apparently differs according to the direction from which it is approached; thus for the $\frac{K}{H}NO_3$ margin, when an experiment is started with a low voltage and this is gradually increased, no margin can be detected at 0·29 volt, whereas starting with 0·4 volt and diminishing, the boundary does not entirely disappear until a potential fall of 0·17 volt is reached. This is due to the fact that with a lower potential fall a longer time is required before the stationary condition is reached than with a higher one, and if before an experiment is started a little diffusion takes place at the point where gelatine and aqueous solutions are in contact, it is possible that the margin may have travelled right through the tube before it has become visible.

The production of a refraction boundary in the case that the slower ion precedes the faster, has been noticed for a few cases, *e.g.*, $\frac{Li}{Mg}Cl$, $\frac{Li}{Cu}Cl$, and $\frac{Mg}{Cu}Cl$. In the majority of instances, however, no boundary is obtainable under these circumstances. But if at the beginning of such an experiment the concentrations of the two solutions are proportional to the transport numbers of the respective cations (or anions), then a stable boundary results which travels with the current. Such a ratio of concentration is automatically brought about during electrolysis, when the slower ion follows the faster, and once this condition is established the margin so produced may be made to move backwards by altering the direction of the current, and this may be repeated a number of times without losing the boundary, although its velocity has been proved by experiment to be less in the backward than in the forward direction. The stability of this margin is easily understood from the fact that the potential slope in the two solutions is so adjusted that the velocities of the two ions is the same. This is, however, only strictly true for what may be called the reverse direction, for the first moments after reversal of the current, since the change in potential slope no longer hinders diffusion between the two solutions, but aids it, and hence the resistances become altered, and so also the potential fall and the velocity. The boundaries that have been investigated in this direction are $\frac{Li}{K}$ Cl and Cu $\frac{Ac}{SO_4}$, and both of these are quite permanent when the current is reversed, and are not lost even after 2 hours, their velocities becoming, however, steadily less.

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* For the investigation of acids and bases it is impossible to employ gelatine solutions in the cathode and anode cells respectively, as these are immediately destroyed by the H and OH ions; for these, therefore, a partition of earthenware was substituted at the one end, and in order to prevent as far as possible movements of the solution through the membrane, the open limb of the apparatus (E, fig. 1) is dispensed with.

The Influence of Hydrolysis in the Indicator.

It is necessary that the indicator solution should not be such as has undergone hydrolysis; when this is the case, the H and OH ions overlap the boundary, and entering the intermediate solution, thus reduce the resistance and so also the velocity at that end.

Provided an experiment in which one of the indicators undergoes hydrolysis is not carried so far that the H or OH ions reach the second boundary, the ratio of the velocities remains perfectly constant, but is quite different to the ratio obtained with an indicator which is not hydrolysed.

This is clearly seen by a comparison of the transport number of copper sulphate as obtained with aluminium sulphate, and with cadmium sulphate:

Indicators =	$Al_2(SO_4)_3$ and $NaC_2H_3O_2$.	$CdSO_4$ and $NaC_2H_3O_2$.
$CuSO_4$ 0·5 N	0·749	—
1·0 N	0·842	0·660
2·0 N	—	0·730

The influence of the H ions formed by hydrolysis of the $Al_2(SO_4)_3$ solution, in reducing the velocity of the $\frac{Al}{Cu}SO_4$ margin, is strikingly shown.

The Gelatine Solutions.

For the preparation of the indicator jellies the best commercial gelatine was employed, which had been purified by diffusion in distilled water in the manner described by LOBRY DE BRUYN ('Rec. Trav. Chim.,' 1900, 19, 236). Gelatine solutions containing half an equivalent weight in grammes of indicator salt were used for the $\frac{n}{2}$ solutions, containing two equivalent weights for the N and 2 N solutions. The strength of the jellies in gelatine was always, where possible, 12 per cent.

Certain regularities in the behaviour of the different jellies used may be noted.

Good firm jellies suitable for use in the experiments are obtained with the following salts, using 12 per cent. gelatine: Potassium chloride, bromide, fluoride, and chromate, sodium acetate, the sulphates of magnesium, copper, and cadmium, and half-normal silver nitrate.

To obtain a sufficiently solid jelly of lithium or magnesium chloride, it is necessary to use 20 per cent. gelatine.

0·5 N potassium iodide and 2·0 N cadmium bromide, do not form solid jellies even with 23 and 25 per cent. of gelatine.

It is not possible to prepare a 0·5 N aluminium sulphate jelly containing 12 per cent. gelatine, as it becomes coagulated and semi-solid even at 100°; with 5 per cent. gelatine a good solution is obtained, which solidifies to a very fine jelly of high melting point.

By the electrolytic formation of the following gelatine solutions, the jellies are melted even with very small heating by the current:*

Copper or magnesium nitrate formed by the passage of NO_3 anion into 12 per cent. jellies of 2 N copper or magnesium sulphate. In only one experiment the copper nitrate jelly so formed was not melted.

The passage of Br ions into a lithium chloride jelly results in its melting with a current density which would be quite safe, with no formation of lithium bromide.

The entrance of bichromate ions into a lithium chloride jelly invariably occasions the melting of the latter; and H or OH ions, so dilute as 0·2 N, immediately destroy any jelly they enter.

From these observations, which are entirely qualitative, the statement seems to be justified that the influence of the salt on the melting point of a jelly depends on the nature of the ions, that, among anions the lowering effect increases as we pass down the series SO_4, Cl, Br, to NO_3, Cr_2O_7 and I, and similarly for the cations as we pass down the series K, Na, Cu, Cd, Li, and Mg.

The sulphates of all these readily form jellies of high melting point; with the chlorides the melting point falls as we pass from potassium and sodium to Mg, and so for the others. A quantitative investigation of the influence of salts on the melting points of gelatine solutions should yield instructive and interesting results.

As indicators, normal solutions of the following salts have been used:—

Cation Indicators.

$CuSO_4$, with copper anode, to minimise the formation of H ions.

$CdSO_4$, with cadmium anode.

LiCl, with Li_2CO_3 suspended in the solution.

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* This difficulty of the melting of the gelatine would be overcome by substituting an earthenware partition, as has been done for the acids and alkalies.

$MgCl_2$, with MgO, suspended in the solution.
KCl, containing dissolved K_2CO_3.
$AgNO_3$, with silver anode.
$Al_2(SO_4)_3$, with $Al_2(OH)_6$ suspended in solution.

Anion Indicators.

NaAc, with HAc to prevent formation of OH ions.
KF, containing a small quantity of HAc.
K_2CrO_4, with $K_2Cr_2O_7$.

The current was obtained from a battery of thirty storage cells, and the total voltage used in different experiments varied from 20 to 70 volts.

The temperature of the experiments was that of the room; it was constant to $0^\circ\cdot5$ throughout any experiment, but varied in different experiments between 14° and 19°.

The Accuracy of the Measurements.

The apparatus having been prepared as already described, shortly after the current is started, the boundaries having advanced into the tube; their rate of motion is measured by means of a cathetometer, a small gas jet when placed behind the tube being seen to be cut by a dark line at the point where the two solutions are in contact. Each tube has etched on it a horizontal line, which serves as a fixed point from which measurements are made. At frequent intervals the time, the current, and the distance moved over are measured; the ratio of the distances moved over by the anion and the cation boundaries gives directly $\frac{V}{U}$, from which $\frac{V}{U+V}$ is at once obtained.

The cathetometer employed was capable of giving readings correct to the tenth of a millimetre, but from various causes the accuracy with which the position of the margin can be read is about one-third of this. The difference in the values of p obtained in the various readings of the same experiment amounts in some cases to 2 per cent., but the error here arises in nearly every case from the fact that in the earlier measurements, where the distance moved over by the boundary is often less than 0·5 centim., a small error in reading the cathetometer has a very much greater influence than later when the distance is more, hence it is found always that the last few readings agree much better among themselves than do the earlier ones.

All the readings from an experiment are therefore averaged in the following manner:—

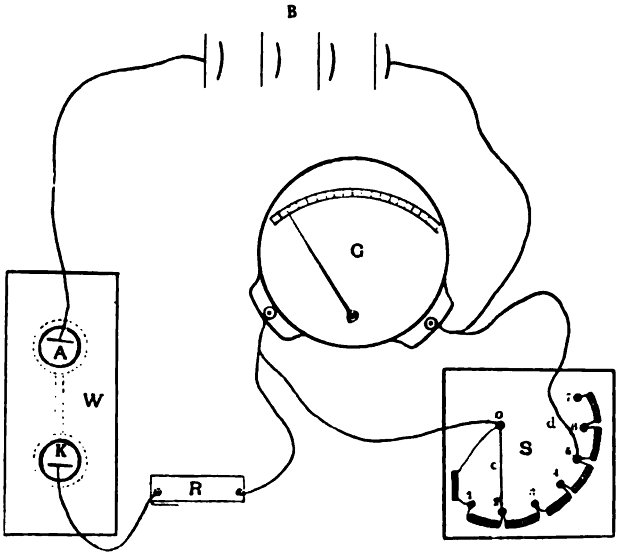
Each value for p is multiplied by the distance moved over by one of the boundaries since the beginning, the numbers thus obtained are added together and divided by the sum of the multiplicands. When averaged in this manner, it is found that for salts of simple type, the difference in transport number of the same salt, as determined in different experiments, is exceedingly small: thus six experiments with 2 N, KCl gave 0·492, 0·490, 0·494, 0·488, 0·488, and 0·489, and two experiments with N, KBr gave 0·473 and 0·474.

For salts of less simple type than these, only two duplicate measurements have been made; these are, for magnesium sulphate 0·5 N, 0·693 and 0·694, and for a 0·2 N solution of the same salt 0·649 and 0·646.

For the measurement of the current, a galvanometer with the resistance of about 100 ohms was employed; it was far too sensitive for the amount of current used, and was therefore always connected with a shunt. Provided the total deflection of the needle amounted to at least 70 scale divisions, readings correct to 1 part in 200 could be made; to ensure accuracy in the measurements, therefore, it was necessary that, whatever the strength of the current, the deflection should be not less than about 50 scale divisions.

Since the current varied greatly in different experiments, a series of shunts was prepared by the use of which widely varying values could be given to the scale divisions. The apparatus is shown in fig. 4, where G is the galvanometer and S the box of coils, one or all of which can be used as a shunt. In the figure the three coils between 2 and 5 are connected with the binding screws of the galvanometer, and by moving the connecting wire *c* and the wire *d* to different mercury cups, a large number of different values can be given to the shunt. The value in amperes of each scale division was determined by direct measurement when the various coils were connected with the galvanometer, and by this means it was possible, however the amount of current varied, to ensure in all cases a deflection of from 80 to 100

Fig. 4.



scale divisions. Fig. 4 shows, also diagrammatically, the arrangement of the whole apparatus for an experiment. B represents the battery. W is the water-bath, in which is immersed the apparatus, of which A and K are the anode and cathode cells respectively. R is a very large and easily varied resistance placed in circuit, by means of which the voltage between A and K can be varied as required.

EXPERIMENTAL RESULTS.

The independence of the margin velocity on the nature of the indicator, provided the latter fulfils the required conditions shown by MASSON, has been confirmed by the measurement of magnesium sulphate with three pairs of indicators—copper sulphate and potassium chromate, cadmium sulphate and sodium acetate, copper sulphate and sodium acetate—and by the fact that practically the same result has been obtained for the velocity of the hydrogen ion when measured with silver nitrate or with potassium chloride as cation indicator.

The negative case has been proved by a large number of experiments, all of which show that, when the faster ion follows the slower, in cases where a boundary is produced at all, the ratio of its velocity to that of the boundary at the opposite end of the solution is quite inconstant.

Table III. contains the results obtained during the research.

In column 1 are given the formulæ of the salts measured. In the second column is given the concentration of each salt.

In column 3, under $\frac{V}{U}$, is given the ratio of the velocity of the anion to that of the cation, this being calculated directly from the observation in the manner previously described.

In columns 4, 5, and 6, under $\frac{V}{U+V}$, are given for comparison the values of the anion transport number. Under S are given the numbers obtained by the author, under (MASSON) those obtained by MASSON by direct measurement in gelatine, and under (HITTORF, &c.) are given those obtained by different investigators by the indirect method of HITTORF. The latter numbers are taken from KOHLRAUSCH and HOLBORN ('Leitvermögen der Electrolyte').

A comparison of the figures in the fourth and fifth columns shows that for potassium and sodium chloride there is a very close agreement between the numbers obtained by direct measurements in water and in gelatine, whilst for lithium chloride and magnesium sulphate no such agreement exists. For 2 N magnesium sulphate there is given in brackets (column 5) the number 0·688, obtained by the author for the value of p in gelatine; this measurement being made for purposes of comparison. For this salt the gelatine results agree among themselves, as also do the aqueous; whereas the two sets of measurements show no such agreement as

TABLE III.

Salt.	N.	$\frac{V}{U}$	$\frac{V}{U+V}$			Indicators.	
			S.	(MASSON.)	(HITTORF, &c.)	Cation.	Anion.
KCl. . .	0·5	0·96	0·490	0·495	—	LiCl	NaAc
	1·0	0·955	0·488	0·490	0·515	—	—
	2·0	0·955	0·489	0·483	—	—	—
NaCl . .	0·5	1·48	0·597	0·598	0·626	LiCl	NaAc
	1·0	1·45	0·591	0·595	0·637	—	—
	2·0	1·44	0·590	0·587	0·642	—	—
KBr . .	0·1	0·935	0·483	—	—	LiCl	NaAc
	0·5	0·920	0·478	—	0·513	—	—
	1·0	0·900	0·473	—	—	—	—
	2·0	0·880	0·468		—	—	—
NaBr . .	0·5	1·47	0·595	—	—	LiCl	NaAc
LiCl. . .	0·5	2·52	0·716	0·687	0·73	$CdSO_4$	NaAc
	1·0	3·02	0·751	0·680	0·745	—	—
KOH . .	0·574	2·71	0·730	—	0·738	LiCl	KBr
$AgNO_3$. .	1·15	0·943	0·486	—	0·495	$CuSO_4$	KF
$BaCl_2$ . .	0·5	1·36	0·576	—	0·615	MgCl	NaAc
	1·0	1·62	0·619	—	0·640	—	—
	2·0	1·73	0·633	—	0·657	—	—
$SrCl_2$ . .	0·5	1·67	0·625	—	—	$MgCl_2$	NaAc
	1·0	1·98	0·665	—	—	—	—
	2·0	2·44	0·709	—		$CuSO_4$	—
$CaCl_2$ . .	0·5	2·14	0·681	—	0·675	$CdSO_4$	NaAc
	1·0	2·30	0·697	—	0·686	—	—
	2·0	2·51	0·715	—	0·700	$CuSO_4$	—
$MgCl_2$ . .	0·5	2·39	0·705	—	0·690	$CdSO_4$	NaAc
	1·0	2·60	0·722	—	0·709	—	—
	2·0	2·84	0·740	—	0·729	$CuSO_4$	—
$MgSO_4$. .	0·184	1·82	0·646	—	0·660	$CdSO_4$	NaAc
	0·5	2·26	0·693	0·684	0·700	—	—
	1·0	2·50	0·715	0·703	0·740	—	—
	2·0	2·80	0·737	0·693	0·750	—	—
	—	—	—	(0·688)	—	—	—
$CuSO_4$ . .	1·0	1·94	0·66	—	0·696	$CdSO_4$	NaAc
	2·0	2·71	0·73	—	0·720	—	—
K_2CrO_4. .	0·5	0·807	0·447	—	0·512	LiCl	NaAc
	2·0	0·677	0·403	—	—	—	—
KFe'''Ox .	0·603	0·495	0·331	—	—	LiCl	NaAc

do those for potassium chloride. The explanation of this behaviour is rendered more difficult by the fact that the transport number for magnesium sulphate has been determined by Hittorf's method, using a gelatine partition to separate the anode and cathode portions of the solution. Two experiments made in this manner gave 0·732 and 0·744 for the anion transport number, numbers which agree well with the older determinations and with the figures in column S. It is worth pointing out that whilst *p* for these salts, when determined in water, shows a considerable change with concentration, in gelatine on the other hand *p* is approximately constant.

TABLE IV.

Group I.				Group II.			
	n = 0·5.	n = 1·0.	n = 2·0.		n = 0·5.	n = 1·0.	n = 2·0.
Li	0·716	0·751	—	Mg	0·705	0·722	0·740
Na.	0·597	0·591	0·590	Ca.	0·681	0·697	0·715
K	0·490	0·488	0·489	Sr	0·625	0·665	0·709
				Ba.	0·576	0·619	0·633

A point that is shown with great clearness by a consideration of the figures in the fourth column is the periodicity of the transport number of the same anion with different cations belonging to the same group. Table IV. shows the periodicity in question for the chlorides of the elements of Groups I. and II. as far as these have been studied. Tables III. and IV. bring out very clearly the difference, referred to in the earlier part of this paper, between salts of the type of potassium chloride and those of the type of barium chloride and magnesium sulphate. For the former, *p* is practically independent of the concentration; for the latter, *p* changes with the concentration.

A comparison of the figures in the fourth and sixth columns shows that for potassium and sodium chlorides the values obtained by the two methods are both nearly constant, although not coincident with one another. For the other salts a general agreement is seen to hold between the two sets of figures; the variation in *p* is also in the same sense and of approximately the same magnitude.

In referring to the individual experiments it may be mentioned that for the measurement of potassium hydroxide only one gelatine partition was used.

The experiment with silver nitrate is probably the least accurate of all those tabulated, as the potential fall employed for the anion margin was very near the inferior limit for a good boundary, in order to prevent melting of the jelly on entrance of the NO_3 ion.

Many unsuccessful attempts have been made to measure solutions of sodium bromide and silver nitrate of different concentrations to those given in the table. In the former case, the $\frac{\text{Li}}{\text{Na}}$ Br and in the latter the $\frac{\text{F}}{\text{NO}_3}$ Ag boundary could not be made to move with a constant velocity or without "washing" under any of the varied conditions tried. For the first salt, no other indicator has as yet been used; for the second, no other anion whose silver salt was soluble has suggested itself for trial.

In Table V. is given the ratio of the current, as measured by the galvanometer, to that calculated from the margin velocities. For potassium and sodium chlorides, potassium and sodium bromide, potassium hydroxide, and 0·5 N lithium chloride, this ratio is, as required by theory, within the limits of experimental error, equal to unity. On the other hand, this is not the case for all the other salts examined, the closest agreement with theory being shown by the more dilute solutions; for example, 0·2 N magnesium chloride and 0·2 N magnesium sulphate. Only for one salt have duplicate measurements been made, namely, for 0·5 magnesium sulphate, but here the results are in accordance.

TABLE V.

Salt.	N.	$\frac{C\eta}{AN(U+V)}$.	Salt.	N.	$\frac{C\eta}{AN(U+V)}$.	Salt.	N.	$\frac{C\eta}{AN(U+V)}$.
KCl . .	0·5	0·984	$BaCl_2$.	0·5	0·956	$MgSO_4$.	0·183	0·980
	2·0	1·009		1·0	0·944		0·5	0·949
NaCl. .	1·0	0·989		2·0	0·956		0·5	0·951
	2·0	1·004	$SrCl_2$. .	0·5	0·870		1·0	0·977
KBr . .	0·1	0·978		1·0	0·916		2·0	0·956
	0·5	1·000		2·0	1·080		2·0	0·807*
	1·0	1·031	$CaCl_2$.	0·5	1·04		2·0	0·814†
	2·0	0·998		1·0	1·03	$CuSO_4$.	1·0	1·06
NaBr .	0·5	1·001		2·0	0·973		2·0	0·965
LiCl . .	0·5	1·006	$MgCl_2$.	0·5	1·01	K_2CrO_4 .	0·5	0·965
	1·0	1·070		1·0	1·05		2·0	0·910
KOH .	0·57	1·02		2·0	0·968			

* MASSON, in gelatine. † STEELE, in gelatine.

MASSON has found that for this salt the ratio as measured in gelatine is considerably less than 1. Under 2·0 N, $MgSO_4$, are given in the table the values found by the author in aqueous and in gelatine solution, and also MASSON'S number found in gelatine; a difference is here to be noted corresponding with that already pointed out for the transport number.

For the calculation of the current the average is taken, this being obtained from the area of the time-current curve, and for the velocities the total distance moved over divided by the time in seconds.

TABLE VI.

Salt.	N.	Conductivity.		$U = xu$.		$V = xr$.	
		Measured.	Calculated.	KOHLRAUSCH.	Found.	KOHLRAUSCH.	Found.
KCl . .	0·5	102·3	104·5	0·000512	0·000553	0·000543	0·000529
	2·0	92·6	91·0	0·000466	0·000483	0·000494	0·000458
NaCl. .	1·0	74·4	74·5	0·000285	0·000318	0·000485	0·000452
	2·0	64·8	64·6	0·000250	0·000274	0·000418	0·000395
KBr . .	0·5	105·7	104·5	0·000542	0·000568	0·000553	0·000516
	1·0	102·0	99·5	0·000522	0·000542	0·000532	0·000484
	2·0	97·4	97·3	0·000500	0·000538	0·000510	0·000471
KOH .	0·576	192·0	190·0	0·000530	0·000535	0·001450	0·001435
LiCl . .	0·5	70·3	67·0	0·000196	0·000191	0·000535	0·000483
	1·0	62·8	58·8	0·000173	0·000141	0·000480	0·000450
$BaCl_2$.	0·5	77·6	75·2	0·000310	0·000330	0·000494	0·000450
	1·0	70·3	71·5	0·000264	0·000283	0·000465	0·000457
	2·0	60·3	60·7	0·000213	0·000231	0·000411	0·000398
$SrCl_2$. .	0·5	80·4	81·0	0·000312	0·000316	0·000512	0·000524
	1·0	73·6	75·2	0·000255	0·000261	0·000507	0·000519
	2·0	54·1	50·9	0·000164	0·000152	0·000396	0·000374
$CaCl_2$.	0·5	74·7	67·6	0·000252	0·000224	0·000521	0·000476
	1·0	67·8	64·2	0·000220	0·000201	0·000482	0·000464
	2·0	58·0	55·0	0·000180	0·000162	0·000420	0·000408
$MgCl_2$.	0·5	71·0	64·2	0·000229	0·000196	0·000508	0·000468
	1·0	63·0	56·9	0·000189	0·000163	0·000463	0·000427
	2·0	53·0	51·6	0·000149	0·000139	0·000400	0·000396
$MgSO_4$.	0·18	44·0	45·4	0·000155	0·000167	0·000301	0·000304
	0·5	35·4	36·8	0·000111	0·000117	0·000257	0·000264
	1·0	28·9	29·4	0·000078	0·000087	0·000221	0·000217
	2·0	21·4	23·1	0·000054	0·000061	0·000168	0·000178
$CuSO_4$.	1·0	25·8	22·7	0·000082	0·000080	0·000186	0·000155
	2·0	20·1	19·9	0·000058	0·000055	0·000150	0·000151
NaOH .	0·2	—	—	—	—	0·00152	0·00158
KOH .	0·576	192·0	190·0	0·000540	0·000535	0·00145	0·001435
HNO_3 .	0·2	—	—	—	—	0·00280	0·00282*
	0·2	—	—			0·00280	0·00272†

* With $AgNO_3$ as cation indicator margin = $\substack{\text{Ag}\\ \text{H}}\text{NO}_3$.

† With KNO_3 as cation indicator margin = $\substack{\text{K}\\ \text{H}}\text{NO}_3$.

The correctness of the assumption that the velocity of the boundary is in reality that of the preceding ion is capable of being tested by the calculation of the conductivity of the intermediate solution from the observed velocity. For this purpose it is necessary to first reduce the latter to that which would be produced by a potential fall of 1 volt per centimetre. From the sum of the velocities thus obtained the conductivity is obtained by multiplication by ϵ, where ϵ is the quantity of electricity carried by 1 ion = 96,500 coulombs. Table VI. contains a comparison of the conductivities so calculated with those given by KOHLRAUSCH. In the last four columns are given the values of the velocities, calculated from KOHLRAUSCH's tables, at the concentrations indicated in the second column, and those reduced from the author's measurements.

Since the sum of the velocities is proportional to the conductivities, the sum of U and V must show the same agreement with the sum of U and V from KOHLRAUSCH's figures, as is shown in columns 3 and 4; that U and V singly do not show such an agreement is due to the fact that the velocity $U = \frac{\mu}{\epsilon} p$, and hence if the Hittorfian p is used in the calculation, a different value for U will be obtained than if the author's p is used. In columns 5 and 7 the p that is used is HITTORF's; in columns 6 and 8 that obtained during the present research, and hence the want of agreement. The fact that by the use of the latter the figures in question would be brought into much greater concordance, speaks strongly in favour of their greater accuracy.

The Existence of Complex Ions.

With the three electrolytic measurements that have been considered, it has been found that it is only in the case of a few salts of the most simple type that experiment and theory are in agreement. In the case of all other salts, in the first place the transport number, whether measured by the older method of HITTORF or by the direct method, described in the present paper, is not independent of the concentration; secondly, from the measurements of the conductivity it is not possible to assign any specific ionic velocity to such ions as Mg, which is constant in different salts of the same cation; and finally, the current as measured by the galvanometer is not the same as that calculated from the observed velocity of the margin.

For the change in transport number two explanations only seem possible, since in the majority of instances the influence of hydrolysis is practically negligible.

The first of these is that with change in concentration there occurs a variation in the specific velocity occasioned by the electrolytic friction, and that the influence of the latter is greater for some ions than for others, and usually greater for the cation than for the anion.

If we assume that at a given total concentration the specific mobilities of the ions are represented by u and v, and the ionic concentration by c, then for the transport number we have

$$p = \frac{cv}{c(u+v)} = \frac{v}{u+v}.$$

For any other total concentration n the mobilities are u_1 and v_1, and the ionic concentration c_1:

$$p_n = \frac{c_1 v_1}{c_1(u_1+v_1)} = \frac{v_1}{u_1+v_1}.$$

In order that p_n should be greater than p it is necessary either that v_1 increase faster than u_1, that u_1 diminish faster than v_1, or that u_1 decrease and v_1 increase with increase of concentration; the latter case may be excluded from its great improbability. Further, since in all cases the fluidity of a solution decreases with addition of salt, it is not probable that the ionic mobility will be greater in the more viscous solution.*

Assuming, then, that no increase in u_1 or v_1 takes place, for a variation in p u_1 must decrease very much more rapidly than v_1. To take the case of barium chloride, the anion transport number changes between $n = 0{\cdot}01$ and $n = 2{\cdot}0$ from 0·56 to 0·66.

In the dilute solution $\quad u = \frac{0{\cdot}44}{0{\cdot}56} v = 0{\cdot}786\, v,$

in the stronger solution $\quad u_1 = \frac{0{\cdot}34}{0{\cdot}66} v_1 = 0{\cdot}514\, v,$

and hence $\quad \frac{u}{u_1} = 1{\cdot}53 \frac{v}{v_1},$

or between these concentrations u has diminished in velocity 1·53 times as much as v.

For calcium chloride a similar calculation gives the relation

$$\frac{u}{u_1} = 2{\cdot}94 \frac{v}{v_1}.$$

Such a large difference in the influence of concentration change on the actual velocities of the two ions seems hardly to be expected.

It is also difficult to explain by any change in u and v the fact that if we consider solutions up to 0·1 N of potassium nitrate and chloride and barium nitrate and chloride, in the case of the three former no change in p occurs; whilst for the latter it does; this would mean, either that the Ba ion has its velocity diminished more than the anion in solution of the chloride but not in the nitrate, or, that the NO_3 ion is diminished in velocity in the barium salt, but not in the potassium salt. Further, the variation in p for potassium sulphate points to the fact that the potassium ion is retarded more than the sulphate, but not more than the chloride. Any of these conclusions are of course possible, but improbable, and the more so since all the facts are far more simply explained by the supposition that complex ions exist in certain salt solutions.

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* The conclusions of JAHN ('Zeitschrift für Phys. Chem.,' 33, 545, 1900; 35, 1, 1900) point to the fact that with increasing concentration, up to $\frac{N}{30}$ the velocity of the ions increases in solutions of potassium, sodium, and hydrogen chloride, the ratio of the velocities, however, remaining constant. (See also ARRHENIUS, *ibid.*, 36, 28, 1901, and SACKUR, 'Zeit. für Electrochemie,' 1901, No. 34.)

The variation in p with N, which is found by direct measurement in aqueous solution, seems to point to the correctness of the assumption of a variation in u and v. But for this class of salts it has been demonstrated for a few cases, and is probably true for all, that a change in concentration of the solution occurs within the anion boundary. Such a change does not occur with potassium chloride, and it is not clear how it can be brought about in a system containing only simple ions. And here again a more reasonable explanation is afforded by the theory of complex ions.

The form of the equation given by MASSON for the relation between current and margin velocity is not altered by supposing a variation in u and v, hence no explanation is afforded by this assumption of the divergence from unity of the ratio of the current as measured by the galvanometer to that calculated from the observed velocities.

The difficulty in assigning values to the specific ionic velocities finds here, also, a possible explanation. For a given solution the conductivity is $\lambda = \epsilon c (u + v)$, where ϵ is the quantity of electricity carried by one monad ion, and u, v, and c have the same signification as before. The molecular conductivity $\mu = \frac{\lambda}{n} = \epsilon \frac{c}{n} (u + v)$, where n is the total concentration of the solution.

For infinite dilution, $c = n$ and $\mu_\infty = \epsilon (u + v_\infty)$, where u and v_∞ are the specific velocities at infinite dilution. Hence $\frac{\mu}{\mu_\infty} = \frac{c}{n} \cdot \frac{u + v}{u_\infty + v_\infty}$. Here $\frac{c}{n}$ = the ratio of ionic to total concentration, or the coefficient of ionization.

Therefore
$$x = \frac{\mu}{\mu_\infty} = \frac{u_\infty + v_x}{u + v},$$

and since
$$\mu_\infty = \epsilon (u_x + v_\infty) \text{ therefore } x = \frac{\mu}{\epsilon (u + v)},$$

or the coefficient of ionization is given by the ratio of the molecular conductivity to the product of ϵ into the sum of the specific velocities at the same concentration, and is only equal to $\frac{\mu}{\mu_x}$ when $(u + v) = (u_x + v_\infty)$.

From what has been already said, the probability is that if u and v change, $u + v$ is less than $(u_x + v_\infty)$, and hence the real value of x is greater than $\frac{\mu}{\mu_\infty}$.

Now, taking as an example solutions of calcium chloride, if we assume that v is not diminished in velocity, u only being affected, and assign such a value to x that the absolute velocity of the chlorine ions at the various concentrations $= 0·000690$ centim. second, the value given by KOHLRAUSCH for the specific velocity, then we find that for

$$N = 0·5,\ x = 0·756\ ;\ N = 1·0,\ x = 0·697\ ;\ \text{and}\ N = 2·0,\ x = 0·610.$$

For the corresponding solutions of sodium chloride, the figures are

0·737, 0·678, and 0·590,

or, solutions of calcium chloride are more dissociated than solutions of sodium chloride of the same concentration. If we assume, instead of this, that the ratio $\frac{\mu}{\mu_\infty}$ gives the correct values for x, and calculate the specific ionic velocities for the anion and the cation at different concentrations, we obtain the following figures for solutions of calcium chloride :—

$N = 0·5$ . . .	$u = 0·000401$	$v = 0·000830$
1·0 . . .	386	845
2·0 . . .	372	867

v for Cl from KCl or Na Cl at all concentrations $= 0·000690$.

Thus we find that, unless calcium chloride is much more dissociated than sodium chloride, the velocity of the anion steadily increases, and that of the cation steadily decreases with rise in concentration; and the same is shown by all the chlorides referred to in this paper whose transport number increases with increasing N.

The assumption of a variation in specific ionic velocity does not in itself seem to be sufficient to afford a probable explanation of the difficulties in question.

The second explanation that we shall consider is one that was advanced in 1859 by HITTORF, who says ('Pogg. Ann.,' 106, p. 385, 1859): "Die Verhältnisse, welche von der Zunahme des Wassers abhängen, und bei den Verbindungen der Metalle aus der Magnesia-gruppe auf die Ueberführungen so wesentlich einwirken, müssen bei den Kalium und Ammoniumsalzen so gut wie fehlen. Dadurch werden wir auf chemische Veränderungen der Constitution unserer Electrolyte, die mit der wachsenden Menge des Wassers eintreten, hingewiesen." In the case of the chlorides of zinc and cadmium, and of cadmium iodide, the change in p is so great that the ratio $\frac{V}{U+V} > 1$, as measured by HITTORF, or there is more current carried by the anion than the total current—a conclusion which is obviously absurd. To explain these cases, he supposes that ionization takes place not only into the simple ions $\overset{++}{\text{Cd}}$ and $\overset{-}{\text{I}}$, but also, at least partly, into complexes of the nature of $(\mathrm{Cd}\,\overset{-}{\mathrm{I}}_2\,\overset{-}{\mathrm{I}}_2{}')$, and that a portion of the current being carried by these, the change in concentration in the neighbourhood of the anode becomes increased by the amount of neutral salt carried by this complex anion. Referring again to the variation in p for other salts, he says (*ibid.*, p. 546-7) :

"Die bedeutende Abhängigkeit der Ueberführungszahlen von der Concentration der Lösung erklärt sich in derselben Weise, wie bei dem Doppelsalze (ICd + IK). Mit der Zunahme des Wassers zerfallen die Doppelatome in immer wachsender Zahl in die einfachen, der Strom wird daher immer mehr von den einfachen geleitet, welche bei stark Verdünnung allein vorhanden sind.

"Nur durch diese Deutung vermag ich den Thatsachen gerecht zu werden, und stehe nicht an, dieselbe auf das Verhalten sämmtlicher Salze, welche zur Magnesiagruppe gehören, zu übertragen. Die schon früher für einen Theil derselben angegebenen Ueberführungen sind ebenfalls in hohem Grade von der Concentration abhängig und würden, wenn noch concentrirtere Lösungen in hinreichender Ausdehnung untersucht werden könnten, für das Anion ebenfalls die Einheit übersteigen."

More recently **Bredig** ('Zeit. f. Phys. Chemie,' vol. 13, 262, 1894) has pointed out the possibility of explaining this difficulty by means of complex ions, and **Noyes** (*loc. cit.*) assumes the existence of complex anions $\mathrm{Ba}\overset{-}{\mathrm{Cl}}_3$ or $\mathrm{Ba}\overset{--}{\mathrm{Cl}}_4$ in solutions of barium chloride so dilute as $\frac{\mathrm{N}}{10}$.

The most natural assumption that can be made as to the manner in which such a salt as magnesium chloride would ionize, is that there would be first formed the two ions $\mathrm{Mg}\overset{+}{\mathrm{Cl}}$ and $\overset{-}{\mathrm{Cl}}$, and that on further dilution the former of these would dissociate into the simple ions $\overset{++}{\mathrm{Mg}}$ and $\overset{-}{\mathrm{Cl}}$. **Noyes** ('Zeitschr. f. Physik. Chemie,' 9, 618), from a study of the solubilities of such salts, concludes that this is the case, but the formation of such ions (complex cations) would lead to a change in p exactly the opposite of that which actually occurs, and it is possible that his results could be as well explained by the supposition of complex anion, *e.g.*, $(\mathrm{Mg}\overset{-}{\mathrm{Cl}}_2\overset{-}{\mathrm{Cl}}_2)$. The tendency towards the formation of complex ions is discussed fully by **Abegg** and **Bodländer** ('Zeitschr. f. Anorg. Chemie,' vol. 20, 471, 1899), who show that these are formed always by the combination of a free electrically charged ion with an electrically neutral molecule, the latter may be of the nature of a salt or not. As an example of the former, the complex $\mathrm{K_3Fe(Cu)_6}$ is formed of the simple cation K and the complex anion $\mathrm{Fe(Cn)_6}'''$, which latter is formed from the neutral part $\mathrm{Fe(Cn)_3}$ and the simple anion Cn^-.

Similarly, the complex anion of the periodides $\overset{-}{\mathrm{I}}_3$ is formed by the union of the anion $\overset{-}{\mathrm{I}}$ with the neutral molecule I_2. This definition also includes the case of the double salts, which differ only from the ferrocyanides and others in the greater tendency of the complex to dissociate into its component parts when dissolved in water.

It is only with the weaker ions that this tendency to the formation of stable complexes is manifested, and according to the theory put forward by **Abegg** and **Bodländer**, this tendency is a measure of the electro-affinity or the ability of an atom to combine with unit charge of electricity to form an ion. "The weaker an ion is, the more it seeks to form complexes by the addition of a neutral part." The nature of the complex that will be formed, whether cation or anion, depends on the question whether the simple anion or cation has the greater electro-affinity.

"The weaker of the two ions will combine with the neutral molecule to form the complex."

Thus the complexes that are formed in solutions of the chlorides of the alkaline earth group are anions, the comparatively strong cations showing less tendency to combine with the neutral salt than does the chlorine. On the other hand, in solutions of silver nitrate, where the cation is a very weak one, and the anion one of the strongest known, complexes, if formed at all, should be cations, and, as a matter of fact, **Hittorf's** measurements show that for this salt p changes in the opposite direction to that for the other salts considered.

With the very strong ions K and Na there is shown no tendency towards the formation of such complex bodies, and consequently no change in transport number with increasing concentration, whereas with the weaker ions of the second group, a very marked change in p occurs, and for the still weaker ions, such as Cu and Zn, the change in p is still more marked.

If now we apply this explanation to our three difficulties, we shall find that it is able to afford at least a qualitative explanation without involving any improbable conclusions.

If we assume that in a given solution dissociation takes place in such a manner that there are formed in addition to the simple ions only one species of complex (for example, complex anions) then, if c is the ionic concentration of the cations, and c' that of the complex anions, $(c - c')$ is the ionic concentration of the simple anions, and if u, v, and v' are the specific ionic velocities of the cation, simple anion, and complex anion respectively, the total amount of current carried is proportional to

$$cu + (c - c')\,v + c'v',$$

the amount of current carried by the anions to the anode is equal to

$$(c - c')\,v + c'v,$$

and hence

$$p = \frac{(c - c')\,v + c'v'}{cu + (c - c')v + c'v'} = \frac{cv + c'(v' - v)}{c(u + v) + c'(v' - v)}.$$

Since v will be most probably greater than v', this equation seems to show that, by the formation of complexes, p should become diminished and not increased; but in the Hittorfian experiments, whilst the current, as determined by the silver voltameter, is correctly represented by $c(u+v)+c'(v'-v)$, the numerator in the above fraction is determined by a measurement of concentration change, and the amount of this will depend on the type of complex that is formed.

If u, v, and v' represent the velocity of that quantity of matter which carries 1 unit charge of electricity, u and v will be associated with the passage from one part of the solution to another of single equivalents, but this is not the case with v'. If m is the number of monad anions into which the complex would fall if completely dissociated, then the change in concentration that would be produced by the movements of the complex is proportional to mv'; hence the increase in concentration at the anode is proportional to $(c-c')v+c'mv'$, and the Hittorfian

$$p=\frac{cv+c'(mv'-v)}{c(u+v)+c'(v'-v)}=\frac{v+\alpha(mv'-v)}{u+v+\alpha(v'-v)},$$

where $\alpha=c'/c$, or the ratio of complex to total anions.

Now since, as we have before remarked, v' is probably less than v, the denominator becomes diminished, and unless m is very small, $mv'>v$, and so the numerator increases with formation of complexes, or the anion transport number as determined in this manner increases with formation of complex anions, that is, with concentration, and the change is in the opposite direction if complex cations are present. If the above equation is put into the form

$$p=\frac{v+\alpha(v'-v)+\alpha v'(m-1)}{v+\alpha(v'-v)+u},$$

it is seen at once that for the case $p>1$ it is only necessary that $\alpha v'(m-1)$ should be $>u$, a relation which is fulfilled if either α or m is very large. In the case of the cadmium and zinc salts the anions are probably more complex than in the magnesium group.

For the relation of the coefficient of ionization to μ we have the following. Let us define by the coefficient of ionization the ratio of total cations or anions to the total number of molecules; then, if c is the ionic concentration of the cations and n the total concentration, $c/n=x$.

The conductivity of the solution is

$$\lambda=\epsilon(c(u+v)-c'(v-v')),$$

or if $\frac{c'}{c}=\alpha$,

$$=\epsilon c[(u+v)-\alpha(v-v')].$$

For any other concentration

$$\lambda_1=\epsilon c_1[(u+v)-\alpha_1(v-v')].$$

Since the molecular conductivity $\mu = \frac{\lambda}{n}$, we get

$$\frac{\mu}{\mu_1} = \frac{\frac{c}{n}\left[(u + v) - \alpha(v - v')\right]}{\frac{c_1}{n_1}\left[(u + v) - \alpha_1(v - v')\right]}.$$

At infinite dilution $c = n$ and $\alpha = 0$,

and

$$\frac{\mu}{\mu_\infty} = \frac{x\left[(u + v) - \alpha(v - v')\right]}{u + v}$$

$$= x - \alpha x\frac{v - v'}{u + v},$$

or

$$x = \frac{\mu}{\mu_\infty} + \alpha x\frac{v - v'}{u + v}.$$

Hence, only in the case that $v = v'$ or $\alpha = 0$ do we obtain the true coefficient of ionization from the conductivities.

Further, from the conductivities and transport number we are not able to determine the values of the specific ionic velocity for any one species of ion, unless the transport number is determined at such a dilution that no complexes exist. With other values of p we obtain instead

$$\mathrm{U} = xu - \alpha xv'(m - 1)$$

and

$$\mathrm{V} = xv + \alpha x(mv' - v),$$

and therefore the apparent velocity of the anion becomes increased and that of the cation decreased by increasing concentration, and consequent increase in α. To determine the specific ionic velocities, therefore, of such cations as Ba, Ca, Cu, &c., it is necessary to know the value of p at very great dilutions, much greater than any at which this important physical constant has been hitherto determined.

It is of interest to point out that the values for p recently determined by **NOYES** for 0·02 N solution of barium chloride and nitrate are such as give to Ba the same absolute velocity in solutions of the two salts

$$\mu_{\infty} \cdot \epsilon \times (1 - p) \text{ for } BaCl_2 = 0{\cdot}000564$$

$$\text{for } BaNO_3 = 0{\cdot}000562.$$

Since the total amount of current crossing unit area of the conducting medium in unit time under a potential fall of π volts per centim. $= \epsilon\pi c\left[(u + v) - \alpha(v - v')\right]$, the amount of current passing across a section of area A in a solution of concentration N is given by

$$\mathrm{C} = \mathrm{A}\epsilon\pi x\left[(u + v) - \alpha(v - v')\right]\mathrm{N},$$

since

$$x = \frac{c}{\mathrm{N}}.$$

Thus the equation in the form given by MASSON no longer holds good for solutions which contain complex ions, a conclusion which is verified by the irregular values for the ratio $\frac{C}{An\epsilon(U + V)}$ shown for these salts in Table V.

The conclusions arrived at by KOHLRAUSCH, and given in the early part of this paper, as to the movements during electrolysis of a portion of an electrolyte which differs from the remainder in concentration, these movements being conditioned by a change in transport number, receive a simple explanation by the assumption of complex ionization. Let us imagine such a solution having initially a section which differs from the remainder in concentration, and first let us assume no complexes to be present, that is, p constant. Then since only the simple ions carry the current, that portion of the salt which is not ionized remains stationary (neglecting diffusion), and no movement is brought about. Next let us assume complexes to be present, that is, p variable; then, in addition to the motion of the completely ionized salt, a portion at least of the remainder of the salt is carried by the current, and movements of the section take place. If the complexes are cations, that is, p decreases with increasing concentration, the movement is in the positive direction; if, on the other hand, p increases, or the complexes are anions, in the negative direction, in complete agreement with KOHLRAUSCH'S conclusions.

This behaviour of such a section explains at once the change in p that is found when the velocities of the two boundaries are measured; for, assuming that the margins are those between indicator and simple ion, there occurs simultaneously and independently the migration of the complex, by means of which the whole column of salt solution (cations and anions) is carried along and in the positive or negative direction, according as the complex ions are cations or anions, and hence the apparent velocity of the one boundary is diminished and that of the other increased. The influence of the presence of such complexes on the velocity of the margin between two solutions is a question of considerable importance, and one that is worthy of strict mathematical investigation. It is, however, not considered by either KOHLRAUSCH or WEBER in the papers previously cited.

It is possible that, at the anion boundary, where the indicator is following two ions which differ in velocity, the concentration of the solution within the margin may become slightly altered.

Such a change undoubtedly occurs at the anion boundary of solutions of magnesium sulphate and copper sulphate, and the change is in the direction of a diminution of concentration at this point. This corresponds with an increased resistance, and accordingly a higher potential fall, and hence a greater velocity for this margin, a result which would also help to explain the increase in p with more concentrated solutions.

Experimental evidence of the changes referred to has been obtained in the following manner:—

The apparatus previously described, and shown in fig. 2, is filled with the solution to be measured, and the cells C and C′ with the indicator jellies are placed in position for the investigation of the anion margin. The cell C′ contains the anion indicator. After the current is started the resistance of the solution between the points *a* and *b* is measured.*

If the concentration is the same right up to the margin, the resistance will remain constant up to the time when the margin reaches the wire *b*, when it will suddenly increase, and continue to do so until both the points *a* and *b* are well within the indicator.

In fig. 5 are drawn the curves for the conductivity between the points *a* and *b* and for the neighbourhood of the anion margin for solutions of magnesium and copper sulphates. As abscissæ are plotted the position of the margin at the time of observation, the position of the platinum points being indicated by two small circles on each curve. As ordinates are plotted the conductivity of the column of liquid lying between the two points.

The similarly obtained curve for potassium chloride is also shown. Following this curve from left to right, it is seen that the conductivity, and so also the concentration, is constant as long as the two points are within the solution, a sudden and great fall in conductivity taking place after the point *b* is passed by the boundary. This fall continues, and the concentration is not constant again until the two points are both well within the indicator solution. The curves for the magnesium and copper sulphates show no such constancy in conductivity within the boundary. On the contrary, it is seen that just as the margin approaches the point *b* a very considerable change is indicated.

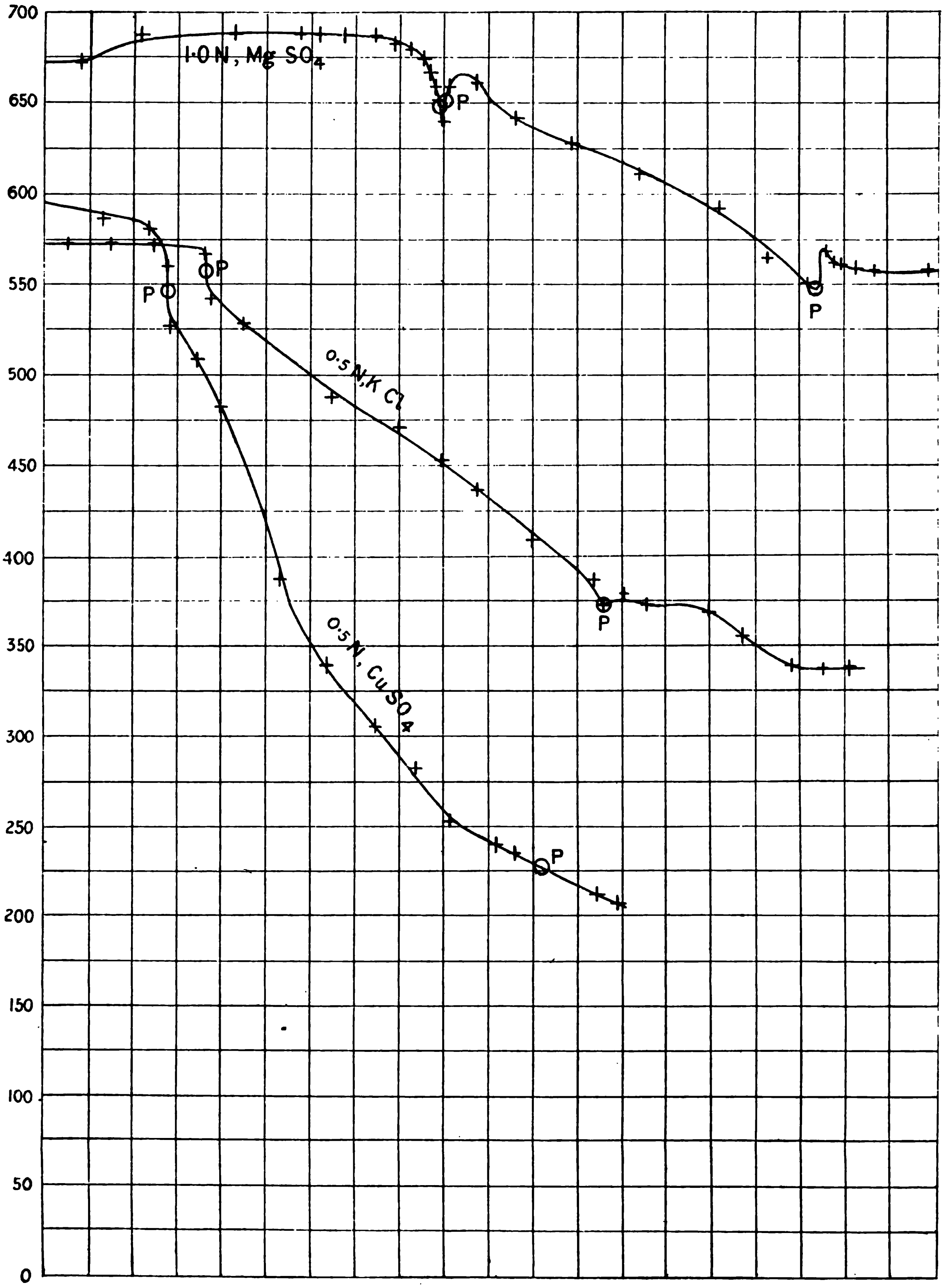
The cation boundary of magnesium sulphate has been examined by the same method, but only a very slight conductivity change was shown; this was, however, in the opposite direction to that detected at the anion end.

Further and very striking evidence of the dilution that occurs at the anion margin for the solution of copper sulphate, is obtained by simple observation of the solution during an experiment. Using sodium acetate as indicator there results a system in which a dark-blue solution of copper acetate lies over the lighter blue solution of the sulphate; a perfect boundary is produced, and one at which the refraction and colour margins are absolutely coincident; but the solution immediately beneath the indicator is much lighter in colour than that a ½ centim. lower, thus indicating a diminution in copper concentration. This difference in colour has been

---

* It is necessary that the platinum points should be so near together that the difference in potential between them is not sufficient to overcome the decomposition tension of water and so to cause an evolution of hydrogen and oxygen gases, which would result in a mixing of the two solutions. A very curious phenomenon has been noticed in measuring the resistances in this apparatus: in all cases the resistance as measured during the passage of the current is less than that found when it is measured with no current passing, and this is the case whether the latter is sent in the one direction or in the opposite.

Fig. 5.


700
650
600
550
500
450
400
350
300
250
200
150
100
50
0
1·0 N, Mg SO4
0·5 N, K Cl
0·5 N, Cu SO4
P
P
P
P
P
P

noticed in every experiment which has been made with copper sulphate as intermediate solution, and it is always much more noticeable in the more concentrated solutions.

A few experiments have been carried out which show that complex ions of a different kind to those hitherto considered do undoubtedly exist. The most striking of these is the measurement of the transport number of a solution of potassium ferric oxalate; for this salt it was possible to obtain a very good margin with the acetic ion as anion indicator. At the other end of the tube, where the complex entered the cation indicator, there is formed a solution of lithium ferric oxalate, and the ion penetrated the lithium chloride jelly for about 1·0 centim. on its way to the anode as a complex anion.

In another experiment, evidence of the constitution of the periodides was obtained. The apparatus shown in fig. 1 was filled with a 0·5 N solution of potassium iodide containing iodine, the indicators being sodium acetate and lithium chloride.

At the cathode end, where the $\frac{\mathrm{Ac}}{\mathrm{I}}$ Na boundary was formed, a precipitation of iodine occurred, doubtless on account of the fact that the I_3 ion being slower than the I ion, some of the former would be left behind in a portion of the system, where the equilibrium between KI, I_2, and KI_3 was disturbed, and therefore nearly the whole of the iodine was thrown out of the solution. The existence of complex I_3 ions is conclusively proved by the behaviour of the solution at the anode end; here a $\frac{\mathrm{Li}}{\mathrm{K}}$ I margin travels away from the gelatine solution of lithium chloride, and simultaneously the I ions enter the latter, forming lithium iodide, but not only I ions enter the jelly, but I_3 also, and the jelly for a length of about 1 centim. becomes coloured a deep red by the solution of LiI_3 that is produced. The entrance of the iodine is far too rapid to be explained by diffusion, and a blank experiment that was carried out showed that in the same time the distance covered by diffusion of the iodine amounted to only about 1 millim., and showed no such sharp line of demarcation between the coloured and colourless portions of the system.

Previous evidence of the existence of the compound KI_3 is given by LE BLANC and NOYES ('Zeitschrift für Phys. Chem.,' 13, 359, 1894, and 20, 19, 1896), JAKOWKIN (*ibid.*, 6, 385, 1890), and NOYES and SIEDENSTICKER (*ibid.*, 27, 357, 1898), whilst DAWSON ('Chem. Soc. Jl.,' 79, 238, 1900) discusses the dissociation of the compound KI_3 into the ions K and I_3.

Yet another series of experiments may be described which tend to show in the same manner the presence of complexes in solutions of copper sulphate. When a solution of this salt is followed by the three indicator ions Cd, Mg, or Li, a great difference in behaviour is to be noticed. Cd as indicator fulfils the condition that it is slower than the Cu, and consequently the solution behind the boundary remains quite colourless; with the other two ions this condition is not fulfilled. When Mg follows Cu the latter falls behind the margin, and the colour does not entirely disappear until a point about 2 centims. behind the boundary is reached; no copper, however, is found to enter the $MgSO_4$ jelly.

With Li as indicator the copper also lags behind the boundary, but as has been previously mentioned, the colour shows no sign of entirely fading out, and when there exists a column 6 centims. long of Li_2SO_4 solution, the blue colour of the copper left behind is perfectly uniform, but lighter in colour than the measured solution. On examining the lithium chloride jelly after the experiment, it was found that copper had entered it for a distance of between 3–3·5 millims., thus indicating the passage of the copper to the anode as complex anion. This behaviour is possibly associated with the fact that the double sulphates of copper belong to the class $M'_2Cu(SO_4)_2$, and hence a complex $Cu(SO_4)_2$ will be stable in the presence of a monad cation such as Li, but not if the cation is of the type of Mg.

The method described in the preceding pages gives a simple and accurate means of determining the transport number of the simplest type of salts; in its present form it is applicable only to more concentrated solutions. It is hoped to modify it in such a manner as to permit of the measurement of more dilute solutions. For salts of the dyad metals, more doubt attaches to the accuracy of the method, in consequence of our want of exact knowledge as to the mechanism of the changes that may take place at the margin, on account of the presence of complexes.

The existence of these may be considered to be established with a probability amounting almost to certainty, but the evidence is, as yet, qualitative only; by the solution of the problem, as to exactly how the margin velocity is influenced by the presence of complexes, and their dissociation outside the margin, it is possible that a means of determining quantitatively the proportion of complex to simple ions in a given solution may be indicated.

In conclusion, it is the author's very pleasant duty to express his indebtedness to Professor Abegg for the help he has received from him during the course of the work.

IV. *On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load.*

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§ 1. *Object and Aims of the Paper.*

THE usual solution for the extension and compression of elastic bars assumes that the latter are strained under a normal tension or pressure uniformly distributed across the plane ends. In like manner the solution for torsion of such bars assumes that the external forces which cause the torsion consist of a determinate system of tangential stresses, acting across the plane ends.

In both cases the solution is such that the torsion and extension are transmitted throughout the bar *without change of type.* Such terminal conditions of stress, however, do not usually occur in practice, and it accordingly becomes of considerable interest to find out how the results obtained for such a theoretical system of loading are modified, if at all, when we consider applied external stresses which give a closer representation of every-day mechanical conditions.

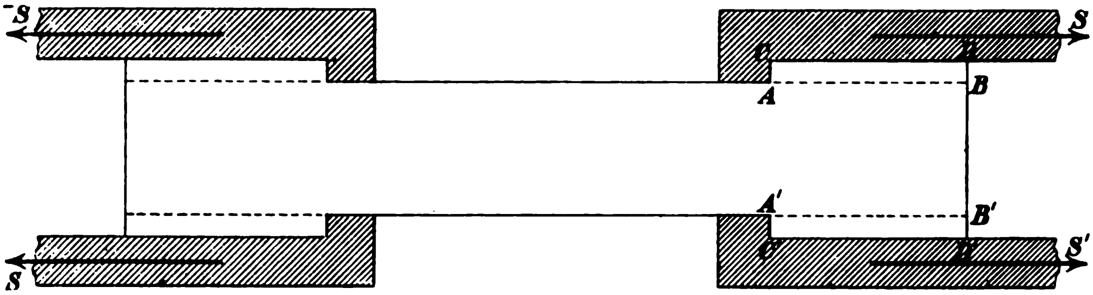
The present paper is an attempt towards the solution of this problem in three cases, which appear of especial practical interest.

The first case is that of a bar which is subjected to a determinate system of normal radial pressures and of axial shears all over the curved surface, the radial pressures being symmetrical about the mid-section and the shears having their sign changed. Thus the cylinder is subjected to a total axial pull, due to the shears, and also to a given transverse pressure. The plane ends are free from stress, except for a self-equilibrating system of radial shears, which will have little or no effect at points at some distance from the ends.

A special case worked out is that where the normal pressure is zero throughout, but a determinate axial shear, which has been taken constant, is made to act over two equal rings on the surface of the cylinder.

This will give us valuable information about a system of stress which often occurs in practice, in testing machines, for example, in which a specimen is pulled apart by means of pressures applied to the inner rims of projecting collars (see fig. 1). The shaded parts of the figure represent the "grips," and if S be the total pull applied, this is transmitted to the test piece by means of pressure applied along CA, C′A. Now consider the thinner cylinder in the middle ideally produced inside the thicker ends. It is in equilibrium under the stresses, radial and tangential, between the inner core and the hollow cylinder produced by the revolution of ABDC.

Fig. 1.



But what are these radial and tangential stresses? If we consider the equilibrium of the outer hollow cylinder only, we see that the resultant of the stresses across AB, A′B′ must exactly balance the pull S, however applied. The radial stress will probably be small, as it has no external traction to balance, and the longitudinal shears are therefore equivalent to S. Thus the thin cylinder inside is really stretched, not by normal traction over the flat ends, but by longitudinal shears over the curved surface, and a careful investigation will show that, in every *practical* case, extension is obtained by the application of an axial shear to the curved surface of the cylinder, *never* of tractions to the flat ends. The general effects of such a distribution appear, therefore, of great practical interest.

The second problem discussed is that of a cylinder of moderate length, which is compressed between two rough rigid planes in such a way that the terminal cross-sections are constrained to remain plane, but are not allowed to expand, their perimeter being kept fixed. By adding a suitable uniform distribution of pressure to a load system of this type, we can obtain the solution for a cylinder constrained in such a way that its ends expand by a definite amount. These two problems are of importance with reference to the behaviour of a block of stone or masonry when tested between millboard or metal planes, which practically hinder the block from expanding, and when tested between sheets of lead, which, on the other hand, favour the expansion of the block. The widely divergent results obtained for the strength of the same material when tested by these two methods have troubled many elasticians. UNWIN ('Testing of Materials of Construction') is of opinion that the results obtained when sheet lead is used are unreliable; whereas Professor PERRY, in his 'Applied Mechanics,' states that the true strength of the material is the one given by the lead experiments, and should be usually taken as half the published strength.

Finally, the third problem treated is that of a cylinder subjected to transverse shears over the parts of the curved surface near the ends, these shears being equivalent to a torsion couple. This is really the analogue, for torsion, of the first problem for tension, and corresponds to the case of a bar "gripped" as explained above, and twisted. This again is the method by which torsion is practically produced in most cases—almost always in laboratory experiments.

§ 2. *Method of Solution Adopted. Historical References.*

The method adopted has been to obtain symmetrical solutions of the equations of elasticity in cylindrical co-ordinates and to express the typical term in the form

$$\begin{matrix}\cos\\ \sin\end{matrix}\{kz\} \times f(r),$$

r, ϕ, z being the usual cylindrical co-ordinates.

The expressions for the strains and stresses, over any coaxial cylinder, are therefore series of sines and cosines of multiples of z. The arbitrary constants of the coefficients are determined by comparison with the coefficients of the FOURIER's series which express the applied stresses at the external boundary.

This method is not a new one. It has been indicated by LAMÉ and CLAPEYRON ("Mémoire sur l'équilibre intérieur des corps solides homogènes," 'Crelle's Journal,' vol. 7), but it has been for the first time worked out with any completeness by Professor L. POCHHAMMER ("Beitrag zur Theorie der Biegung des Kreiscylinders," 'Crelle's Journal,' vol. 81, 1876). Professor POCHHAMMER obtains the general solution of the elastic equations for an infinite circular cylinder subject to any system of surface loading, repeated at regular intervals. This he applies to the case of a built-in beam. The solution is not restricted to be symmetrical about the axis of the cylinder, but is perfectly general. The complete accurate expressions are, however, quite unwieldy; but, as the result of expanding the functions involved to the first two or three terms, Professor POCHHAMMER obtains far more manageable expressions, which he is eventually able to identify with those previously given by NAVIER and DE SAINT VENANT for more special cases of loading. It is to be noted, however, that POCHHAMMER restricts himself solely to the case of bending, and that his approximations depend upon the ratio of diameter to length being a small quantity.

The same general expressions have been independently arrived at by Mr. C. CHREE ("The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-ordinates, their Solution and Application," 'Camb. Phil. Trans.,' vol. 14). Here, again, the solutions are not restricted to be symmetrical. The symmetrical terms, however, agree with the solutions of the present paper, but the latter are obtained by a process slightly different from that of Mr. CHREE. Mr. CHREE has also given a solution of the symmetrical case proceeding in powers of r and z. Using each form of solution *independently*, it is not possible to satisfy the condition that there shall be no stress at all on the curved surface; this is effected in the second problem of this paper, by means of a combination of the two types of solution.

In the paper referred to, Mr. CHREE, like Professor POCHHAMMER, has not, so far as I am aware, applied his general solution to the problems of tension and compression. He does give one example of torsion, which he obtains by applying an arbitrary system of cross-radial shears across the flat ends. Such a system, we have seen, would not usually correspond to what occurs in practice.

Mr. CHREE has written several other papers ("On some Compound Vibrating Systems," 'Camb. Phil. Trans.,' vol. 15, Part II.; "On Longitudinal Vibrations," 'Quarterly Journal of Mathematics,' 1889; "Longitudinal Vibrations in Solid and Hollow Cylinders," 'Phil. Mag.,' 1899; "On Long Rotating Circular Cylinders," 'Camb. Phil. Soc. Proc.,' vol. 7, Part VI., &c.), which deal with the solutions of the equations of elasticity in cylindrical co-ordinates, with special application to vibrations and rotating shafts; but I cannot find that he has anywhere returned to the statical problem and its solution by means of sine and cosine expansions.

[*October* 3, 1901.—Professor SCHIFF ('Journal de Liouville,' Série 3, vol. 9, 1883) has attempted the solution of the problem of the cylinder compressed between parallel planes, which is one of those treated of in the present paper. His solution is expressed in a series, not of circular functions, but of hyperbolic sines and cosines of nz, the successive values of n being obtained as roots of a certain transcendental equation. This enables him to satisfy the conditions at the curved surface, but the arbitrary coefficients are finally determined by the conditions over the plane ends. He assumes both the radial shear and the molecular rotation in a diametral plane to be given by known functions, $f(r)$ and $F(r)$, over the plane ends, and from these he succeeds in obtaining the coefficients. As he has only a single set of the latter left to carry out the identification, his functions $f(r)$ and $F(r)$ are not really independent. Theoretically only the shear $f(r)$ should be required, and in a practical problem even this is unknown, the total pressure being all that is given. The actual distribution of this pressure does not appear to enter into Professor SCHIFF's solution. Also the fact that the values of n are roots of a transcendental equation singularly complicates the solution from a numerical point of view, and Professor SCHIFF appears to have made no attempt to translate his results into numbers.]*

It has therefore appeared worth while to apply the solutions involving circular functions of z to problems such as those sketched above.

Of each of these I have given a concrete numerical example. Indeed, the greater part of the work has been spent on these numerical examples. The labour of calculation has in most cases been considerable, owing to the slow convergence of many of the series involved, which has necessitated special methods of approximation.

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* Since writing the above, I find that the problem of the circular cylinder under a symmetrical strain has been considered by J. THOMAE in two papers ("Über eine einfache Aufgabe aus der Theorie der Elasticität," 'Leipzig Berichte,' vols. 37–38). The author has used expansions in sines and cosines of kz, but, as far as I can make out, the only problem he considers is that of the vertical pillar under its own weight.

§ 3. *General Solution for a Symmetrical Strain.*

Let r, ϕ, z be the usual cylindrical co-ordinates; also, following the notation of TODHUNTER and PEARSON'S 'History of Elasticity,' let $\widehat{st}$ denote the stress, parallel to ds, across an element of surface perpendicular to dt, s, t standing for any two of the letters r, ϕ, z.

Let u, v, w denote the radial, cross-radial, and longitudinal displacements respectively, then we have (LAMÉ, 'Leçons sur l'Elasticité'), if u, v, w are independent of ϕ:

$$(\lambda + 2\mu)\frac{d^2u}{dr^2} + (\lambda + 2\mu)\frac{d}{dr}\left(\frac{u}{r}\right) + \mu\frac{d^2u}{dz^2} + (\lambda + \mu)\frac{d^2w}{drdz} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(rv)}{dr}\right) + \frac{d^2v}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

$$(\lambda + \mu)\left(\frac{d^2u}{drdz} + \frac{1}{r}\frac{du}{dz}\right) + \mu\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) + (\lambda + 2\mu)\frac{d^2w}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (3).$$

$$\left.\begin{array}{ll} \widehat{rr} = (\lambda + 2\mu)\dfrac{du}{dr} + \lambda\dfrac{u}{r} + \lambda\dfrac{dw}{dz} & \widehat{rz} = \mu\left(\dfrac{dw}{dr} + \dfrac{du}{dz}\right) \\ \widehat{zz} = \lambda\dfrac{du}{dr} + \lambda\dfrac{u}{r} + (\lambda + 2\mu)\dfrac{dw}{dz} & \widehat{\phi z} = \mu\dfrac{dv}{dz} \\ \widehat{\phi\phi} = (\lambda + 2\mu)\dfrac{u}{r} + \lambda\dfrac{du}{dr} + \lambda\dfrac{dw}{dz} & \widehat{r\phi} = \mu\left(\dfrac{dv}{dr} - \dfrac{v}{r}\right) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

λ and μ being the elastic constants of LAMÉ.

We see from the above that $\widehat{\phi z}$ and $\widehat{r\phi}$ depend only on v, the other stresses only on u and w. Also the equation (2) contains v only, (1) and (3) contain u and w only. The solution for transverse displacements is therefore absolutely independent of the solution for radial and longitudinal displacements.

Let us now denote the operators $\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\right)$ by ϑ^2, and $\frac{d}{dz}$ by D. Differentiate (1) with regard to z and (3) with regard to r, and remember that the order of the symbols D and ϑ is indifferent, then we find

$$(\vartheta^2 + D^2)v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$$\left((\lambda + 2\mu)\vartheta^2 + \mu D^2\right)\frac{du}{dz} + (\lambda + \mu)D^2\frac{dw}{dr} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

$$(\lambda + \mu)\vartheta^2\frac{du}{dz} + \left(\mu\vartheta^2 + (\lambda + 2\mu)D^2\right)\frac{dw}{dr} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Eliminate either du/dz or dw/dr between (6) and (7): it is found that either of these quantities satisfies the partial differential equation

$$\checkmark\ (\vartheta^2 + D^2)^2\, y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The whole problem of the determination of the elastical equilibrium of the circular cylinder under any symmetrical system of stress depends therefore on the solution of this differential equation.

§ 4. *Solution of the Differential Equation.*

The differential equation

$$(\vartheta^2 + D^2)\, y = 0$$

is really identical with LAPLACE's equation in cylindrical co-ordinates, namely,

$$\frac{1}{r}\frac{d}{dr}\frac{r\,dV}{dr} + \frac{1}{r^2}\frac{d^2V}{d\phi^2} + \frac{d^2V}{dz^2} = 0.$$

Suppose V independent of ϕ and differentiate with regard to r, we have

$$(\vartheta^2 + D^2)\, dV/dr = 0.$$

If therefore V be such a solution of LAPLACE's equation, $y = dV/dr$ will be a solution of the given differential equation. For our purpose, however, it will be simpler to proceed from the equation itself.

Assume a typical solution

$$y_1 = R_1 \,.\, Z_1,$$

where R_1 is a function of r only, Z_1 a function of z only.

We find easily

$$\frac{d^2R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} - \left(\frac{1}{r^2} + k^2\right)R_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\frac{d^2Z_1}{dz^2} + k^2 Z_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

The solutions of (9) are of the form

$$I_1(kr) \quad \text{and} \quad K_1(kr),$$

where

$$I_n(x) = \sum_0^{\infty} \frac{x^{n+2s}}{2^{n+2s}\Pi(s)\,\Pi(s+n)},$$

$$K_n(x) = (-1)^n \frac{1.3.(2n-1)}{x^n} \int_0^{\infty} \frac{\cos(x\sinh\phi)}{\cosh^{2n}\phi}\, d\phi.$$

(See GRAY and MATHEWS, 'Bessel's Functions,' pp. 66–7.)

Consider now the equation

$$(\vartheta^2 + D^2)^2 y = 0.$$

Let y_1 be any solution of the equation

$$(\vartheta^2 + D^2)\, y = 0.$$

Then if y_2 be a solution of

$$(\vartheta^2 + D^2)\, y = y_1$$

it is also a solution of

$$(\vartheta^2 + D^2)^2 y = 0.$$

Now if $y_1 = R_1 Z_1$ what is the condition that we can obtain a second product solution $y_2 = R_2 Z_2$?

We have, substituting

$$\frac{\vartheta^2 R_2}{R_2} + \frac{D^2 Z_2}{Z_2} = \frac{R_1}{R_2} \cdot \frac{Z_1}{Z_2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11),$$

or function of r only + function of z only = product function in r and z.

If (11) is to be identically satisfied, this product function must be a function of r only or of z only.

Case (i.). $Z_2 = a Z_1$ where a is a constant.

We find

$$\vartheta^2 R_2 - k^2 R_2 = \frac{1}{a} R_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12),$$

or, since R_1 is a solution of

$$(\vartheta^2 - k^2)\, R_1 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13),$$

which is the same as (9), R_2 is a solution of

$$(\vartheta^2 - k^2)^2 R_2 = 0,$$

which is not at the same time a solution of (13).

Now the solutions of this equation are

$$I_1(kr), \quad K_1(kr), \quad \frac{d}{dk} I_1(kr), \quad \frac{d}{dk} K_1(kr).$$

But

$$\frac{d}{dk} I_1(kr) = r I_1'(kr) = r I_0(kr) - \frac{1}{k} I_1(kr),$$

and similarly

$$\frac{d}{dk} K_1(kr) = r K_0(kr) - \frac{1}{k} K_1(kr).$$

The four independent integrals are therefore

$$I_1(kr), \quad K_1(kr), \quad r I_0(kr), \quad r K_0(kr),$$

and therefore the required values of R_2 are $r I_0$, $r K_0$.

Case (ii.). $R_2 = bR_1$; we find, using (13),

$$(D^2 + k^2) Z_2 = \frac{1}{b} Z_1,$$

and therefore Z_2 is a solution of $(D^2 + k^2)^2 Z_2 = 0$, which is not at the same time a solution of

$$(D^2 + k^2) Z_2 = 0.$$

The possible values of Z_2 are $z \cos kz$, $z \sin kz$.

Hence the possible sets of product functions satisfying the equation

$$(\vartheta^2 + D^2)^2 y = 0$$

are as follows:—

$$\left.\begin{aligned} y = {} & A \cos(kz + \alpha) I_1(kr) \\ & B \cos(kz + \beta) K_1(kr) \\ & C \cos(kz + \gamma) r I_0(kr) \\ & D \cos(kz + \delta) r K_0(kr) \\ & E z \cos(kz + \epsilon) I_1(kr) \\ & F z \cos(kz + \theta) K_1(kr) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (14).$$

§ 5. *Solution under given conditions of Surface-loading; the first problem.*

Let us now consider first the case of a circular cylinder under the following system of stress:

$\widehat{rr}/\mu$ = a given even function of $z\ (= f(z))$ over the curved surface $r = a$,

$\widehat{rz}/\mu$ = a given odd function of $z\ (= \psi(z))$ over the curved surface $r = a$,

$\widehat{zz} = 0$ over the plane ends $z = \pm c$.

Since du/dz, dw/dr are both solutions of (8) we may have them composed of a series of terms as follows:

$$\frac{du}{dz} = \Sigma \left\{ \begin{aligned} & A_1 \cos(kz + \alpha_1) I_1(kr) + C_1 \cos(kz + \gamma_1) r I_0(kr) \\ & + E_1 z \cos(kz + \epsilon_1) I_1(kr) \end{aligned} \right\} \quad \ldots \quad (15).$$

$$\frac{dw}{dr} = \Sigma \left\{ \begin{aligned} & A_2 \cos(kz + \alpha_2) I_1(kr) + C_2 \cos(kz + \gamma_2) r I_0(kr) \\ & + E_2 z \cos(kz + \epsilon_2) I_1(kr) \end{aligned} \right\} \quad \ldots \quad (16).$$

No K-functions have been introduced in this case, as they lead to infinite terms at the axis.

Also the conditions of the problem require that u shall be an even function of z and w an odd function of z. Hence

$$\alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = -\pi/2, \ \epsilon_1 = \epsilon_2 = 0.$$

Integrating (15) and (16) we have

$$u = \chi(r) + \Sigma\left\{-\frac{A_1}{k}\cos kz I_1(kr) - \frac{C_1}{k}\cos kz . rI_0(kr) + \frac{E_1}{k}\left(z\sin kz + \frac{\cos kz}{k}\right)I_1(kr)\right\},$$

$$w = \theta(z) + \Sigma\left\{\frac{A_2}{k}\sin kz I_0(kr) + \frac{C_2}{k}\sin kz . rI_1(kr) + \frac{E_2}{k} z\cos kz\, I_0(kr)\right\}.$$

To find the relations between the constants we must substitute in equations (1) and (3). We then find the following relations:

$$\vartheta^2\chi(r) = 0 \quad . \ . \ . \ . \ (17), \qquad D^2\theta(z) = 0 \quad . \ . \ . \ . \ (18),$$

$$(A_1 - A_2)(\lambda + \mu)k^2 + 2k\{C_1(\lambda + 2\mu) - \mu E_1 - (\lambda + \mu)E_2\} = 0 \quad . \ . \ (19),$$

$$(A_1 - A_2)(\lambda + \mu)k^2 + 2k\{C_1(\lambda + \mu) + \mu C_2 - (\lambda + 2\mu)E_2\} = 0 \quad . \ . \ (20),$$

$$C_1 = C_2 = C, \text{ say} \quad . \ . \ . \ (21), \qquad E_1 = E_2 = E, \text{ say} \quad . \ . \ . \ (22).$$

In virtue of equations (21) and (22), (19) and (20) reduce to the single equation

$$(A_1 - A_2)(\lambda + \mu)k + 2(\lambda + 2\mu)(C - E) = 0 \quad . \ . \ . \ . \ . \ (23).$$

Also from (17) and (18)

$$\chi(r) = u_0 r, \qquad \theta(z) = w_0 z,$$

remembering that w is odd in z and that u is not to be infinite when $r = 0$.

For the stresses $\widehat{rr}$, $\widehat{zz}$, $\widehat{rz}$ we find from (4), after some obvious reductions,

$$\widehat{rr} = 2(\lambda + \mu)u_0 + \lambda w_0$$
$$+ \Sigma\left[\begin{array}{l} \dfrac{-\mu}{+2\mu}\{(2\lambda + 3\mu)A_1 + \mu A_2\}I_0(kr)\cos kz \\ + 2\mu\left\{\left(A_1 - \dfrac{E}{k}\right)\dfrac{I_1(kr)}{kr}\cos kz + Ez\sin kz\left(I_0(kr) - \dfrac{I_1(kr)}{kr}\right)\right\} - CrI_1(kr)\cos kz \end{array}\right] \quad . \ . \ . \ (24).$$

$$\widehat{zz} = 2\lambda u_0 + (\lambda + 2\mu)w_0$$
$$+ \Sigma\left[\begin{array}{l} \left\{(\lambda + 2\mu)A_2 - \lambda A_1 - \lambda\dfrac{2C}{k} + 2(\lambda + \mu)\dfrac{E}{k}\right\}I_0(kr)\cos kz \\ + 2\mu\{CrI_1(kr)\cos kz - EI_0(kr)z\sin kz\} \end{array}\right] \quad . \ . \ . \ . \ (25).$$

$$\widehat{rz} = \mu\Sigma\{(A_1 + A_2)I_1(kr)\sin kz + 2CrI_0(kr)\sin kz + 2EI_1(kr)z\cos kz\} \quad . \ . \ . \ (26).$$

But clearly if $\widehat{zz}$ is to be zero all over the plane ends we must have

$$u_0 = -\frac{\lambda + 2\mu}{2\lambda} w_0 \quad \ldots \ldots \quad (27),$$

$$k = \frac{(2n+1)\pi}{2c} \quad \ldots \ldots \quad (28),$$

and

$$\mathrm{E} = 0 \quad \ldots \ldots \quad (29).$$

The expressions for the displacements and stresses then reduce to the following, writing $kr = \rho$ for shortness :

$$u = u_0 r - \Sigma \left\{ \mathrm{A}_1 \mathrm{I}_1(\rho) + \frac{\mathrm{C}}{k} \rho \mathrm{I}_0(\rho) \right\} \frac{\cos kz}{k} \quad \ldots \ldots \quad (30),$$

$$w = w_0 z + \Sigma \left\{ \mathrm{A}_2 \mathrm{I}_0(\rho) + \frac{\mathrm{C}}{k} \rho \mathrm{I}_1(\rho) \right\} \frac{\sin kz}{k} \quad \ldots \ldots \quad (31),$$

$$\widehat{rr} = 2(\lambda + \mu) u_0 + \lambda w_0$$

$$+ \Sigma \left\{ -\frac{\mu}{\lambda + 2\mu} [(2\lambda + 3\mu) \mathrm{A}_1 + \mu \mathrm{A}_2] \mathrm{I}_0(\rho) + 2\mu \left[\frac{\mathrm{A}_1}{\rho} - \frac{\mathrm{C}\rho}{k} \right] \mathrm{I}_1(\rho) \right\} \cos kz. \quad . \quad (32),$$

$$\widehat{zz} = \Sigma \left\{ \frac{\mu}{\lambda + 2\mu} [(3\lambda + 4\mu) \mathrm{A}_2 - \lambda \mathrm{A}_1] \mathrm{I}_0(\rho) + 2\mu \frac{\mathrm{C}}{k} \rho \mathrm{I}_1(\rho) \right\} \cos kz \,. \quad \ldots \ldots \quad (33),$$

$$\widehat{rz} = \mu \Sigma \left\{ (\mathrm{A}_1 + \mathrm{A}_2) \mathrm{I}_1(\rho) + \frac{2\mathrm{C}}{k} \rho \mathrm{I}_0(\rho) \right\} \sin kz \,. \quad \ldots \ldots \quad (34),$$

where

$$\frac{2\mathrm{C}}{k} = -\frac{\lambda + \mu}{\lambda + 2\mu} (\mathrm{A}_1 - \mathrm{A}_2) \quad \ldots \ldots \quad (35).$$

Over the surface of the cylinder $r = a$, we find

$$(\widehat{rr}/\mu)_{r=a} = -\frac{3\lambda + 2\mu}{\lambda} w_0$$

$$+ \Sigma \left\{ \begin{array}{l} \mathrm{A}_1 \left[-(1+\gamma) \mathrm{I}_0(\alpha) + \left(\frac{2}{\alpha} + \gamma\alpha \right) \mathrm{I}_1(\alpha) \right] \\ + \mathrm{A}_2 [-(1-\gamma) \mathrm{I}_0(\alpha) - \gamma\alpha \mathrm{I}_1(\alpha)] \end{array} \right\} \cos kz \quad . \quad . \quad (36),$$

$$(\widehat{rz}/\mu)_{r=a} = \Sigma \left\{ \begin{array}{l} \mathrm{A}_1 [\mathrm{I}_1(\alpha) - \gamma\alpha \mathrm{I}_0(\alpha)] \\ + \mathrm{A}_2 [\mathrm{I}_1(\alpha) + \gamma\alpha \mathrm{I}_0(\alpha)] \end{array} \right\} \sin kz \,. \quad \ldots \ldots \quad (37),$$

where γ is written for $\frac{\lambda + \mu}{\lambda + 2\mu}$ and α for ka.

Now $f(z)$ being an even function, we can expand it in a FOURIER's series between the limits $\pm c$ in the form

$$\left.\begin{aligned} f(z) - f(c) &= \sum_0^\infty a_n \cos \frac{\overline{2n+1}\,\pi z}{2c} \\ \text{where} \qquad a_n &= \frac{1}{c}\int_{-c}^{+c} \{f(z) - f(c)\} \cos \frac{\overline{2n+1}\,\pi z}{2c}\, dz \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38),$$

and $\psi(z)$ being an odd function can be expanded in the form

$$\left.\begin{aligned} \psi(z) &= \sum_0^\infty b_n \sin \frac{\overline{2n+1}\,\pi z}{2c} \\ \text{where} \qquad b_n &= \frac{1}{c}\int_{-c}^{+c} \psi(z) \sin \frac{\overline{2n+1}\,\pi z}{2c}\, dz \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).$$

Now since $\qquad (\widehat{rr})_{r=a} = \mu f(z), \ (\widehat{rz})_{r=a} = \mu\psi(z),$

we have, comparing (36) and (37) with (38) and (39),

$$-\frac{3\lambda + 2\mu}{\lambda} w_0 = f(c)$$

$$\mathrm{A}_1\left(-\{1+\gamma\}\mathrm{I}_0(\alpha) + \left\{\frac{2}{\alpha} + \gamma\alpha\right\}\mathrm{I}_1(\alpha)\right) + \mathrm{A}_2(-\{1-\gamma\}\,\mathrm{I}_0(\alpha) - \gamma\alpha\mathrm{I}_1(\alpha)) = a_n$$

$$\mathrm{A}_1(\mathrm{I}_1(\alpha) - \gamma\alpha\mathrm{I}_0(\alpha)) + \mathrm{A}_2(\mathrm{I}_1(\alpha) + \gamma\alpha\mathrm{I}_0(\alpha)) = b_n,$$

whence

$$\mathrm{A}_1 = -\tfrac{1}{2}\frac{a_n(\alpha\mathrm{I}_1(\alpha) + \gamma\alpha^2\mathrm{I}_0(\alpha)) + b_n(\gamma\alpha^2\mathrm{I}_1(\alpha) + (1-\gamma)\alpha\mathrm{I}_0(\alpha))}{\gamma\alpha^2\mathrm{I}_0^2(\alpha) - (1+\gamma\alpha^2)\mathrm{I}_1^2(\alpha)} \quad . \quad . \quad . \quad (40),$$

$$\mathrm{A}_2 = \tfrac{1}{2}\frac{a_n(\alpha\mathrm{I}_1(\alpha) - \gamma\alpha^2\mathrm{I}_0(\alpha)) + b_n((1+\gamma)\alpha\mathrm{I}_0(\alpha) - (2+\gamma\alpha^2)\mathrm{I}_1(\alpha))}{\gamma\alpha^2\mathrm{I}_0^2(\alpha) - (1+\gamma\alpha^2)\mathrm{I}_1^2(\alpha)} \quad . \quad . \quad (41),$$

$$w_0 = -\frac{(2\gamma - 1)}{4\gamma - 1} f(c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42),$$

and therefore from (35) and (27)

$$\mathrm{C} = \frac{(2n+1)\pi}{2c}\,\tfrac{1}{2}\,\frac{a_n\alpha\gamma\mathrm{I}_1(\alpha) + b_n(\gamma\alpha\mathrm{I}_0(\alpha) - \gamma\mathrm{I}_1(\alpha))}{\gamma\alpha^2\mathrm{I}_0^2(\alpha) - (1+\gamma\alpha^2)\mathrm{I}_1^2(\alpha)} \quad . \quad . \quad . \quad . \quad . \quad (43),$$

$$u_0 = \frac{1}{2(4\gamma - 1)} f(c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44).$$

§ 6. *Consideration of the Approximate Expressions to which the Results of the last Section lead, when the Ratio of Diameter to Length is small.*

If we can treat the diameter of the cylinder as small compared with its length, we can obtain a first approximation by following the method of Professor POCHHAMMER ('Crelle,' vol. 81), and expanding A_1, A_2, C in powers of α, which is then a small quantity, provided the index n is not too large.* If we do this we find

$$A_1 = - a_n \frac{2\gamma + 1}{4\gamma - 1} + \frac{b_n}{\alpha} \frac{2\gamma - 2}{4\gamma - 1},$$

$$A_2 = - a_n \frac{2\gamma - 1}{4\gamma - 1} + \frac{b_n}{\alpha} \frac{2\gamma}{4\gamma - 1},$$

$$\frac{C}{k} = a_n \frac{\gamma}{4\gamma - 1} + \frac{b_n}{\alpha} \frac{\gamma}{4\gamma - 1},$$

and, expanding $I_0(kr)$ and $I_1(kr)$ in powers of r, and dropping all the terms except the first (which is really equivalent to a second approximation, since the indices go up two at a time), we find

$$u = \frac{r}{2(4\gamma - 1)} \left\{ f(c) + \Sigma \left(a_n - \frac{4\gamma - 2}{\alpha} b_n \right) \cos kz \right\}$$

$$= \frac{r}{2(4\gamma - 1)} \left\{ f(c) + f(z) - f(c) + \frac{4\gamma - 2}{a} \int_c^z \psi(z)\, dz \right\},$$

using the Fourier expansions (38) and (39)

$$= \frac{r}{4\gamma - 1} \frac{1}{2\mu} \left\{ (\widehat{rr})_{r=a} - (2\gamma - 1) \frac{1}{\pi a^2} \int_z^c (\widehat{rz})_{r=a}\, dz \times 2\pi a \right\}.$$

Now $\int_z^c (\widehat{rz})_{r=a}\, dz \times 2\pi a$ is equal to the total longitudinal pull exerted on the bar by all the forces on one side of the cross-section considered. It represents, in other words, the total tension at that cross-section. Denoting it by $\pi a^2 Q$, where Q is the mean tension at that cross-section,

$$u = r \left\{ (\widehat{rr})_{r=a} \times \frac{\lambda + 2\mu}{2\mu(3\lambda + 2\mu)} - Q \frac{\lambda}{2\mu(3\lambda + 2\mu)} \right\} \quad . \quad . \quad . \quad . \quad . \quad (45),$$

which shows that the radial displacement is exactly the same as if the only forces on a thin lamina between two cross-sections were an external radial tension $(\widehat{rr})_{r=a}$ and a uniform tension Q across the plane faces.

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* For the analytical restrictions necessary in such a case, see § 28.

In like manner it can be shown that

$$\left.\begin{aligned} w &= w_0 z + \Sigma\left(-a_n\frac{2\gamma-1}{4\gamma-1}+\frac{b_n}{\alpha}\frac{2\gamma}{4\gamma-1}\right)\frac{\sin kz}{k} \\ &= -\frac{2\gamma-1}{4\gamma-1}\left\{zf(c)+\int_0^z (f(z)-f(c))\,dz\right\}+\frac{2\gamma}{4\gamma-1}\frac{1}{\alpha}\int_z^c dz\int_0^z \psi(z)\,dz \\ &= -\frac{2\gamma-1}{4\gamma-1}\int_0^z \frac{(\widehat{rr})_{r=a}}{\mu}\,dz+\frac{\gamma}{4\gamma-1}\int_0^z \frac{Q}{\mu}\,dz \\ &= \int_0^z\left\{-\frac{\lambda}{3\lambda+2\mu}\frac{(\widehat{rr})_{r=a}}{\mu}+\frac{\lambda+\mu}{3\lambda+2\mu}\frac{Q}{\mu}\right\}dz \\ &= \int_0^z s_z\,dz \end{aligned}\right\} \quad (46),$$

s_z being the stretch parallel to the axis in a cylinder which is under a tension Q across its plane faces and a radial tension $(\widehat{rr})_{r=a}$.

Thus the longitudinal and radial displacements are, to a first approximation, the same as if the cylinder were supposed made up of any number of thin circular laminæ, piled up on top of each other, the longitudinal tension in any lamina being uniform and giving a total tension equal to the total pull of all the external forces acting on the cylinder on one side of the section considered.

Further, the shearing stress $\widehat{rz}$ at a point inside is found to the same approximation to be given by

$$\frac{\widehat{rz}}{\mu} = \Sigma(kr)\frac{b_n}{\alpha}\sin kz = \frac{r}{a}\Sigma b_n \sin kz = \frac{r}{a}\frac{(\widehat{rz})_{r=a}}{\mu}$$

so that, in the parts of the cylinder to which external shearing stress is applied, and in these only, there is shearing stress inside the cylinder, which shearing stress is proportional to the distance from the axis.

The other stresses, $\widehat{rr}$, $\widehat{\phi\phi}$, $\widehat{zz}$, are found to the same approximation to be all constants for any given value of z.

$$\widehat{zz} = 2\mu\Sigma\frac{b_n}{\alpha}\cos kz = \frac{2\mu}{a}\int_z^c \psi(z)\,dz = Q,$$

$$\widehat{rr} = \widehat{\phi\phi} = \mu f(z).$$

It follows from the above that the action of any radial pressure will be purely local, and also that, whatever the manner in which the cylinder is "gripped" and the pull is applied, the stress in the portions of the bar between the points of application of the pull reduces practically to a uniform tension.

The above results are somewhat remarkable as tending to show how very restricted is the effect of local stresses, provided they leave no total resultant, and how, when they do leave a total resultant, the effect of this resultant is practically independent of the manner in which it is applied. This is the celebrated "principle of the equivalence of statically equipollent loads," which was first enunciated by DE SAINT-VENANT on general physical principles, and has been considerably confirmed by BOUSSINESQ's researches on the effect of small local surface actions.

It is to be borne in mind, of course, that the solution obtained in § 5, although making $\widehat{zz} = 0$ over the flat ends, does not at the same time ensure $\widehat{rz} = 0$. In other words, we have a determinate system of radial shears over the flat ends, but from symmetry this system must be self-equilibrating. The disturbances due to it will therefore, by the above principle, be purely local, and, provided we remove the ends sufficiently far from the parts of the beam which we desire to study, no trouble need arise on account of all the conditions not being strictly satisfied.

§ 7. *Numerical Problem. Expressions for Strains and Stresses.*

Let us now return to the exact expressions and apply them to the case of a comparatively short cylinder.

Suppose that $\widehat{rr} = 0$ all over the curved surface and that in some way, as described in § 1, a shear $\widehat{rz}$, which we shall take uniform and equal to S, is made to act along two rings upon the curved surface, so that

$$(\widehat{rz})_{r=a} = 0 \text{ when } -b+e < z < b-e$$

$$z < -b-e, \quad z > b+e$$

$$(\widehat{rz})_{r=a} = \text{S when } b-e < z < b+e$$

$$(\widehat{rz})_{r=a} = -\text{S when } -b-e < z < -b+e.$$

We have then

$$u_0 = w_0 = 0,$$

$$a_n = 0,$$

$$\mu b_n = \frac{8\text{S}}{(2n+1)\pi} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c}.$$

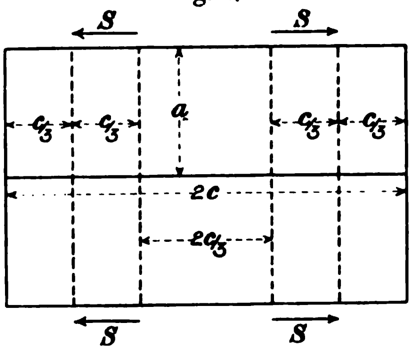
The expressions for the constants, stresses, and displacements then reduce to

$$\left.\begin{aligned} \text{A}_1 &= -\frac{4\text{S}}{(2n+1)\pi\mu} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} \frac{\gamma\alpha^2\text{I}_1 + (1-\gamma)\alpha\text{I}_0}{\gamma\alpha^2\text{I}_0^2 - (1+\gamma\alpha^2)\text{I}_1^2} \\ \text{A}_2 &= \frac{4\text{S}}{(2n+1)\pi\mu} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} \frac{(1+\gamma)\alpha\text{I}_0 - (2+\gamma\alpha^2)\text{I}_1}{\gamma\alpha^2\text{I}_0^2 - (1+\gamma\alpha^2)\text{I}_1^2} \\ \frac{\text{C}}{k} &= \frac{4\text{S}}{(2n+1)\pi\mu} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} \frac{\gamma\alpha\text{I}_0 - \gamma\text{I}_1}{\gamma\alpha^2\text{I}_0^2 - (1+\gamma\alpha^2)\text{I}_1^2} \end{aligned}\right\} \quad . \quad . \quad (47).$$

$$\left.\begin{aligned}
\frac{\widehat{rr}}{\mu} &= \Sigma\left[-\left\{(1+\gamma)\,\mathrm{A}_1+(1-\gamma)\,\mathrm{A}_2\right\}\mathrm{I}_0(\rho)+2\left(\frac{\mathrm{A}_1}{\rho}-\frac{\mathrm{C}}{k}\rho\right)\mathrm{I}_1(\rho)\right]\cos kz \\
\frac{\widehat{\phi\phi}}{\mu} &= \Sigma\left[-\frac{2\mathrm{A}_1}{\rho}\,\mathrm{I}_1(\rho)-\frac{1-\gamma}{\gamma}\,\frac{2\mathrm{C}}{k}\,\mathrm{I}_0(\rho)\right]\cos kz \\
\frac{\widehat{zz}}{\mu} &= \Sigma\left[\left\{(2\gamma+1)\,\mathrm{A}_2-(2\gamma-1)\,\mathrm{A}_1\right\}\mathrm{I}_0(\rho)+\frac{2\mathrm{C}}{k}\,\rho\mathrm{I}_1(\rho)\right]\cos kz \\
\frac{\widehat{rz}}{\mu} &= \Sigma\left[(\mathrm{A}_1+\mathrm{A}_2)\,\mathrm{I}_1(\rho)+\frac{2\mathrm{C}}{k}\,\rho\mathrm{I}_0(\rho)\right]\sin kz \\
u &= -\Sigma\left(\mathrm{A}_1\mathrm{I}_1(\rho)+\frac{\mathrm{C}}{k}\,\rho\mathrm{I}_0(\rho)\right)\frac{\cos kz}{k} \\
w &= \Sigma\left(\mathrm{A}_2\mathrm{I}_0(\rho)+\frac{\mathrm{C}}{k}\,\rho\mathrm{I}_1(\rho)\right)\frac{\sin kz}{k}
\end{aligned}\right\} \quad (48).$$

In the above α is the argument of the I-functions, unless the argument is written. To simplify the expressions we shall take $\pi a = 2c$, so that the length is about three times the radius. This makes $\alpha = 2n + 1$ ($n = 0, 1, 2, \ldots$). Further, suppose $e = c/6$, $b = c/2$, so that the cylinder is divided into 5 zones, as shown in fig. 2.

Fig. 2.



The middle one from $-c/3$ to $+c/3$, unstressed; two rings from $c/3$ to $+2c/3$ and $-c/3$ to $-2c/3$ over which a uniform shear is acting; finally, the outer rings $2c/3$ to c and $-2c/3$ to $-c$, which are unstressed. Also, in order to simplify still more, we shall suppose POISSON'S ratio to have the value $1/4$, or $\gamma = 2/3$.

It may be objected, it is true, that in many actual materials POISSON'S ratio is not $1/4$. But this is not really an objection, because the object of this investigation is not so much to find out the *absolute values* of strains and stresses in any given material, as to calculate the *alterations* in these values as deduced from the hypothesis of uniform stress, and this we can best do by taking a value for POISSON'S ratio which is, on the whole, well within the limits indicated by practical results, and which makes the arithmetic somewhat easier.

If we do this and calculate the values of the constants, we find that for the first 10 terms they come to the following values :

TABLE of Constants.

$\frac{\mu\pi A_1}{4S}$.	$\frac{\mu\pi A_2}{4S}$.	$\frac{\mu\pi C}{4Sk}$.
− ·272602	+ ·205790	+ ·159464
− ·142172	− ·0359051	+ ·0354223
+ ·0323529	+ ·0163035	− ·00534980
+ ·00492450	+ ·00308747	− ·000612345
− ·000534505	− ·000374718	+ ·0000532623
− ·0000285886	− ·0000214593	+ ·00000237642
− ·00000413531	− ·00000325082	+ ·000000294829
− ·00000162159	− ·00000131798	+ ·000000101204
+ ·000000315996	+ ·000000263390	− ·0000000175353
+ ·0000000448505	+ ·0000000381292	− ·00000000224045

From these I have calculated the coefficients of the **Fourier's** series for the stresses and strains for $r = 0$, $r = {\cdot}2a$, $r = {\cdot}4a$, and $r = {\cdot}6a$. For higher values of r the convergence becomes slower and the expressions more difficult to handle. In the case of the stresses and strains at the boundary $r = a$, special methods of approximation have to be resorted to.

The expressions for the strains and stresses are :

$$u = \frac{8Sc}{\mu\pi^2}\left[\begin{array}{l} -{\cdot}00482\cos\frac{\pi z}{2c} + {\cdot}00380\cos\frac{3\pi z}{2c} - {\cdot}00230\cos\frac{5\pi z}{2c} \\ -{\cdot}00043\cos\frac{7\pi z}{2c} + {\cdot}00006\cos\frac{9\pi z}{2c} + \ldots \end{array}\right] (r = {\cdot}2a),$$

$$u = \frac{8Sc}{\mu\pi^2}\left[\begin{array}{l} -{\cdot}01075\cos\frac{\pi z}{2c} + {\cdot}01412\cos\frac{3\pi z}{2c} - {\cdot}00541\cos\frac{5\pi z}{2c} \\ -{\cdot}00130\cos\frac{7\pi z}{2c} + {\cdot}00023\cos\frac{9\pi z}{2c} + {\cdot}00002\cos\frac{11\pi z}{2c} + \ldots \end{array}\right] (r = {\cdot}4a),$$

$$u = \frac{8Sc}{\mu\pi^2}\left[\begin{array}{l} -{\cdot}01897\cos\frac{\pi z}{2c} + {\cdot}02014\cos\frac{3\pi z}{2c} - {\cdot}00991\cos\frac{5\pi z}{2c} \\ -{\cdot}00330\cos\frac{7\pi z}{2c} + {\cdot}00084\cos\frac{9\pi z}{2c} + {\cdot}00011\cos\frac{11\pi z}{2c} \\ +{\cdot}00004\cos\frac{13\pi z}{2c} + {\cdot}00005\cos\frac{15\pi z}{2c} - {\cdot}00002\cos\frac{17\pi z}{2c} \\ -{\cdot}00001\cos\frac{19\pi z}{2c} + \ldots \end{array}\right] (r = {\cdot}6a),$$

and in like manner for w and the stresses.

To save space, the coefficients of the series may be exhibited in tabular form as follows :—

	$\widehat{zz} \times \frac{\pi}{4S}$.				$rr \times \frac{\pi}{4S}$.			
Coefficient of	$r = 0$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.	$r = 0$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.
$\cos \pi z/2c$ . .	+ ·57104	+ ·58318	+ ·62014	+ ·68364	+ ·11314	+ ·10923	+ ·09721	+ ·07616
$\cos 3\pi z/2c$.	− ·03639	− ·02640	+ ·01004	+ ·09557	+ ·10675	+ ·10983	+ ·11683	+ ·11921
$\cos 5\pi z/2c$.	+ ·02726	+ ·02846	+ ·02810	+ ·00614	− ·02700	− ·03253	− ·04981	− ·07754
$\cos 7\pi z/2c$.	+ ·00556	+ ·00732	+ ·01181	+ ·01457	− ·00431	− ·00679	− ·01547	− ·03650
$\cos 9\pi z/2c$.	− ·00070	− ·00113	− ·00298	− ·00692	+ ·00048	+ ·00099	+ ·00354	+ ·01242
$\cos 11\pi z/2c$.	− ·00004	− ·00009	− ·00036	− ·00136	+ ·00003	+ ·00007	+ ·00040	+ ·00209
$\cos 13\pi z/2c$.	− ·00001	− ·00002	− ·00011	− ·00068	—	+ ·00002	+ ·00013	+ ·00096
$\cos 15\pi z/2c$.	—	− ·00001	− ·00010	− ·00089	—	+ ·00001	+ ·00011	+ ·00119
$\cos 17\pi z/2c$.	—	—	+ ·00004	+ ·00058	—	—	− ·00004	− ·00074
$\cos 19\pi z/2c$.	—	—	+ ·00001	+ ·00027	—	—	− ·00001	− ·00033

	$\widehat{\phi\phi} \times \pi/4S$			
Coefficient of	$r = 0$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.
$\cos \pi z/2c$	+ ·11314	+ ·11291	+ ·11218	+ ·11091
$\cos 3\pi z/2c$	+ ·10675	+ ·10998	+ ·11998	+ ·13760
$\cos 5\pi z/2c$	− ·02700	− ·02980	− ·03927	− ·05916
$\cos 7\pi z/2c$	− ·00431	− ·00528	− ·00907	− ·01922
$\cos 9\pi z/2c$	+ ·00048	+ ·00068	+ ·00159	+ ·00489
$\cos 11\pi z/2c$. . . .	+ ·00003	+ ·00004	+ ·00015	+ ·00065
$\cos 13\pi z/2c$. . . .	—	+ ·00001	+ ·00004	+ ·00025
$\cos 15\pi z/2c$. . . .	—	—	+ ·00003	+ ·00026
$\cos 17\pi z/2c$. . . .	—	—	− ·00001	− ·00014
$\cos 19\pi z/2c$. . . .	—	—	—	− ·00006

	$w \times \mu\pi^2/8Sc$.				$\widehat{rz} \times \pi/4S$.		
Coefficient of	$r = 0$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.
$\sin \pi z/2c$ . . .	+ ·20579	+ ·21106	+ ·22712	+ ·25475	+ ·05771	+ ·11909	+ ·18801
$\sin 3\pi z/2c$ . .	− ·01197	− ·01085	− ·00655	+ ·00418	− ·00944	− ·00878	+ ·01915
$\sin 5\pi z/2c$ . .	+ ·00326	+ ·00352	+ ·00403	+ ·00323	+ ·01395	+ ·02861	+ ·03569
$\sin 7\pi z/2c$ . .	+ ·00044	+ ·00058	+ ·00103	+ ·00163	+ ·00444	+ ·01219	+ ·02464
$\sin 9\pi z/2c$ . .	− ·00004	− ·00007	− ·00019	− ·00050	− ·00082	− ·00310	− ·00955
$\sin 11\pi z/2c$ . .	—	—	− ·00002	− ·00007	− ·00007	− ·00037	− ·00171
$\sin 13\pi z/2c$ . .	—	—	− ·00001	− ·00003	− ·00002	− ·00012	− ·00081
$\sin 15\pi z/2c$ . .	—	—	—	− ·00003	− ·00001	− ·00010	− ·00104
$\sin 17\pi z/2c$ . .	—	—	—	+ ·00002	—	+ ·00004	+ ·00066
$\sin 19\pi z/2c$ . .	—	—	—	+ ·00001	—	+ ·00001	+ ·00030

Using the above values I have calculated the stresses and strains for the points $r = 0$, $a/5$, $2a/5$, $3a/5$, and $z = \pm (0, c/10, 2c/10, \&c.)$. These are tabulated on pp. 171–173.

§ 8. *Calculation of the Stresses on the Outer Surface of the Cylinder.*

Along the outer surface $r = a$, $\rho = \alpha$, and we have the following expressions for the stresses $\widehat{\phi\phi}$ and $\widehat{zz}$, and the displacements u and w: $\widehat{rz}$ and $\widehat{rr}$ of course are known.

Consider for example the stresses $(\widehat{zz})_{r=a}$ and $(\widehat{\phi\phi})_{r=a}$.

They are

$$(\widehat{zz})_{r=a} = \frac{4S}{\pi}\sum_0^\infty \frac{6\gamma\alpha I_0^2 - (4\gamma + 2) I_0 I_1 - 2\gamma\alpha I_1^2}{(\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2)(2n+1)} \sin\frac{\overparen{2n+1}\pi c}{2c} \sin\frac{\overparen{2n+1}\pi b}{2c} \cos\frac{\overparen{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad (49),$$

and

$$(\widehat{\phi\phi})_{r=a}$$
$$= \frac{4S}{\pi}\sum_0^\infty \frac{2\gamma\alpha I_1^2 + 4(1-\gamma) I_0 I_1 - 2(1-\gamma)\alpha I_0^2}{(\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2)(2n+1)} \sin\frac{\overparen{2n+1}\pi c}{2c} \sin\frac{\overparen{2n+1}\pi b}{2c} \cos\frac{\overparen{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad (50).$$

Now, when α is fairly large (say > 10), I_0 and I_1 may be replaced by their semi-convergent expansions:

$$I_0(\alpha) = \sqrt{\frac{1}{2\pi\alpha}}\, e^\alpha \left(1 + \frac{1^2}{8\alpha} + \frac{1^2.3^2}{2!(8\alpha)^2} + \frac{1^2.3^2.5^2}{3!(8\alpha)^3} + \ldots\right)$$

$$I_1(\alpha) = \sqrt{\frac{1}{2\pi\alpha}}\, e^\alpha \left(1 - \frac{3}{8\alpha} - \frac{3.5}{2!(8\alpha)^2} - \frac{3.5.21}{3!(8\alpha)^3} - \ldots\right)$$

(see Gray and Mathews, 'Bessel's Functions,' p. 68). From which we find that the coefficients of $\cos\frac{\overparen{2n+1}\pi z}{2c}$ in the expansions of $\frac{\pi}{4S}(\widehat{\phi\phi})_{r=a}$ and $\frac{\pi}{4S}(\widehat{zz})_{r=a}$ approximate to the values (remembering $\alpha = 2n + 1$):

$$\left\{\frac{4\gamma - 2}{\gamma}\frac{1}{2n+1} + \frac{(2 - 2\gamma)(3\gamma - 1)}{\gamma^2}\frac{1}{(2n+1)^2}\right\} \sin\frac{\overparen{2n+1}\pi c}{2c} \times \sin\frac{\overparen{2n+1}\pi b}{2c}$$

and

$$\left\{\frac{4}{2n+1} + \frac{2(1-\gamma)}{\gamma}\frac{1}{(2n+1)^2}\right\} \sin\frac{\overparen{2n+1}\pi c}{2c} \sin\frac{\overparen{2n+1}\pi b}{2c}.$$

Now let us write

$$\frac{6\gamma\alpha\mathrm{I}_0^2 - (4\gamma + 2)\,\mathrm{I}_0\mathrm{I}_1 - 2\gamma\alpha\mathrm{I}_1^2}{\gamma\alpha^2\mathrm{I}_0^2 - (1 + \gamma\alpha^2)\,\mathrm{I}_1^2}\,\frac{1}{(2n+1)}\sin\frac{\overparen{2n+1}\pi e}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c}$$

$$= \left\{\frac{4}{2n+1} + \frac{2\,(1-\gamma)}{\gamma}\,\frac{1}{(2n+1)^2}\right\}\sin\frac{\overparen{2n+1}\pi c}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c} + q_n' \quad . \quad . \quad . \quad . \quad . \quad (51),$$

$$\frac{2\gamma\alpha\mathrm{I}_1^2 + 4\,(1-\gamma)\,\mathrm{I}_0\mathrm{I}_1 - 2\,(1-\gamma)\,\alpha\mathrm{I}_0^2}{\gamma\alpha^2\mathrm{I}_0^2 - (1 + \gamma\alpha^2)\,\mathrm{I}_1^2}\,\frac{1}{(2n+1)}\sin\frac{\overparen{2n+1}\pi c}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c}$$

$$= \left\{\frac{4\gamma-2}{\gamma}\,\frac{1}{2n+1} + \frac{(2-2\gamma)(3\gamma-1)}{\gamma^2}\,\frac{1}{(2n+1)^2}\right\}\sin\frac{\overparen{2n+1}\pi c}{2c}\sin\frac{2n+1\pi b}{2c} + p_n' \ . \ (52),$$

so that p_n', q_n' are comparable with the terms of the series $\Sigma\,\dfrac{1}{(2n+1)^3}$, which converge fairly rapidly. We see therefore that $\widehat{zz}$ and $\widehat{\phi\phi}$ are made up of two kinds of terms (*a*), terms of the form $\dfrac{4\mathrm{S}}{\pi}\,\Sigma p_n' \cos\dfrac{2n+1\pi z}{2c}$ and $\dfrac{4\mathrm{S}}{\pi}\,\Sigma q_n' \cos\dfrac{\overparen{2n+1}\pi z}{2c}$, which are absolutely and uniformly convergent series, and (*b*) series, in which the coefficients are the approximate expressions found above. Of the series (*b*), those which have terms containing $1/(2n+1)^2$ or $1/(2n+1)^3$ are absolutely and uniformly convergent. This, however, is not the case with the series formed by taking the leading terms in the approximation, viz. :—

$$\frac{4\mathrm{S}}{\pi}\sum_0^\infty \frac{8}{(2n+1)}\sin\frac{(2n+1)\pi e}{2c}\sin\frac{(2n+1)\pi b}{2c}\cos\frac{(2n+1)\pi z}{2c}$$

and

$$\frac{4\mathrm{S}}{\pi}\sum_0^\infty \frac{4\gamma-2}{\gamma(2n+1)}\sin\frac{(2n+1)\pi e}{2c}\sin\frac{(2n+1)\pi b}{2c}\cos\frac{(2n+1)\pi z}{2c}.$$

For the series

$$\sum_0^\infty \frac{1}{(2n+1)}\sin\frac{\overparen{2n+1}\pi e}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c}\cos\frac{\overparen{2n+1}\pi z}{2c}$$

may be broken up into the sum of four other series, thus :

$$\tfrac{1}{4}\sum_0^\infty \frac{1}{(2n+1)}\cos\frac{\overparen{2n+1}\pi}{2c}(z-b+e) + \tfrac{1}{4}\sum_0^\infty \frac{1}{(2n+1)}\cos\frac{\overparen{2n+1}\pi}{2c}(z+b-e)$$

$$-\tfrac{1}{4}\sum_0^\infty \frac{1}{(2n+1)}\cos\frac{\overparen{2n+1}\pi}{2c}(z+b+e) - \tfrac{1}{4}\sum_0^\infty \frac{1}{(2n+1)}\cos\frac{\overparen{2n+1}\pi}{2c}(z-b-e).$$

Now it is easy to show that

$$\sum_0^\infty \frac{1}{(2n+1)}\cos(2n+1)\,x = \tfrac{1}{2}\log\left(\cot\frac{|x|}{2}\right)$$

where $|x|$ is the numerical value of x.

The series on the left is divergent and $\log\left(\cot \frac{|x|}{2}\right) = \infty$ if $x = 0$. We see, therefore, that, at the points $z = \pm b \pm e$, *i.e.*, wherever the shear $\widehat{rz}$ changes discontinuously, the stresses $\widehat{zz}$ and $\widehat{\phi\phi}$ become infinite.

The meaning of this in practice would be that, as the transition from the stressed to the unstressed surface becomes more abrupt, the tractions in the neighbourhood become dangerously large. And if the shear is applied by means of a projecting rim or collar of material, on which the pull is brought to bear, as in fig. 1, then this rim or collar must not project out of the material at a sharp angle, or in any way which tends to introduce a discontinuous tangential stress over the surface of the cylinder. This is already recognised in practice; test pieces, which are thicker at the ends than in the middle, being made in such a way that the transition from the smaller to the larger diameter is gradual.

The series containing $1/(2n + 1)^2$ can also be evaluated in finite terms :

$$\sum_0^\infty \frac{1}{(2n+1)^2} \sin\frac{\overparen{2n+1}\pi e}{2c} \sin\frac{\overparen{2n+1}\pi b}{2c} \cos\frac{\overparen{2n+1}\pi z}{2c}$$

$$= \frac{\pi}{2c}\int^c \sum_0^\infty \frac{1}{(2n+1)} \sin\frac{\overparen{2n+1}\pi e}{2c} \sin\frac{\overparen{2n+1}\pi b}{2c} \sin\frac{\overparen{2n+1}\pi z}{2c}$$

$$= \frac{\pi^2}{16cS}\int_z^c (\widehat{rz})_{r=a}\, dz$$

$$= 0 \text{ from } z = -c \text{ to } -b-e$$

$$\frac{\pi^2}{16c}(z + b + e) \text{ from } z = -b-e \text{ to } z = -b+e$$

$$\frac{\pi^2 e}{8c} \qquad \text{from } z = -b+e \text{ to } z = b-e$$

$$\frac{\pi^2}{16c}(b + e - z) \text{ from } z = b-e \text{ to } z = b+e$$

$$0 \qquad \text{from } z = b+e \text{ to } z = c.$$

Thus we have only to calculate p_n', q_n' and to sum the corresponding series, the rest of the expressions for the stresses being reducible to finite terms.

For $\gamma = 2/3$, I find the values of p_n', q_n' to be given by :

n	p_n'	q_n'
0	$-\cdot 35130$	$-\cdot 01184$
1	$-\cdot 04818$	$-\cdot 05960$
2	$+\cdot 00838$	$+\cdot 01431$
3	$+\cdot 00217$	$+\cdot 00436$
4	$-\cdot 00061$	$-\cdot 00135$
5	$-\cdot 00011$	$-\cdot 00025$
6	$-\cdot 00006$	$-\cdot 00015$
7	$-\cdot 00010$	$-\cdot 00026$
8	$+\cdot 00008$	$+\cdot 00024$
9	$+\cdot 00006$	$+\cdot 00018$

from which the values of $\widehat{zz}$, $\widehat{\phi\phi}$ can be found when $r = a$. They are tabulated, with the other stresses, upon p. 171.

§ 9. *Calculation of the Displacements on the Outer Surface of the Cylinder.*

In a precisely similar manner we find for the displacements

$$(u)_{r=a} = -\frac{8Sc}{\mu\pi^2}\sum_0^\infty \left\{1 - \frac{\alpha I_0/I_1 - 1}{\gamma\alpha^2 I_0^2/I_1^2 - 1 - \gamma\alpha^2}\right\}\frac{\sin\frac{\overparen{2n+1}\pi e}{2c}}{(2n+1)^2}\sin\frac{\overparen{2n+1}\pi b}{2c}\cos\frac{\overparen{2n+1}\pi z}{2c}$$

$$= -\frac{8Sc}{\mu\pi^2}\sum_0^\infty \frac{1 - \frac{1}{\gamma}}{(2n+1)^2}\sin\frac{\overparen{2n+1}\pi e}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c}\cos\frac{\overparen{2n+1}\pi z}{2c}$$

$$- \frac{8Sc}{\mu\pi^2}\Sigma u_n'\cos\frac{\overparen{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where

$$u_n' = \left(\frac{1}{\gamma} - \frac{\alpha I_0/I_1 - 1}{\gamma\alpha^2 I_0^2/I_1^2 - 1 - \gamma\alpha^2}\right)\frac{1}{(2n+1)^2}\sin\frac{\overparen{2n+1}\pi e}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c}$$

and is of the order $1/(2n+1)^3$

$$(w)_{r=a} = \frac{8Sc}{\mu\pi^2}\sum_0^\infty \frac{(1+\gamma)\alpha I_0^2 - \gamma\alpha I_1^2 - 2I_1 I_0}{(\gamma\alpha^2 I_0^2 - (1+\gamma\alpha^2)I_1^2)(2n+1)^2}\sin\frac{\overparen{2n+1}\pi e}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c}\sin\frac{\overparen{2n+1}\pi z}{2c}$$

$$= \frac{8Sc}{\mu\pi^2}\sum_0^\infty \frac{1}{\gamma}\frac{1}{(2n+1)^2}\sin\frac{(2n+1)\pi e}{2c}\sin\frac{\overparen{2n+1}\pi b}{2c}\sin\frac{\overparen{2n+1}\pi z}{2c}$$

$$+ \frac{8Sc}{\mu\pi^2}\sum_0^\infty \left(\frac{1-\gamma}{\gamma}\right)^2\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi e}{2c}\sin\frac{(2n+1)\pi b}{2c}\sin\frac{(2n-1)\pi z}{2c}$$

$$+ \frac{8Sc}{\mu\pi^2}\sum_0^\infty w_n'\sin\frac{\overparen{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

where

$$w_n' = \left\{\frac{(1+\gamma)\,\alpha I_0^2 - \gamma\alpha I_1^2 - 2I_1I_0}{\gamma\alpha^2 I_0^2 - (1+\gamma\alpha^2)\,I_1^2} - \frac{1}{\gamma} - \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{1}{\alpha}\right\} \frac{1}{(2n+1)^2} \sin\frac{\overline{2n+1}\pi e}{2c} \sin\frac{\overline{2n+1}\pi b}{2c},$$

and is of order $1/(2n+1)^4$.

It so happens that in w the term of order $1/(2n+1)^3$ is evaluable in finite terms, and I have included it.

It is easy to see that

$$\begin{aligned}
&\sum_0^\infty \frac{1}{(2n+1)^3} \sin\frac{(2n+1)\pi e}{2c} \sin\frac{(2n+1)\pi b}{2c} \sin\frac{\overline{2n+1}\pi z}{2c} \\
&= -\frac{\pi^3 eb}{16c^3} \text{ from } z = -c \text{ to } z = -b-e \\
&= -\left\{2eb - \tfrac{1}{2}(b+e+z)^2\right\} \frac{\pi^3}{32c^2} \text{ from } z = -b-e \text{ to } z = -b+e \\
&= \frac{\pi^3 ez}{16c^2} \text{ from } z = -b+e \text{ to } z = b-e \\
&= \left\{2eb - \tfrac{1}{2}(b+e-z)^2\right\} \frac{\pi^3}{32c^2} \text{ from } z = b-e \text{ to } z = b+e \\
&= \frac{\pi^3 eb}{16c^2} \text{ from } z = b+e \text{ to } z = +c\,.
\end{aligned}$$

The leading series in w cannot, however, be evaluated so easily. It is seen to depend upon the evaluation of the series

$$\begin{aligned}
\sum_0 \frac{\sin \overline{2n+1}x}{(2n+1)^2} &= \int_0^x \tfrac{1}{2}\log\left(\cot\frac{x}{2}\right) dx \\
&= \frac{x}{2}\log\left(\cot\frac{x}{2}\right) + \int_0^x \frac{x}{2}\operatorname{cosec} x\, dx.
\end{aligned}$$

As series of this kind are frequently turning up in investigations like the present, I have tabulated below the values of $\frac{1}{2}\int_0^x \frac{x}{\sin x}\,dx$ and also of $\sum_0^\infty \frac{\sin \overline{2n+1}x}{(2n+1)^2}$ for values of x ranging from 0 to $\pi/2$ at intervals of $\pi/40$. Intermediate values are then obtained by interpolation when required.

TABLE of $\frac{1}{2}\int_0^x \frac{x}{\sin x}\,dx = f(x)$.

x.	$f(x)$.	x.	$f(x)$.	x.	$f(x)$.	x.	$f(x)$.
$\pi/40$	·039283	$6\pi/40$	·238572	$11\pi/40$	·450873	$16\pi/40$	·690354
$2\pi/40$	·078648	$7\pi/40$	·279605	$12\pi/40$	·496043	$17\pi/40$	·743248
$3\pi/40$	·118174	$8\pi/40$	·321246	$13\pi/40$	·542417	$18\pi/40$	·798291
$4\pi/40$	·157947	$9\pi/40$	·363596	$14\pi/40$	·590147	$19\pi/40$	·855760
$5\pi/40$	·198050	$10\pi/40$	·406766	$15\pi/40$	·639400	$20\pi/40$	·915963

TABLE of $\sum_0^\infty \frac{\sin \overline{2n+1}x}{(2n+1)^2}$.

x.	Σ.	x.	Σ.	x.	Σ.	x.	Σ.
$\pi/40$	·16639	$6\pi/40$	·57475	$11\pi/40$	·78536	$16\pi/40$	·89109
$2\pi/40$	·27830	$7\pi/40$	·62754	$12\pi/40$	·81379	$17\pi/40$	·90202
$3\pi/40$	·36959	$8\pi/40$	·67442	$13\pi/40$	·83839	$18\pi/40$	·90978
$4\pi/40$	·44740	$9\pi/40$	·71602	$14\pi/40$	·85938	$19\pi/40$	·91442
$5\pi/40$	·51513	$10\pi/40$	·75288	$15\pi/40$	·87690	$20\pi/40$	·91596

We have thus the means of evaluating all those parts of the expressions which give rise to the most slowly convergent of the series employed.

Taking $\gamma = 2/3$, the values found for u_n', w_n' are tabulated below :—

n.	u_n'	w_n'.
0	− ·13933	+ ·03040
1	− ·01331	− ·00634
2	+ ·00192	+ ·00098
3	+ ·00043	+ ·00022
4	− ·00011	− ·00005
5	− ·00002	− ·00001
6	− ·00001	− ·00000
7	− ·00001	− ·00001
8	+ ·00001	+ ·00001
9	+ ·00001	+ ·00000

Using these and the expressions given above for the finite terms, we can find the values of the displacements on the outer surface of the cylinder.

§ 10. *Numerical Values of the Stresses and Displacements.*

The numerical values obtained in this way are tabulated below ; I have given the stresses in the form of ratio (stress) /Q, where Q is the uniform tension which would produce a pull equal to that due to the shear S.

TABLE of Stresses.

$\widehat{rr}/\mathrm{Q}$.

z.	$r = 0$	$r = (\cdot 2)\,a$	$r = (\cdot 4)\,a$	$r = (\cdot 6)\,a$	$r = a$
0	·22990	·21985	·18587	·11786	·00000
$c/10$ . . .	·22600	·21860	·19244	·13540	·00000
$2c/10$ . . .	·20818	·20842	·20227	·18098	·00000
$3c/10$ . . .	·17064	·17487	·18746	·21264	·00000
$4c/10$ . .	·10688	·10931	·12179	·15054	·00000
$5c/10$ . . .	·02697	·02239	·01531	·00700	·00000
$6c/10$ . . .	− ·04561	− ·05754	− ·08614	− ·13427	·00000
$7c/10$ . . .	− ·09035	− ·10187	− ·13595	− ·18782	·00000
$8c/10$ . . .	− ·08893	− ·09947	− ·12221	− ·14045	·00000
$9c/10$ . . .	− ·05430	− ·05936	− ·06827	− ·06636	·00000
c	·00000	·00000	·00000	·00000	·00000

$\widehat{\phi\phi}/\mathrm{Q}$.

z	$r = 0$	$r = (\cdot 2)\,a$	$r = (\cdot 4)\,a$	$r = (\cdot 6)\,a$	$r = a$
0	·22990	·22924	·22568	·21396	·12363
$c/10$ . . .	·22600	·22631	·22618	·22146	·15364
$2c/10$ . . .	·20818	·21209	·21990	·23357	·22157
$3c/10$ . . .	·17064	·17486	·18827	·21854	·42904
$4c/10$ . . .	·10688	·10865	·11808	·14158	·25536
$5c/10$ . . .	·02697	·02412	·02049	·01498	·00162
$6c/10$ . . .	− ·04561	− ·05310	− ·07023	− ·10563	− ·24868
$7c/10$ . . .	− ·09035	− ·09706	− ·11924	− ·16352	− ·41155
$8c/10$ . . .	− ·08893	− ·09653	− ·11404	− ·14443	− ·18289
$9c/10$ . . .	− ·05430	− ·05836	− ·06709	− ·07983	− ·07880
c	·00000	·00000	·00000	·00000	·00000

$\widehat{zz}/\mathrm{Q}$.

z.	$r = 0$.	$r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.	$r = a$.
0	·68906	·71895	·81048	·96162	1·11724
$c/10$ . . .	·67272	·70006	·78586	·93696	1·16333
$2c/10$ . . .	·63120	·65168	·71983	·85919	1·34405
$3c/10$ . . .	·58195	·59404	·63659	·73710	2·02246
$4c/10$ . . .	·53943	·54503	·56451	·61686	1·36800
$5c/10$ . . .	·50302	·50502	·50829	·50813	·47865
$6c/10$ . . .	·45713	·45662	·44745	·39955	− ·40866
$7c/10$ . . .	·38411	·38062	·35965	·27810	− 1·05552
$8c/10$ . . .	·27795	·27204	·24438	·15199	− ·35747
$9c/10$ . . .	·14532	·14077	·12060	·06040	− ·13448
c	·00000	·00000	·00000	·00000	·00000

$\widehat{rz}/Q$.

z.	$r = 0$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.	$r = a$.
0	·00000	·00000	·00000	·00000	·00000
$c/10$ . . .	·00000	·02148	·05127	·08883	·00000
$2c/10$ . . .	·00000	·03347	·08205	·15547	·00000
$3c/10$ . . .	·00000	·03262	·08123	·16519	·00000
$4c/10$ . . .	·00000	·02571	·06262	·13159	·95493
$5c/10$ . . .	·00000	·02495	·05701	·11897	·95493
$6c/10$ . . .	·00000	·03718	·08085	·14879	·95493
$7c/10$ . . .	·00000	·05812	·12259	·20528	·00000
$8c/10$ . . .	·00000	·07753	·15596	·23109	·00000
$9c/10$ . . .	·00000	·08891	·16969	·21918	·00000
c	·00000	·09231	·17214	·20992	·00000

In the above tables it is to be remembered that $\widehat{rr}$, $\widehat{zz}$, $\widehat{\phi\phi}$ are all even functions of z; $\widehat{rz}$ is an odd function of z. At the points $r = a$, $z = \pm \frac{c}{3}$, $\widehat{zz}$ and $\widehat{\phi\phi}$ are both $= + \infty$, while at the points $r = a$, $z = \pm \frac{2c}{3}$, $\widehat{zz}$ and $\widehat{\phi\phi}$ are both $= - \infty$.

The displacements u and w have been compared with the corresponding total elongation and lateral contraction w_0 and u_0 of the same cylinder under a uniform tension Q over its plane ends.

TABLE of Displacements.

w/w_0.

z.	$r = 0$.	$r = (\cdot 4)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.	$r = a$
0	·00000	·00000	·00000	·00000	·00000
$c/10$ . . .	·05693	·06005	·06988	·08685	·10972
$2c/10$ . . .	·11132	·11693	·13489	·16752	·22900
$3c/10$ . . .	·16235	·16949	·19259	·23676	·38253
$4c/10$ . . .	·21132	·21915	·24461	·29489	·59238
$5c/10$ . . .	·26013	·26829	·29467	·34706	·67152
$6c/10$ . . .	·30896	·31741	·34425	·39538	·68809
$7c/10$ . . .	·35493	·36360	·39027	·43724	·57421
$8c/10$ . . .	·39299	·40162	·42700	·46684	·51756
$9c/10$ . . .	·41809	·42646	·45002	·48254	·49745
c	·42684	·43506	·45774	·48725	·49196

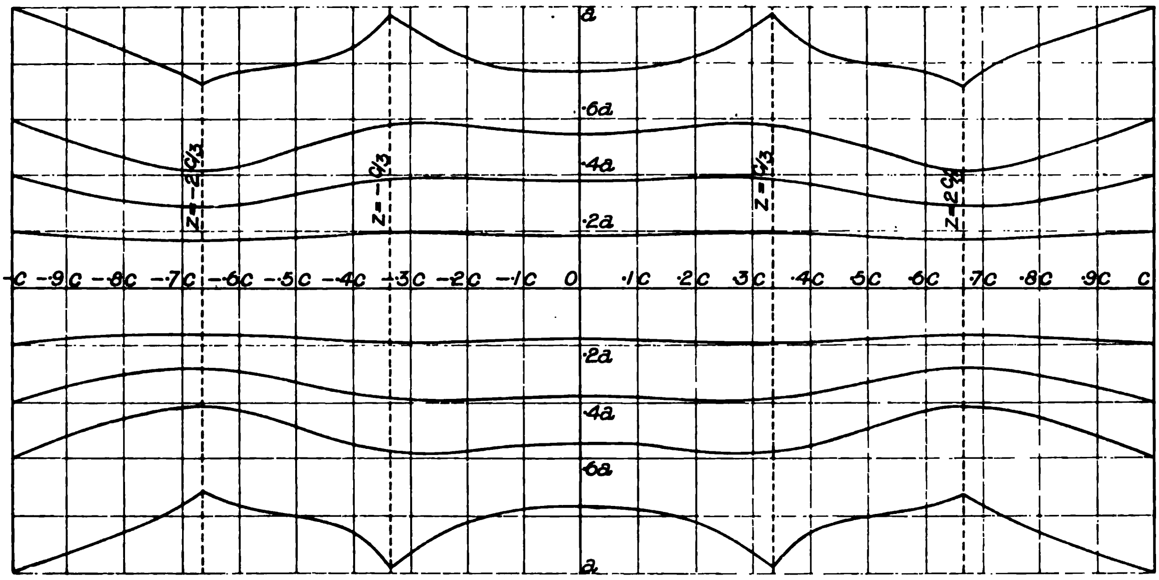
u/u_0.

z.	$r = 0$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.	$r = a$.
0	·0000	·0449	·0375	·1341	·5789
$c/10$ . . .	·0000	·0388	·0294	·1120	·5488
$2c/10$ . . .	·0000	·0262	·0170	·0635	·4578
$3c/10$ . . .	·0000	·0202	·0284	·0454	·1847
$4c/10$ . . .	·0000	·0314	·0856	·1206	·3709
$5c/10$ . . .	·0000	·0575	·1767	·2731	·4963
$6c/10$ . . .	·0000	·0838	·2569	·4127	·5861
$7c/10$ . . .	·0000	·0934	·2803	·4466	·5907
$8c/10$ . . .	·0000	·0790	·2313	·3535	·3741
$9c/10$ . . .	·0000	·0446	·1283	·1880	·1807
c	·0000	·0000	·0000	·0000	·0000

§ 11. *Discussion of the Results.*

The numerical results tabulated above are illustrated by the curves contained in Diagrams 1–6. Diagram 1 shows the radial shift, of course enormously exaggerated, u_0 on the diagram being taken as numerically equal to 2/5ths of the radius of the cylinder. For convenience in plotting, the horizontal and vertical scales are not the same, thus $a/5$ and $c/10$ are represented by the same length on the diagram, although their actual ratio is $4/\pi$. The same arrangement has been adhered to in Diagram 2.

Diagram 1.—Distortion of a Cylinder extended by Shearing Stress applied to the Curved Surface (Radial Shifts).

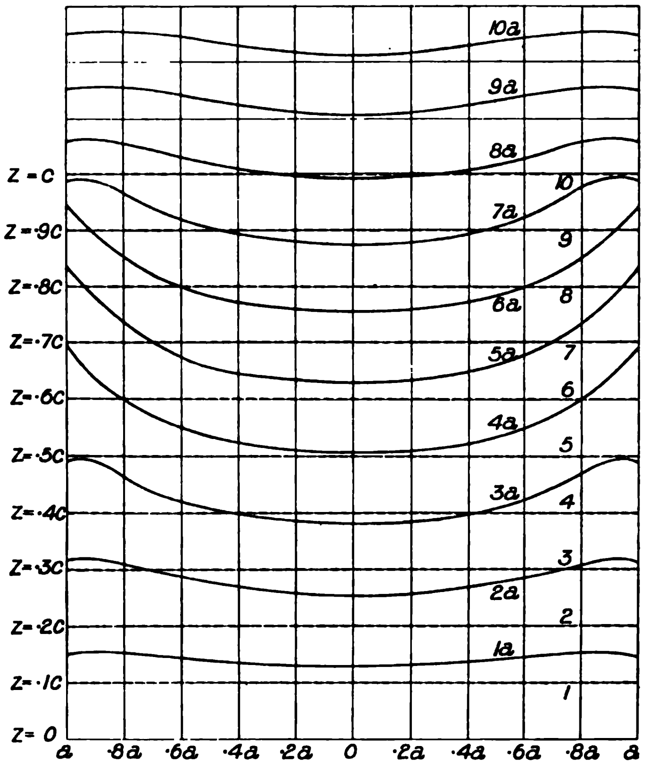


($u_0 = 2a/5$ on the scale of the diagram.)

From Diagram 1 we see at once that a discontinuous change in the slope of the deformed outer surface of the cylinder occurs at the points $z = \pm c/3, \pm 2c/3$, between which the uniform shearing stress is applied. Referring to equation (53) we see that at those points du/dz changes abruptly by the value $-\frac{1}{4}S/\mu$, where S is the abrupt increase in the shear. This result is exhibited in the curves referred to, and we notice that the effect of shear, applied to the outer surface of a cylinder, is to depress that part of the surface towards which the shear is acting. In fact the greatest contraction throughout the cylinder occurs near the points $z = \pm 2c/3$ and appears due to this effect.

Diagram 2.—Distortion of the Cross-sections of a Cylinder under Shearing Stress applied to the Curved Surface.

($w_0 = c/2$ on the scale of the diagram.)

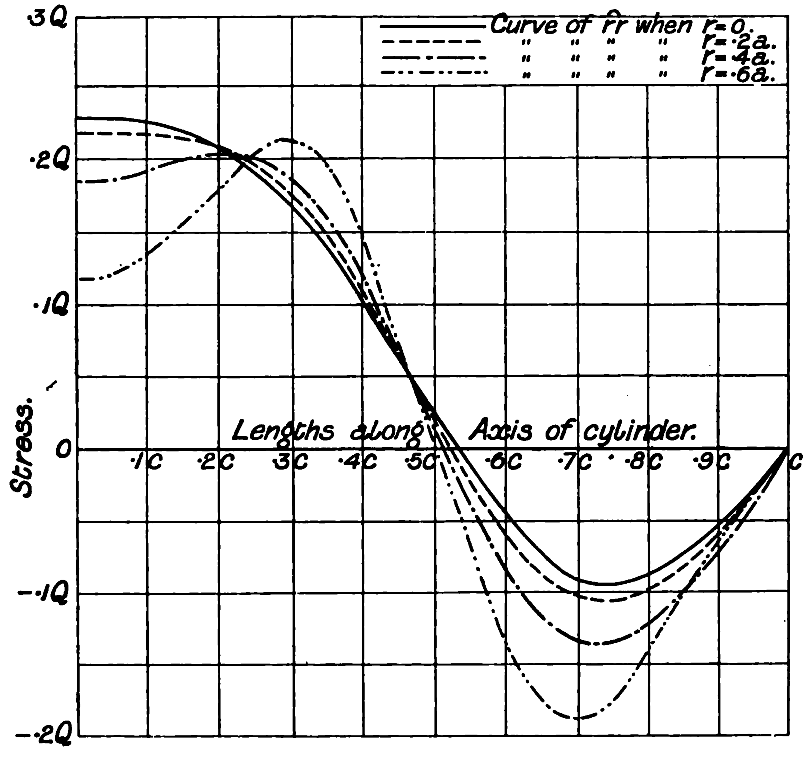


1, 2, 3, 4, 5, 6, 7, 8, 9, 10 undistorted cross-sections.
1a, 2a, 3a, 4a, 5a, 6a, 7a, 8a, 9a, 10a distorted cross-sections.

Near the ends the cylinder broadens out again, as we should expect, though it is to be noted that the distorted generators meet the plane ends obliquely, which should not be the case if the condition of no stress over the plane ends were accurately fulfilled. This we know is not so: there is a system of finite shear over the plane ends, as is easily seen on referring to the table of $\widehat{rz}$ on p. 172. This system of shears is, however, self-equilibrating. The shear is zero at the centre and at the circumference, and its greatest value does not exceed about 1/4 of the laterally applied shear. Its effects, at some distance inside the cylinder, will therefore be small compared with the effects of the large and unbalanced lateral distribution of shear.

We notice that, for so short a bar, the lateral contraction is very much less than the contraction we should expect according to the "uniform tension" theory. In fact it never amounts to 60 per cent. of that contraction. For points deeper in the material, the contraction is much smaller than this. Thus, for $r = (\cdot 2)\, a$, the lateral contraction is 22 per cent. and for $r = (\cdot 4)\, a$ it is 9 per cent. of what it should be on the "uniform tension" hypothesis. This seems due to the fact, in itself extremely remarkable, that there are considerable radial and cross-radial tensions inside the material. Indeed, referring to Diagram 3, we see that the radial tension amounts to about 1/5th of the mean tension Q which would give the same total pull, and which has been consistently taken as the unit of comparison. These tensions are changed to pressures after passing the ring of shear, which is in accordance with the general compressive effect mentioned above.

Diagram 3.—Showing Stress $\widehat{rr}$ for the Cylinder under a Shearing Pull.



It may be noticed that the shape of the successive curves on Diagram 3 suggests that, as we approach the outer skin, the two bumps on either side of $z = \cdot 5c$ would lead to infinities, or at all events, to discontinuities in the stress. In other words, that, though we have chosen our constants so as to make $(\widehat{rr})_{r=a}$ formally zero, yet the limit of $(\widehat{rr})$ as given by the series is not zero when r approaches a. This would suggest that the series for $\widehat{rr}$, considered as a series of I-functions, behaves at $r = a$ in much the same way as a discontinuous Fourier series whose general term is $\sin nz$ behaves at $z = \pi$. In fact, if we differentiate $\widehat{rr}$ in the usual way with regard to r and then put $r = a$, we get a divergent series.

It is easily seen, however, that no discontinuity really occurs except at the points where the shear is applied discontinuously. The general term in $\widehat{rr}$ is of the form (dropping irrelevant factors):

$$\frac{\cos \widehat{2n+1}u}{(2n+1)}\left\{\frac{\gamma\alpha[\alpha \mathrm{I}_1(\alpha)\mathrm{I}_0(\rho)-\rho \mathrm{I}_0(\alpha)\mathrm{I}_1(\rho)]+(1-\gamma)[\mathrm{I}_1(\alpha)\mathrm{I}_0(\rho)-\frac{\alpha}{\rho}\mathrm{I}_0(\alpha)\mathrm{I}_1(\rho)] - \frac{\mathrm{I}_1(\rho)}{\rho}[\gamma\alpha^2\mathrm{I}_1(\alpha)-\gamma\rho^2\mathrm{I}_1(\rho)]}{\gamma\alpha^2\mathrm{I}_0^2(\alpha)-(1+\gamma\alpha^2)\mathrm{I}_1^2(\alpha)}\right\},$$

where $u = \frac{\pi}{2c}(z \pm b \pm e)$.

Now, looking at the semi-convergent expansions for I_0 and I_1 we find, putting $a - d = r$ and $\rho = \alpha - \delta$ where $\delta = \frac{\widehat{2n+1}\pi d}{2c}$ and d is small,

$$\frac{\mathrm{I}_0(\alpha-\delta)}{\mathrm{I}_0(\alpha)} = \epsilon^{-\delta}\left(1+\tfrac{1}{2}\frac{\delta}{\alpha}\right)\left[1+\tfrac{1}{8}\frac{\delta}{\alpha^2} + \text{terms of order } \delta/\alpha^3 \text{ and higher terms in } \delta/\alpha\right],$$

$$\text{and } \frac{\mathrm{I}_1(\alpha-\delta)}{\mathrm{I}_1(\alpha)} = \epsilon^{-\delta}\left(1+\tfrac{1}{2}\frac{\delta}{\alpha}\right)\left[1-\tfrac{3}{8}\frac{\delta}{\alpha^2} + \text{higher terms}\right],$$

where ϵ = base of Napierian system of logarithms.

The coefficient of $\cos(2n + 1)u$ then reduces to the form

$$\frac{1}{(2n+1)}\left[\begin{array}{l}\gamma\alpha\left\{\begin{array}{l} + \mathrm{I}_0(\alpha)\mathrm{I}_1(\alpha)\alpha \times \frac{\delta}{2\alpha^2} \times \epsilon^{-\delta}\left(1 + \text{terms in } \frac{\delta}{\alpha} + \text{terms in } \frac{\delta}{\alpha^2} + \text{\&c.} + \text{terms in } \frac{1}{\alpha} + \ldots\right) \\ + \mathrm{I}_0(\alpha)\mathrm{I}_1(\alpha)\delta\epsilon^{-\delta}\left(1+\tfrac{1}{2}\frac{\delta}{\alpha} + \text{terms in } \frac{\delta}{\alpha^2} \text{ and } \frac{\delta}{\alpha^3} + \text{higher terms}\right)\end{array}\right\} \\ + (1-\gamma)\mathrm{I}_1(\alpha)\mathrm{I}_0(\alpha)\left\{\epsilon^{-\delta}\delta\left(\text{terms of order } \frac{1}{\alpha^2}\right) - \frac{\delta}{\alpha}\epsilon^{-\delta}(\text{a finite term})\right\} \\ - \frac{\mathrm{I}_1(\rho)}{\rho}\gamma\alpha^2\mathrm{I}_1(\alpha)\left\{\delta\epsilon^{-\delta}\left(\text{terms of order } \frac{1}{a^2}\right) + \frac{2\delta}{a}\epsilon^{-\delta}(\text{a finite term})\right\}\end{array}\right]$$

$$\div\ (\gamma\alpha^2\mathrm{I}_0^2(\alpha)-(1+\gamma\alpha^2)\mathrm{I}_1^2(\alpha)).$$

Now, if we remember that $\gamma\alpha^2 I_0^2(\alpha) - (1 + \gamma\alpha^2) I_1^2(\alpha)$ is of order $\gamma\alpha I_0^2(\alpha)$, we see that the successive terms in the coefficient of $\cos \widehat{2n+1}u$ are of the orders

$$\frac{\delta\epsilon^{-\delta}}{(2n+1)^2}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^4}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^3}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^3}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^2},$$

respectively. Also in considering discontinuities, we need only consider the terms towards infinity, for the terms at the beginning can introduce no discontinuity.

But clearly the series

$$\Sigma \frac{\delta\epsilon^{-\delta}}{(2n+1)^3} \cos \widehat{2n+1}\, u, \qquad \Sigma \frac{\delta\epsilon^{-\delta}}{(2n+1)^4} \cos \widehat{2n+1}\, u,$$

are of the order d multiplied by a series, which is finite and continuous up to and including the value $d = 0$. They tend therefore to the limit 0 with d, and can introduce no discontinuity in the stress.

The same will be seen to hold of the series

$$\Sigma \frac{\delta\epsilon^{-\delta}}{(2n+1)^2} \cos \widehat{2n+1}\, u, \quad \text{provided } u \neq 0.$$

If however $u = 0$, we have to deal with the series

$$\frac{\pi d}{2c} \Sigma \frac{1}{(2n+1)} \epsilon^{-(2n+1)\frac{\pi d}{2c}}.$$

The series under the sign of summation is divergent if $d = 0$. If, however, d is small, but still finite, the series can be summed, and we have the expression equal to

$$\frac{\pi d}{4c} \log \left(\frac{1 + \epsilon^{-\frac{\pi d}{2c}}}{1 - \epsilon^{-\frac{\pi d}{2c}}} \right).$$

This tends to zero when d is small, provided

$$\frac{\pi d}{4c} \log \left(1 - \epsilon^{-\frac{\pi d}{2c}}\right), \quad i.e., \quad \frac{\pi d}{4c} \log \frac{\pi d}{2c} \quad \text{tend to zero,}$$

which is known to be the case. Hence this series again can never introduce a discontinuity in the stress.

Now consider the series

$$\Sigma \frac{\delta\epsilon^{-\delta}}{2n+1} \cos \widehat{2n+1}\, u = \frac{\pi d}{2c} \Sigma \epsilon^{-\delta} \cos \widehat{2n+1}\, u.$$

This is not of the same form. The series under the Σ is sometimes oscillatory, and sometimes divergent, but is never convergent, if d is put equal to zero.

But if d be small but still finite, the series

$$\Sigma\epsilon^{-\frac{\overline{2n+1}\pi d}{2c}}\cos\widehat{2n+1}u = \frac{\left(\epsilon^{\frac{\pi d}{2c}} - \epsilon^{\frac{3\pi d}{2c}}\right)\cos u}{1+\epsilon^{4\left(\frac{\pi d}{2c}\right)} - 2\epsilon^{2\left(\frac{\pi d}{2c}\right)}\cos 2u}.$$

As d approaches the limit 0, this series also approaches the limit 0. Hence, *à fortiori*, this series multiplied by d approaches the limit 0, and the stress is continuous.

This holds provided $u \pm 0$. But if $u = 0$ the series in question $= \frac{\epsilon^{\pi d/2c}}{1-\epsilon^{\pi d/2c}}$. The limit of $\frac{\pi d}{2c}\frac{\epsilon^{\pi d/2c}}{1-\epsilon^{\pi d/2c}}$ when $d = 0$ is $-\frac{1}{2}$. But when $d =$ absolute zero in the series $d\Sigma e^{-\delta}\cos\widehat{2n+1}u$ the series $= 0$ identically.

We have therefore in these cases a finite discontinuity in the stress. This takes place at the points $u = 0$, *i.e.*, $z = \pm b \pm c$, where the shear $\widehat{rz}$ varies discontinuously. At all other points $\widehat{rr}$ approaches the value zero continuously as we move up to the outer surface of the cylinder.

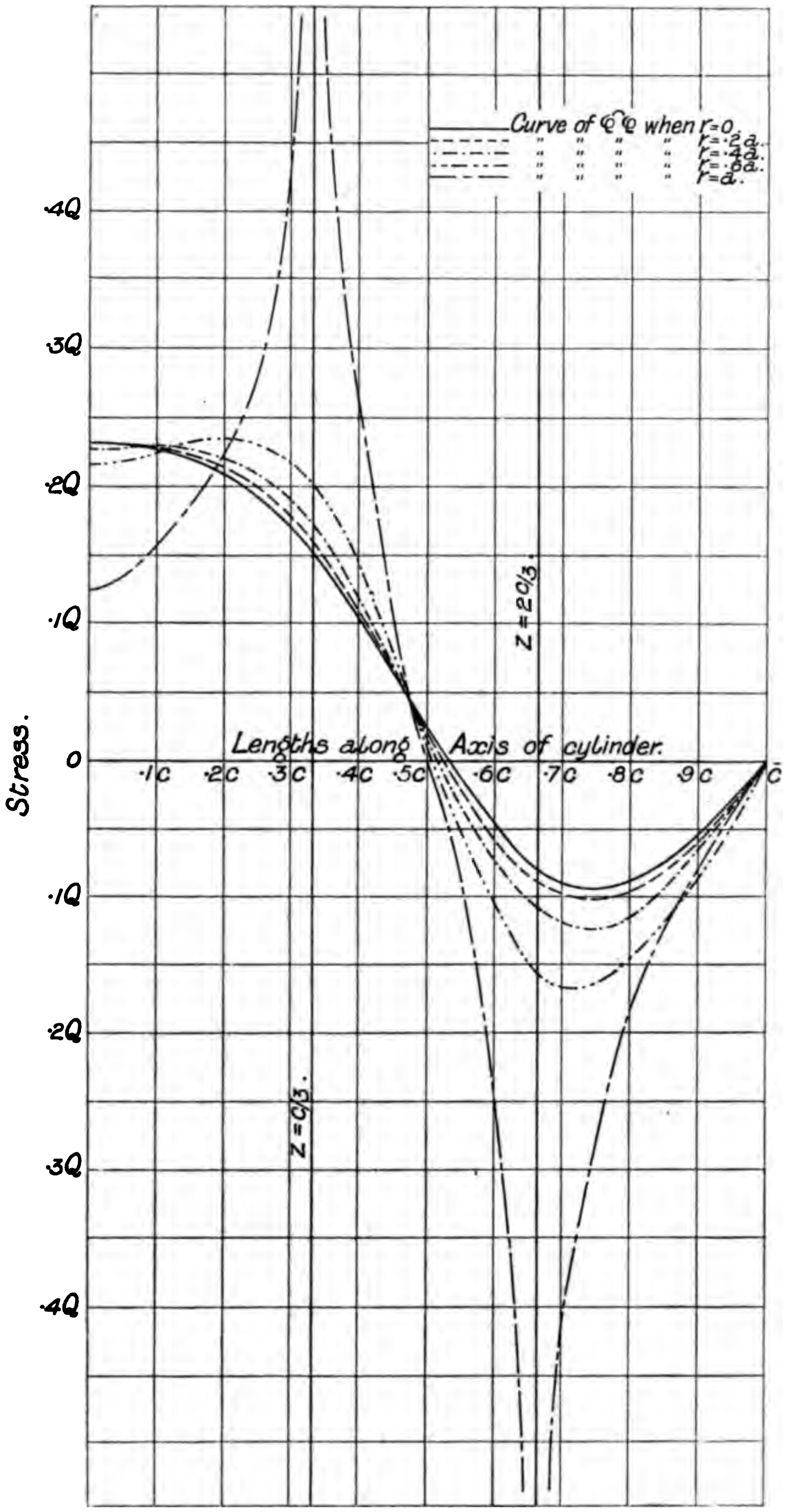
Coming now to the distortion of the cross-sections, this is exhibited in Diagram 2. The displacements are exaggerated, as in Diagram 1, w_0 being taken $= \frac{1}{2}c$. The cross-sections become hollowed out in the middle, the greatest longitudinal extension taking place at the sides. Another noticeable feature is that the cross-sections are slightly curled round the rim, except over the part of the cylinder which is subjected to shear, where they slope up sharply. This follows from the fact that $\frac{dw}{dr} + \frac{du}{dz} = \frac{\widehat{rz}}{\mu}$.

Thus, where $\widehat{rz} = 0$ and $du/dz > 0$ from Diagram 1, it follows that $dw/dr < 0$ or, since $dw/dr > 0$ nearer the centre, a maximum value of w occurs at a comparatively small distance inside the "outer skin" of the cylinder. When, however, $\widehat{rz}$ increases by S, we have seen that du/dz increases by $-\frac{1}{4}\mathrm{S}/\mu$, hence dw/dr increases by $\frac{5}{4}\mathrm{S}/\mu$, and is always positive at the outer surface. At the further end, where S ceases to act, the reverse takes place.

It is now easy to understand why the tension is infinite at the inner end of the shear ring and the pressure infinite at the outer. For if we take two parallel near cross-sections, the one just inside the shear ring and the other just outside, the distorted cross-sections remain sensibly parallel until we approach the outer surface, when they diverge sharply, if near the inside boundary, and converge sharply if at the outside boundary. In the one case we get an infinite extension, in the other an infinite compression. Hence we should expect the stresses $\widehat{zz}$ and $\widehat{\phi\phi}$ to become infinite at these points, and the stress $\widehat{rr}$ to vary infinitely rapidly—and this, we have seen, is what does actually occur.

Further, we see that if we measure the elongation of the outer skin as is done with an extensometer, we shall always get too high a value for the extension. Referring to the table on p. 172, we have the following table of the displacements

Diagram 4.—Showing Stress $\widehat{\phi\phi}$ for the Cylinder under Shearing Pull.

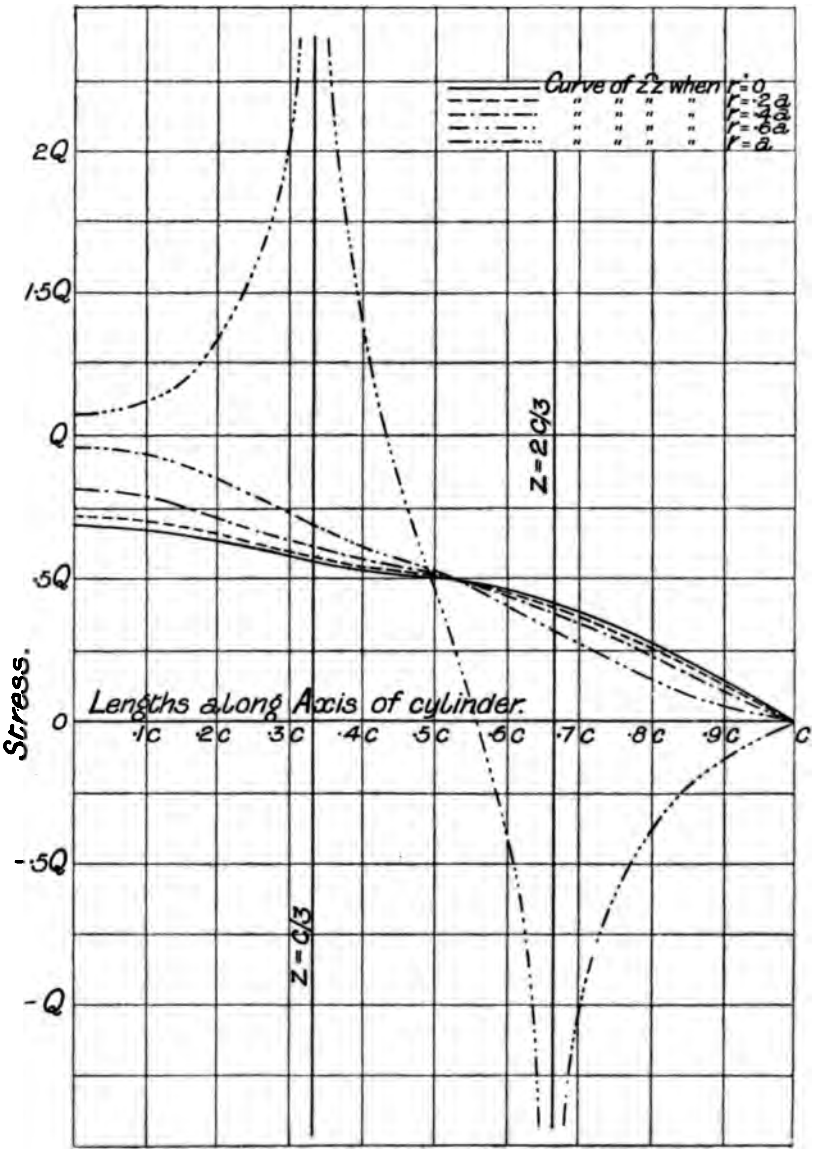


measured by the extensometer, as compared with the displacements calculated from the ordinary theory, over the free length of the bar:—

Displacements.	$z = (\cdot1)c$.	$z = (\cdot2)c$.	$z = (\cdot3)c$.	$z = (\cdot4)c$.	$z = (\cdot5)c$.
Actual	·10972	·22900	·38253	·59238	·67152
Calculated	·10000	·20000	·30000	·40000	·50000
Difference	·00972	·02900	·08253	·19238	·17152
Percentage correction	− 8·86	− 12·66	− 21·57	− 32·48	− 25·54

We see, therefore, that in such a case very large corrections have to be applied to extensometer readings.

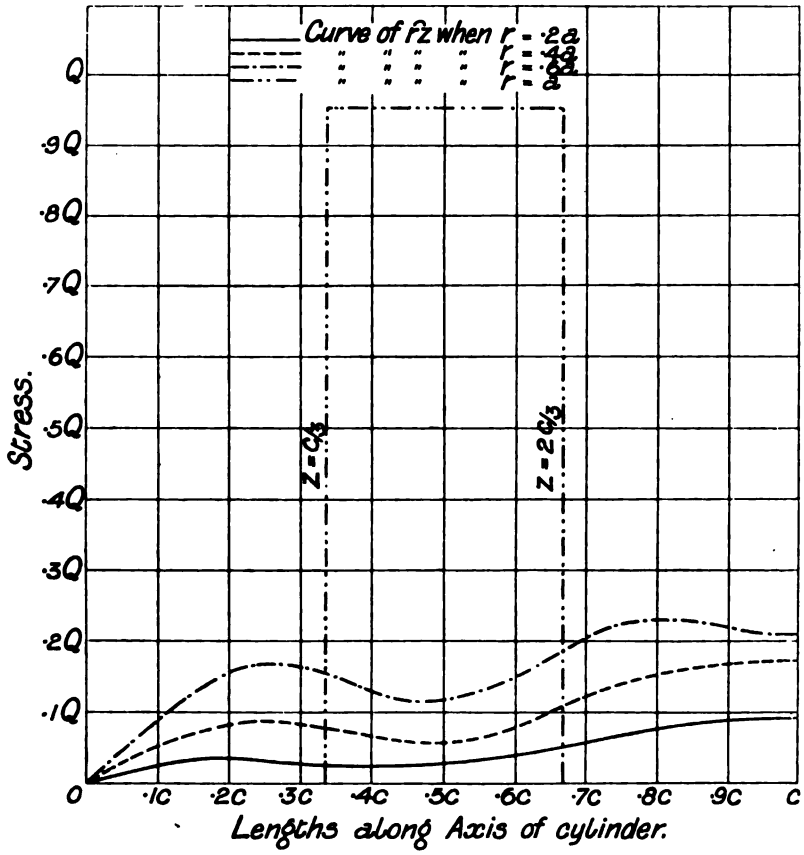
Diagram 5.—Showing Stress $\widehat{zz}$ for the Cylinder under Shearing Pull.



Diagrams 3–6 give curves showing the variations of the stresses, with z, for the values of r equal to 0, $(\cdot 2)\,a$, $(\cdot 4)\,a$, $(\cdot 6)\,a$, a. I have omitted the intermediate value $(\cdot 8)\,a$, because the series used converge in this case inconveniently slowly, and no methods of approximation, such as were employed in the case $r = a$, are here available. Observation of the curves for the smaller values of r will, however, in most cases suggest the process by which they are deformed continuously into the curve corresponding to $r = a$. In Diagram 3, of course, this is not obvious, but here, as has been shown, discontinuous changes occur. In Diagram 6 it is also not quite clear how the curve for $r = (\cdot 6)\,a$ becomes transformed into the rectangle corresponding to $r = a$. The curve for $r = (\cdot 6)\,a$ has, however, already developed a double hump, and its righthandmost ordinate's rate of increase is fast diminishing. This suggests that the two humps will rise and approach each other, ultimately covering the rectangle, whilst the two "tails" will dwindle down to zero.

Remarks of a similar character apply to Diagram 5.

Diagram 6.—Showing Stress $\widehat{rz}$ for the Cylinder under Shearing Pull.

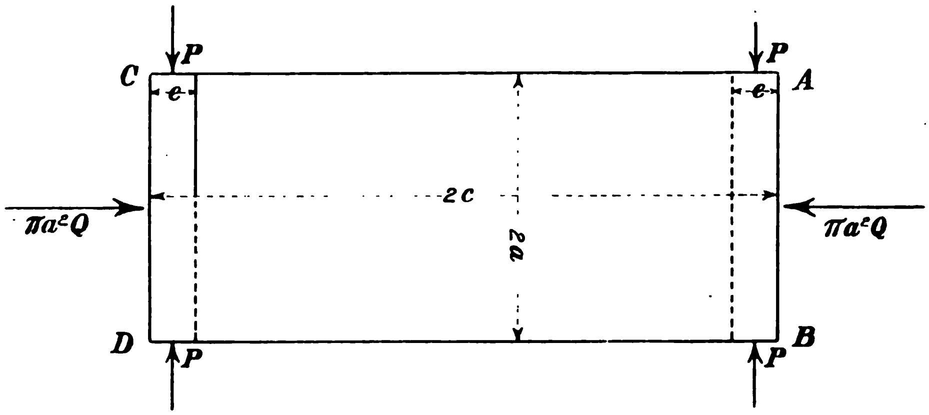


§ 12. *The Second Problem: Case of a Cylinder under Pressure whose Ends are not allowed to expand. (First Method of Constraint.)*

Consider a cylinder (fig. 3) subjected to the following system of load :—

(1.) There is no shear $\widehat{rz}$ along the curved surface $r = a$. Over two rings of breadth e at the ends a radial pressure P is made to act, this pressure being so adjusted that there is no radial shift at the points A, B, C, D ; the breadth e being in the limit to be made indefinitely small.

Fig. 3



(2.) The plane ends AB, CD, are constrained to remain plane, and are subject to a total normal pressure $\pi a^2 Q$.

The above would fit the case of a cylinder compressed between two rigid planes, into which shallow circular depressions had been cut, to fit the ends of the compressed cylinder and prevent them from expanding.

If we return to the expressions for the stresses in the general case, (24), (25), (26), and also to those for the displacements, we find that if w is to be constant for $z = c$

$$kc = n\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

$$C_1 = C_2 = C$$

$$E_1 = E_2 = E = 0$$

$$(A_1 - A_2)\gamma k + 2C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

Also this gives $\widehat{rz} = 0$ over the plane ends, so that we may suppose our rigid constraining plane to be also smooth.

The condition that there is to be no shear over the curved surface now gives

$$(A_1 + A_2) + \frac{2C}{k}\frac{\alpha I_0(\alpha)}{I_1(\alpha)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

(writing $\alpha = ka$, $\rho = kr$ as before).

From (56) and (57)

$$\left.\begin{aligned} A_1 &= -\frac{C}{k}\left(\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right) \\ A_2 &= -\frac{C}{k}\left(\frac{\alpha I_0}{I_1} - \frac{1}{\gamma}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

In what follows, the argument of the I-functions, when not written, will always be assumed to be α.

We find

$$\widehat{rr} = 2(\lambda + \mu)u_0 + \lambda w_0 + \Sigma\frac{2\mu C}{k}\left[\begin{aligned} &\left(1 + \frac{\alpha I_0}{I_1}\right)I_0(\rho) \\ &-\left\{\left(\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right)\frac{I_1(\rho)}{\rho} + \rho I_1(\rho)\right\} \end{aligned}\right]\cos kz \quad . \quad (59).$$

Putting in this $\rho = \alpha$

$$(\widehat{rr})_{r=a} = 2(\lambda + \mu)u_0 + \lambda w_0 + \Sigma\frac{2\mu C}{k}\frac{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2)I_1^2}{\gamma\alpha I_1}\cos kz \quad . \quad . \quad (60).$$

Now expand the given pressure in the form

$$(\widehat{rr})_{r=a} = P\left(a_0 + \sum_1^\infty a_n \cos\frac{n\pi z}{c}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

Where $a_0, a_1, \ldots a_n, \ldots$ are determined, P remains a free constant.

We have at once, comparing coefficients,

$$\frac{2\mu C}{k} = \frac{\gamma\alpha I_1}{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2)I_1^2}Pa_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$2(\lambda + \mu)u_0 + \lambda w_0 = Pa_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

Next we have $u = 0$ when $r = a$, $z = c$

$$0 = u_0 a - \Sigma(-1)^n\left(-\frac{C}{k^2}\left\{\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right\}I_1 + \frac{C}{k^2}\alpha I_0\right)$$

$$u_0 a = -P\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

where $$\zeta = \sum_1^\infty \frac{a_n}{2\mu k}\frac{\alpha I_1^2(-1)^n}{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2)I_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

This condition gives u_0 in terms of P, and hence by (63)

$$\lambda w_0 = \mathrm{P}\left(a_0 + 2(\lambda + \mu)\frac{\zeta}{a}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (66).$$

We have now to find such a value for P that the mean pressure on the plane ends is Q.

$$-\pi a^2 \mathrm{Q} = 2\pi\int_0^a r\widehat{zz}\,dr$$

$$= \pi a^2 (2\lambda u_0 + (\lambda + 2\mu)w_0)$$

$$+ 2\pi\sum_1^\infty \cos\frac{n\pi z}{c}\left\{\left(\{\lambda + 2\mu\}\mathrm{A}_2 - \lambda\mathrm{A}_1 - \lambda\frac{2\mathrm{C}}{k}\right)\int_0^a r\mathrm{I}_0\,dr + 2\mu\mathrm{C}\int_0^a r^2\mathrm{I}_1\,dr\right\},$$

whence, applying the well-known theorem,

$$\frac{d}{dx}(x^n\mathrm{I}_n(x)) = x^n\mathrm{I}_{n-1}(x),$$

we have

$$-\pi a^2\mathrm{Q} = \pi a^2(2\lambda u_0 + (\lambda + 2\mu)w_0)$$

$$+ 2\pi\sum_1^\infty \cos\frac{n\pi z}{c}\left\{\left[(\lambda + 2\mu)\,\mathrm{A}_2 - \lambda\mathrm{A}_1 - \lambda\frac{2\mathrm{C}}{k}\right]\frac{\alpha\mathrm{I}_1}{k} + 2\mu\mathrm{C}\frac{\alpha^2\mathrm{I}_2}{k}\right\}.$$

Using the relation

$$\mathrm{I}_2 + \frac{2}{\alpha}\mathrm{I}_1 - \mathrm{I}_0 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (67),$$

and putting in for A_1, A_2 their values in terms of C, we find that the terms under the Σ vanish identically. Hence

$$\mathrm{Q} = -(2\lambda u_0 + (\lambda + 2\mu)w_0)$$

$$= \mathrm{P}\left(-\frac{\lambda + 2\mu}{\lambda}a_0 - \frac{2\zeta}{a}(3\lambda + 2\mu)\frac{\mu}{\lambda}\right) \quad . \; . \; . \; . \; . \; (68).$$

Now suppose the distribution of stress is such that $\widehat{rr} = 0$ from $z = -(c - e)$ to $z = +(c - e)$ and $\widehat{rr} = -\mathrm{P}$ from $z = -c$ to $z = -(c - e)$, and from $z = c - e$ to $z = c$, we find

$$a_0 = -e/c \quad a_n = -\frac{2(-1)^n}{n\pi}\sin\frac{n\pi e}{c},$$

whence

$$\zeta = -\sum_1^\infty \frac{c}{\mu}\,\frac{\sin\frac{n\pi e}{c}}{n^2\pi^2}\,\frac{\alpha\mathrm{I}_1^2}{\gamma\alpha^2\mathrm{I}_0^2 - (1 + \gamma\alpha^2)\mathrm{I}_1^2} \quad . \; . \; . \; . \; . \; . \; . \; (69).$$

When α is at all large, the terms of ζ are comparable with those of the series

$$-\frac{c}{\mu\gamma}\sum_1^\infty\frac{\sin\frac{n\pi e}{c}}{n^2\pi^2},$$

which is equal to

$$-\frac{c}{\pi^2\mu\gamma}\left\{-\frac{\pi e}{c}\log\left(2\sin\frac{\pi e}{2c}\right)+\int_0^{\pi e/c}\frac{x}{2}\cot\frac{x}{2}\,dx\right\} \quad . \quad . \quad . \quad . \quad . \quad (70).$$

(70) will give the approximate value of ζ whenever $\pi a/c$ is at all large. If $a = 2c$, this method of approximation will already be quite fair.

We see therefore that if e tends to zero, ζ also tends to zero, but ζ/e tends to become logarithmically infinite.

Now from (68)

$$\mathrm{P}a_0 = \frac{-\lambda \mathrm{Q} a_0}{(\lambda+2\mu)a_0+\frac{2\mu\zeta}{a}(3\lambda+2\mu)}$$

$$= \frac{-\lambda \mathrm{Q}}{(\lambda+2\mu)-\frac{2\mu c}{a}\left(\frac{\zeta}{c}\right)(3\lambda+2\mu)}.$$

Hence, since $-\zeta/c$ tends to ∞ when e tends to zero, $\mathrm{P}a_0$ tends to zero when e tends to zero.

And similarly, for any finite value of n, $\mathrm{P}a_n$ tends to zero when e tends to zero.

But if we write down the expressions for the stresses, they are:

$$\left.\begin{aligned}
\widehat{zz} &= -\mathrm{Q}+\Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1(\alpha)}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\rho\mathrm{I}_1(\rho)+\mathrm{I}_0(\rho)\left(2-\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}\right)\right]\cos\frac{n\pi z}{c}\\
\widehat{rr} &= \mathrm{P}a_0+\Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\left(\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}+1\right)\mathrm{I}_0(\rho)-\mathrm{I}_1(\rho)\left\{\frac{\alpha\mathrm{I}_0}{\rho\mathrm{I}_1}+\frac{1}{\gamma\rho}+\rho\right\}\right]\cos\frac{n\pi z}{c}\\
\widehat{\phi\phi} &= \mathrm{P}a_0+\Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\left(\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}+\frac{1}{\gamma}\right)\frac{\mathrm{I}_1(\rho)}{\rho}-\left(\frac{1}{\gamma}-1\right)\mathrm{I}_0(\rho)\right]\cos\frac{n\pi z}{c}\\
\widehat{rz} &= \Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\rho\mathrm{I}_0(\rho)-\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}\mathrm{I}_1(\rho)\right]\sin\frac{n\pi z}{c}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (71).$$

Now the above series are absolutely convergent for all values of r except $r = a$, where indeed they are discontinuous. Leaving the neighbourhood of $r = a$ out of account, we see that for points inside the material, when the space over which the constraining pressure acts is indefinitely reduced, $\mathrm{LP}a_n = 0$ and

$$\widehat{zz} = -\mathrm{Q}$$

$$\widehat{rz} = \widehat{rr} = \widehat{\phi\phi} = 0\,;$$

therefore outside the rim, where plastic deformation may be expected to occur, the stresses are exactly the same as on the ordinary hypothesis.

We come then to the conclusion that this method of preventing the ends from expanding is not adequate, and that to obtain any real effect, we require to make the constraining rim of a certain definite thickness.

In so doing, we are really introducing an additional condition, besides the non-expansion of the ends, the cylinder being now, as it were, built-in. The problem as it stands did not appear of sufficient interest to warrant the expenditure of arithmetical labour upon it, so I have contented myself with stating the algebraical results.

§ 13. *The Second Problem: Constraint effected by Shear over the Terminal Cross-sections. Determination of the Constants.*

Suppose now that we consider our cylinder subject to the following conditions :—

(i.) A total pressure πa^2 Q over the plane ends, the distribution of this pressure being unknown.

(ii.) The ends constrained to remain plane, so that $w =$ const. when $z = \pm c$.

(iii.) The ends not to expand along the perimeter

$$u = 0 \text{ when } r = a,\ z = \pm c.$$

This condition is satisfied by allowing a shear $\widehat{rz}$ over the plane ends, its distribution being, however, unknown.

(iv.) No stress across the curved surface, *i.e.*,

$$\widehat{rr} = 0 \text{ when } r = a,$$

$$\widehat{rz} = 0 \text{ when } r = a.$$

These conditions will represent the state of things which we may expect to hold it the cylinder be compressed between two rigid planes which are sufficiently rough to prevent the expansion of the ends.

Now, in such a case as this, it is obvious that the expressions for the stresses and strains as purely periodic series in z break down, for if we take the expressions (24) and (26) for $\widehat{rr}$ and $\widehat{rz}$ the condition that $w =$ const. when $z = \pm c$ will give us, as before, $E = 0$, and the vanishing of the stresses at the curved surface will give two homogeneous equations of condition between A_1, A_2, and C. These, taken in conjunction with equation (35), give three linear homogeneous equations in A_1, A_2, and C, which are in general inconsistent unless $A_1 = 0$, $A_2 = 0$, $C = 0$, which would destroy the periodic solution altogether.

We have therefore to assume that u and w are made up of two parts. The first part, which I shall denote by U, W, consists of the periodic solution hitherto obtained. The second part is a finite power series in r and z. The resulting expression is a combination of the two types of solution, which are discussed separately by Mr. CHREE ('Camb. Phil. Soc. Trans.,' vol. 14). Either of these two types, taken by itself, is of comparatively restricted application, but by combining the two we are enabled to deal with far more general problems.

Assume therefore

$$u = u_0 r + \frac{u_1 r^3}{3} + \frac{u_2 r^5}{5} + \frac{\mathrm{D} r z^2}{2} + \frac{\mathrm{E} r^3 z^2}{2} + \frac{\mathrm{F} r z^4}{4} + \mathrm{U} \quad . \quad . \quad . \quad . \quad . \quad (72),$$

$$w = w_0 z + \frac{w_1 z^3}{3} + \frac{w_2 z^5}{5} + \frac{\mathrm{D}' r^2 z}{2} + \frac{\mathrm{E}' r^2 z^3}{2} + \frac{\mathrm{F}' r^4 z}{4} + \mathrm{W} \quad . \quad . \quad . \quad . \quad (73).$$

The above power series are the most general expressions of the fifth degree consistent with the conditions that u must be odd in r and even in z, and w must be odd in z and even in r.

In the above we have, as before,

$$\mathrm{U} = \Sigma \left\{ -\frac{\mathrm{A}_1}{k} \mathrm{I}_1 (kr) - \frac{\mathrm{C}}{k} r \mathrm{I}_0 (kr) \right\} \cos kz \quad . \quad . \quad . \quad . \quad (74),$$

$$\mathrm{W} = \Sigma \left\{ \frac{\mathrm{A}_2}{k} \mathrm{I}_0 (kr) + \frac{\mathrm{C}}{k} r \mathrm{I}_1 (kr) \right\} \sin kz \quad . \quad . \quad . \quad . \quad . \quad (75).$$

Consider first of all the condition that w is to be constant when $z = \pm c$. This fixes k:

$$k = n\pi/c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

Further we have

$$\mathrm{F}' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77),$$

$$\tfrac{1}{2}\mathrm{D}' c + \tfrac{1}{2}\mathrm{E}' c^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78).$$

Now remember that u and w have to satisfy the differential equations

$$(\lambda + 2\mu) \frac{d}{dr} \frac{1}{r} \frac{d}{dr} (ru) + \mu \frac{d^2 u}{dz^2} + (\lambda + \mu) \frac{d^2 w}{dr dz} = 0,$$

$$(\lambda + \mu) \frac{1}{r} \frac{d}{dr} r \frac{du}{dz} + \mu \frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) + (\lambda + 2\mu) \frac{d^2 w}{dz^2} = 0.$$

The parts U and W we have seen already will satisfy these equations, provided

$$\mathrm{A}_1 - \mathrm{A}_2 + 2\mathrm{C}/\gamma k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79).$$

Consider therefore only the algebraic terms. Of these $u_0 r$ and $w_0 z$ always satisfy the above equations.

The third order terms require

$$\tfrac{8}{3} u_1 (\lambda + 2\mu) + \mu \mathrm{D} + (\lambda + \mu) \mathrm{D}' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (80),$$

$$2 (\lambda + \mu) \mathrm{D} + 2\mu \mathrm{D}' + (\lambda + 2\mu)\, 2 w_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (81).$$

The fifth order terms give

$$(\lambda + 2\mu)\left(\frac{24}{5}u_2r^3 + 4\mathrm{E}rz^2\right) + \mu\mathrm{E}r^3 + 3\mu\mathrm{F}rz^2 + (\lambda + \mu)(3\mathrm{E}'rz^2) = 0,$$

$$(\lambda + \mu)(4\mathrm{E}r^2z + 2\mathrm{F}z^3) + 2\mu\mathrm{E}'z^3 + (\lambda + 2\mu)[4w_2z^3 + 3\mathrm{E}'r^2z] = 0$$

which imply the four relations

$$(\lambda + 2\mu)\frac{24u_2}{5} + \mu\mathrm{E} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82).$$

$$2(\lambda + \mu)\mathrm{F} + 2\mu\mathrm{E}' + (\lambda + 2\mu)4w_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (83),$$

$$4\mathrm{E}(\lambda + 2\mu) + 3\mu\mathrm{F} + 3\mathrm{E}'(\lambda + \mu) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (84),$$

$$4\mathrm{E}(\lambda + \mu) + 3\mathrm{E}'(\lambda + 2\mu) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85).$$

There is, however, a further relation to be satisfied among these constants, and that is obtained as follows. If we proceed to write down the expressions for $\widehat{rr}$ and $\widehat{rz}$ and to put in them $r = a$, we shall obtain expressions of the form

$$\widehat{rr} = \text{algebraic polynomial in } z + \text{series of cosines of } n\pi z/c,$$

$$\widehat{rz} = \text{algebraic polynomial in } z + \text{series of sines of } n\pi z/c,$$

where the coefficients of cos $n\pi z/c$, sin $n\pi z/c$, contain the two undetermined constants A_1 and A_2.

We may now proceed to expand the two polynomials in series of cosines or sines of $n\pi z/c$. Equating then the coefficient of each cosine and sine to zero, we can make $\widehat{rr}$ and $\widehat{rz}$ zero over the whole of the curved surface, and at the same time we obtain two equations for A_1 and A_2.

But it is clear that, if the Fourier expressions in the second case are to be continuous, then the algebraic polynomial part of $\widehat{rz}$ must reduce to zero when $z = \pm c$, otherwise its expansion in sines of $n\pi z/c$ is discontinuous, and at the perimeter of the flat ends the shear is discontinuous. This introduces infinite stresses at this point which render the solution inconvenient.

Now we have at our disposal *nine* constants; these have already been made to satisfy the seven homogeneous equations (78), (80)–(85), and therefore we are free to make them satisfy an eighth homogeneous equation.

Choose then the constants so as to make (polynomial part of $[du/dz + dw/dr]$ when $r = a$, $z = \pm c$) zero, and we have

$$\mathrm{D}ac + \mathrm{E}a^3c + \mathrm{F}ac^3 + \mathrm{D}'ac + \mathrm{E}'ac^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (86).$$

If now we express all the other constants in terms of the constant E, we find :

$$\left.\begin{aligned} w_1 &= \left[a^2 - \frac{8c^2}{3}\right]\gamma\mathrm{E}, & u_1 &= \tfrac{3}{8}\left[(1-\gamma)a^2 - \tfrac{4}{3}c^2\right]\mathrm{E}, \\ w_2 &= \tfrac{4}{3}\gamma\mathrm{E}, & u_2 &= -\tfrac{5}{24}(1-\gamma)\mathrm{E}, \\ \mathrm{D} &= \{\tfrac{4}{3}(1+\gamma)c^2 - a^2\}\mathrm{E}, & \mathrm{D}' &= \frac{4\gamma c^2}{3}\mathrm{E}, \\ \mathrm{F} &= -\tfrac{4}{3}(1+\gamma)\mathrm{E}, & \mathrm{E}' &= -\tfrac{4}{3}\gamma\mathrm{E} \end{aligned}\right\} \quad . \quad . \quad . \quad (87),$$

and these satisfy the equations and the condition (86).

It is noticeable that a solution can be obtained, in the form

$$u = u_0 r + \frac{u_1 r^3}{3} + \frac{\mathrm{D} r z^2}{2} + \mathrm{U},$$

$$w = w_0 z + \frac{w_1 z^3}{3} + \mathrm{W},$$

which can be made to satisfy all the conditions except (86). If, however, one works out this solution, it is found, as we should expect, to give infinite values for the stresses, all round the perimeter of the plane ends. Thus, though simpler in form, this solution is not really simpler to work with. I have given on pp. 217–219 the expressions for the stresses and displacements obtained from such a solution.

§ 14. *Determination of the Coefficients so as to Satisfy the Conditions at the Curved Surface.*

If we write down the expressions for the stresses, we find

$$\left(\frac{\widehat{rz}}{\mu}\right)_{r=a} = (\mathrm{D}+\mathrm{D}')\,az + \mathrm{E}a^3 z + (\mathrm{E}'+\mathrm{F})\,az^3 + \frac{d\mathrm{U}}{dz} + \frac{d\mathrm{W}}{dr}\,.$$

We have therefore to make

$$-(\mathrm{D}+\mathrm{D}')\,az - \mathrm{E}a^3 z - (\mathrm{E}'+\mathrm{F})\,az^3$$
$$= \sum_1^\infty \left((\mathrm{A}_1+\mathrm{A}_2)\,\mathrm{I}_1(\alpha) + \frac{2\mathrm{C}}{k}\,\alpha \mathrm{I}_0(\alpha)\right)\sin\frac{n\pi z}{c}\,.$$

Now we find easily

$$z = \sum_1^\infty (-1)^{n-1}\frac{2c}{n\pi}\sin\frac{n\pi z}{c}\,,$$

$$z^3 = \sum_1^\infty (-1)^{n-1}\left(\frac{2c^3}{n\pi} - \frac{12c^3}{n^3\pi^3}\right)\sin\frac{n\pi z}{c}\,.$$

Hence if we expand $-(\mathrm{D}+\mathrm{D}')\,az - \mathrm{E}a^3 z - (\mathrm{E}'+\mathrm{F})\,az^3$ in a series of the form $\sum\limits_1^\infty a_n \sin\frac{n\pi z}{c}$ we find

$$a_n = \left[-(\mathrm{D}+\mathrm{D}')\,ac - \mathrm{E}a^3 c - (\mathrm{E}'+\mathrm{F})\,ac^3\right](-1)^{n-1}\times\frac{2}{n\pi}\,.$$
$$+ (-1)^{n-1}\frac{12ac^3}{n^3\pi^3}(\mathrm{E}'+\mathrm{F})\,.$$

The first term is zero in virtue of equation (86), and using the relations (87) we find

$$a_n = (-1)^n \frac{16ac^3}{n^3\pi^3} \times \mathrm{E} \times (2\gamma + 1),$$

whence, comparing coefficients

$$\mathrm{A}_1 + \mathrm{A}_2 + \frac{2\mathrm{C}}{k}\frac{\alpha \mathrm{I}_0}{\mathrm{I}_1} = (-1)^n \frac{16ac^3}{n^3\pi^3}(2\gamma + 1)\mathrm{E} \quad . \quad . \quad . \quad . \quad . \quad (88).$$

This gives us $\widehat{rz}$ consistently zero right up to the plane end.

Next we have

$$\begin{aligned}(\widehat{rr})_{r=a} &= 2(\lambda + \mu)u_0 + \lambda w_0 \\ &+ a^2\left[\tfrac{2}{3}(2\lambda + 3\mu)u_1 + \frac{\mathrm{D}'}{2}\lambda\right] + z^2[(\lambda + \mu)\mathrm{D} + \lambda w_1] \\ &+ \frac{a^2z^2}{4}[(4\lambda + 6\mu)\mathrm{E} + 3\lambda\mathrm{E}'] + a^4\left(\frac{6\lambda + 10\mu}{5}\right)u_2 \\ &+ z^4\left[\frac{\lambda + \mu}{2}\mathrm{F} + \lambda w_2\right] + (\lambda + 2\mu)\left(\frac{d\mathrm{U}}{dr}\right)_{r=a} + \frac{\lambda\mathrm{U}}{a} + \lambda\left(\frac{d\mathrm{W}}{dz}\right)_{r=a}.\end{aligned}$$

Hence we have to make

$$\begin{aligned}&\sum_1^\infty\left\{-\mu[(1+\gamma)\mathrm{A}_1 + (1-\gamma)\mathrm{A}_2]\mathrm{I}_0 + 2\mu\left[\mathrm{A}_1\frac{\mathrm{I}_1}{\alpha} - \frac{\mathrm{C}}{k}\alpha\mathrm{I}_1\right]\right\}\cos\frac{n\pi z}{c} \\ &= -2(\lambda + \mu)u_0 - \lambda w_0 \\ &- a^2\left[\tfrac{2}{3}(2\lambda + 3\mu)u_1 + \frac{\mathrm{D}'\lambda}{2}\right] - a^4\left(\frac{6\lambda + 10\mu}{5}\right)u_2 \\ &- z^2\left\{(\lambda + \mu)\mathrm{D} + \lambda w_1 + \frac{a^2}{2}[(4\lambda + 6\mu)\mathrm{E} + 3\lambda\mathrm{E}']\right\} \\ &- z^4\left[\frac{\lambda + \mu}{2}\mathrm{F} + \lambda w_2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89).\end{aligned}$$

Now we have

$$z^2 = \frac{c^2}{3} + \sum_1^\infty \frac{4c^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi z}{c},$$

$$z^4 = \frac{c^4}{5} + \sum_1^\infty 8c^4(-1)^n\left(\frac{1}{n^2\pi^2} - \frac{6}{n^4\pi^4}\right)\cos\frac{n\pi z}{c}.$$

Hence if the right-hand side of equation (89) be expanded in the form

$$b_0 + \sum_1^\infty b_n \cos\frac{n\pi z}{c},$$

we find

$$\begin{aligned} b_0 = &-2(\lambda+\mu)u_0 - \lambda w_0 \\ &- a^2\left[\tfrac{2}{3}(2\lambda+3\mu)u_1 + \frac{D'}{2}\lambda\right] - a^4\left(\frac{6\lambda+10\mu}{5}\right)u_2 \\ &- \frac{c^2}{3}\left\{(\lambda+\mu)D + \lambda w_1 + \frac{a^2}{2}\left[(4\lambda+6\mu)E + 3\lambda E'\right]\right\} \\ &- \frac{c^4}{5}\left[\frac{\lambda+\mu}{2}F + \lambda w_2\right] \end{aligned}$$

$$\begin{aligned} b_n = &-\frac{(-1)^n 4c^2}{n^2\pi^2}\left\{(\lambda+\mu)D + \lambda w_1 + \frac{a^2}{2}[(4\lambda+6\mu)E + 3\lambda E']\right\} \\ &- \frac{8c^4(-1)^n}{n^2\pi^2}\left\{\frac{\lambda+\mu}{2}F + \lambda w_2\right\} + \frac{48c^4}{n^4\pi^4}(-1)^n\left\{\frac{\lambda+\mu}{2}F + \lambda w_2\right\}, \end{aligned}$$

whence using equations (87) we find after long but otherwise straightforward reductions

$$\begin{aligned} b_0 = &-2(\lambda+\mu)u_0 - \lambda w_0 \\ &+ \mu E\left\{-\tfrac{14}{15}c^4\gamma - \frac{a^4}{12}(2\gamma+1) + \frac{2a^2c^2\gamma}{3}\right\}, \end{aligned}$$

$$b_n = -(-1)^n\frac{4c^2a^2}{n^2\pi^2}\mu E(2\gamma+1) - \frac{96c^4}{n^4\pi^4}(-1)^n E\mu\gamma.$$

Hence, equating coefficients on both sides of equation (89) we obtain the relations

$$2(\lambda+\mu)u_0 + \lambda w_0 = \mu E\left\{-\tfrac{14}{15}c^4\gamma - \frac{a^4}{12}(2\gamma+1) + \frac{2a^2c^2\gamma}{3}\right\} \quad . \quad . \quad (90),$$

$$\begin{aligned} &-[(1+\gamma)A_1 + (1-\gamma)A_2]I_0 + 2\left[A_1\frac{I_1}{\alpha} - \frac{C}{k}\alpha I_1\right] \\ &\qquad = -(-1)^n E \times \frac{4c^2a^2}{n^2\pi^2}\left(2\gamma + 1 + \frac{24c^2\gamma}{a^2n^2\pi^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (91), \end{aligned}$$

and it in (91) we substitute for A_1, A_2 their values in terms of C/k deduced from (79) and (88), we have

$$\begin{aligned} &\frac{2\mu C}{k}\,\frac{\alpha^2\gamma I_0^2 - I_1^2(1+\alpha^2\gamma)}{\alpha\gamma I_1} \\ &\qquad = -(-1)^n\mu E \times \frac{4c^2a^2}{n^2\pi^2}\left[2\gamma + 1 + \frac{4c^2}{a^2n^2\pi^2}(8\gamma+1) - \frac{4cI_0(2\gamma+1)}{an\pi I_1}\right], \end{aligned}$$

or

$$\frac{2\mu C}{k} = -(-1)^n \frac{\mu\gamma E 4a^2}{k^2} \frac{\left\{\left[(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right] I_1 - 4I_0(2\gamma+1)\right\}}{\gamma\alpha^2 I_0^2 - (1+\gamma\alpha^2) I_1^2} \quad . \quad . \quad (92).$$

A_1 and A_2 being known in terms of C, all the constants are now determined, except u_0, w_0, and E.

§ 15. *Determination of the Constants* u_0, w_0, E.

Let us now apply the condition that :

$$\int_0^{2\pi} d\theta \int_0^a r\widehat{zz}dr = -\pi a^2 Q.$$

If we write down the expression for $\widehat{zz}$, using the expressions for A_1 and A_2 in terms of C, we find :

$$\widehat{zz} = 2\lambda u_0 + (\lambda + 2\mu) w_0$$

$$+ \left[\frac{4\lambda}{3} u_1 + (\lambda+2\mu)\frac{D'}{2}\right] r^2 + [\lambda D + (\lambda + 2\mu) w_1] z^2 + \left[2E\lambda + \frac{3E'}{2}(\lambda+2\mu)\right] r^2 z^2$$

$$+ \frac{6\lambda}{5} u_2 r^4 + \left[\frac{\lambda F}{2} + (\lambda + 2\mu) w_2\right] z^4$$

$$+ \sum_1^\infty \frac{2\mu C}{k}\left[\rho I_1(\rho) - I_0(\rho)\left\{\frac{\alpha I_0}{I_1} - 2\right\}\right] \cos\frac{n\pi z}{c}$$

$$+ (2\gamma+1)\mu E \sum_1^\infty \frac{I_0(\rho)}{I_1}(-1)^n \times \frac{16ac^3}{n^3\pi^3} \cos\frac{\pi n z}{c}$$

$$= 2\lambda u_0 + (\lambda + 2\mu) w_0$$

$$+ \mu E \left\{ \begin{array}{l} \left[\frac{2}{3}c^2 + (2\gamma - 1)\frac{a^2}{2}\right] r^2 + \left[a^2 - \frac{4}{3}c^2(2\gamma+1)\right] z^2 - 2r^2 z^2 \\ \qquad - \frac{1}{12}(2\gamma - 1) r^4 + \frac{2}{3}(2\gamma+1) z^4 \end{array} \right\}$$

$$+ \sum_1^\infty \frac{2\mu C}{k}\left[\rho I_1(\rho) - I_0(\rho)\left\{\frac{\alpha I_0}{I_1} - 2\right\}\right] \cos\frac{n\pi z}{c}$$

$$+ (2\gamma+1)\mu E \sum_1^\infty \frac{I_0(\rho)}{I_1}(-1)^n \times \frac{16ac^3}{n^3\pi^3} \cos\frac{n\pi z}{c}\,.$$

Hence

$$\int_0^{2\pi} d\theta \int_0^a \widehat{zz} r\,dr = \pi a^2 (2\lambda u_0 + (\lambda + 2\mu) w_0)$$

$$+ \mu \mathrm{E} \times 2\pi \left\{ \begin{array}{l} \frac{a^4}{4}\left[\frac{(2\gamma-1)}{2} a^2 + \frac{2}{3}c^2\right] - \frac{z^2 a^2}{2}\left[\frac{4}{3}(2\gamma+1)c^2 - a^2\right] - \frac{a^6}{6}\frac{2\gamma-1}{4} - \frac{a^4 z^2}{2} \\ + a^2 z^4 \frac{(2\gamma+1)}{3} + 16a(2\gamma+1)\sum_1^\infty (-1)^n \frac{c^5}{n\pi^5} \frac{\int_0^a \rho \mathrm{I}_0(\rho)\,d\rho}{\mathrm{I}_1(\alpha)} \cos\frac{n\pi z}{c} \end{array} \right\}$$

$$+ 2\pi \sum_1^\infty \frac{2\mu \mathrm{C}}{k^3} \int_0^a \left[\rho^2 \mathrm{I}_1(\rho) - \rho \mathrm{I}_0(\rho)\left\{\frac{\alpha \mathrm{I}_0}{\mathrm{I}_1} - 2\right\}\right] d\rho \cos\frac{n\pi z}{c}.$$

But

$$\int_0^a \left\{\rho^2 \mathrm{I}_1(\rho) - \rho \mathrm{I}_0(\rho)\left[\frac{\alpha \mathrm{I}_0}{\mathrm{I}_1} - 2\right]\right\} d\rho = \alpha^2 \mathrm{I}_2(\alpha) - \alpha \mathrm{I}_1(\alpha)\left\{\frac{\alpha \mathrm{I}_0}{\mathrm{I}_1} - 2\right\} = 0 \text{ by (67),}$$

and

$$\int_0^a \rho \mathrm{I}_0(\rho)\,d\rho = \alpha \mathrm{I}_1(\alpha),$$

so that we have to make

$$2\lambda u_0 + (\lambda + 2\mu) w_0$$

$$+ \mu \mathrm{E} \left\{ \begin{array}{l} \frac{a^2}{4}\left[\frac{2\gamma-1}{2} a^2 + \frac{2}{3}c^2\right] - \frac{a^4}{6}\frac{2\gamma-1}{4} - \frac{(2\gamma+1)}{3}(2z^2c^2 - z^4) \\ \qquad + 16(2\gamma+1)c^4 \sum_1^\infty \frac{(-1)^n}{n^4\pi^4} \cos\frac{n\pi z}{c} \end{array} \right\} = -\mathrm{Q}.$$

Now it is easy to show that

$$\sum_1^\infty \frac{(-1)^n}{n^4} \cos\frac{n\pi z}{c} = -\frac{z^4\pi^4}{48c^4} + \frac{z^2\pi^4}{24c^2} - \frac{7\pi^4}{720},$$

whence finally

$$2\lambda u_0 + (\lambda + 2\mu) w_0 + 2\mu \mathrm{E}\left\{\frac{a^4(2\gamma-1)}{12} + \frac{1}{6}a^2c^2 - \frac{7}{45}(2\gamma+1)c^4\right\} = -\mathrm{Q} \quad . \quad . \ (93).$$

(90) and (93) thus give us already two equations for u_0, w_0, and E. We require a third equation.

This is obtained from the condition that

$$(u)_{\substack{z=c \\ r=a}} = 0.$$

This gives

$$u_0 a + \frac{u_1 a^3}{3} + \frac{u_2 a^5}{5} + \frac{\mathrm{D}ac^2}{2} + \frac{\mathrm{E}a^3c^2}{2} + \frac{\mathrm{F}ac^4}{4}$$

$$+ \Sigma \left\{ - \frac{\mathrm{A}_1}{k}\mathrm{I}_1 - \frac{\mathrm{C}}{k^2}\alpha \mathrm{I}_0 \right\} (-1)^n = 0$$

$$u_0 a + \frac{a^3}{8}\left[(1-\gamma)a^2 - \tfrac{4}{3}c^2\right]\mathrm{E} - \frac{a^5}{24}(1-\gamma)\mathrm{E} + \frac{ac^2}{2}\left[\tfrac{4}{3}(1+\gamma)c^2 - a^2\right]\mathrm{E}$$

$$+ \frac{\mathrm{E}a^3c^2}{2} - \tfrac{1}{3}(1+\gamma)\mathrm{E}ac^4 - (2\gamma+1)\sum_1^\infty \frac{8ac^4}{n^4\pi^4}\mathrm{E} + \sum_1^\infty \frac{\mathrm{C}}{k^2}\frac{\mathrm{I}_1}{\gamma}(-1)^n = 0.$$

Now

$$\sum_1^\infty (-1)^n \frac{\mathrm{C}\mathrm{I}_1}{k\gamma} = -\frac{a^5}{12}\mathrm{E}\zeta,$$

where

$$\zeta = \sum_{n=1}^\infty \frac{24}{\alpha^3}\,\frac{\left[(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right]\mathrm{I}_1^2 - (8\gamma+4)\mathrm{I}_0\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2 - \mathrm{I}_1^2(1+\gamma\alpha^2)} \quad . \quad . \quad . \quad . \quad (94),$$

so that ζ is a known constant.

We then find, putting in for $\sum_1^\infty \frac{1}{n^4}$ its value $\pi^4/90$,

$$u_0 = -\mathrm{E}\left[\left(\frac{1-\gamma-\zeta}{12}\right)a^4 - \tfrac{1}{6}a^2c^2 + \tfrac{1}{3}\left(1+\gamma-\tfrac{4}{15}\{2\gamma+1\}\right)c^4\right] \quad . \quad (95).$$

If now we write

$$\left.\begin{aligned} f &= (2\gamma-1)\frac{a^4}{6} + \tfrac{1}{3}c^2a^2 - \tfrac{14}{45}(2\gamma+1)c^4 \\ g &= \tfrac{1}{12}(1-\gamma-\zeta)a^4 - \tfrac{1}{6}a^2c^2 + \tfrac{1}{3}\left(1+\gamma-\tfrac{4}{15}\{2\gamma+1\}\right)c^4 \\ h &= -\tfrac{14}{15}c^4\gamma - \frac{a^4}{12}(2\gamma+1) + \frac{2a^2c^2\gamma}{3} \end{aligned}\right\} \quad . \quad . \quad (96),$$

so that f, g, h are known constants, then equations (90), (93), (95) may be re-written as follows :—

$$\left.\begin{aligned} 2\lambda u_0 + (\lambda+2\mu)w_0 + \mu f\mathrm{E} &= -\mathrm{Q} \\ u_0 \qquad\qquad\qquad + g\mathrm{E} &= 0 \\ 2(\lambda+\mu)u_0 + \lambda w_0 + \mu h\mathrm{E} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

Solving, we have :

$$\mu\mathrm{E} = \frac{-\mathrm{Q}(2\gamma-1)}{(h + (2\gamma-1)f + 2g(4\gamma-1))} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98),$$

$$u_0 = \frac{\mathrm{Q}g(2\gamma-1)}{\mu(h + (2\gamma-1)f + 2g(4\gamma-1))} \quad . \quad . \quad . \quad . \quad . \quad . \quad (99),$$

$$w_0 = -\frac{Q}{\mu}\,\frac{h(1-\gamma)+2g\gamma}{h+(2\gamma-1)f+2g(4\gamma-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (100).$$

All the constants are therefore absolutely determinate and the solution is complete.

§ 16. *Expressions for the Stresses*

The reduced expressions for the four stresses are given below :

$$\widehat{rr} = \mu E \left(\begin{aligned} & -\left\{\frac{2\gamma+1}{12}a^4 - \frac{2\gamma}{3}a^2c^2 + \tfrac{14}{15}\gamma c^4\right\} + r^2\left\{(1+\gamma)\frac{a^2}{4} - (4\gamma+1)\frac{c^2}{3}\right\} \\ & + z^2 \times 2\gamma[2c^2-a^2] - r^4\left(\frac{\gamma+2}{12}\right) + r^2z^2(4\gamma+1) - 2\gamma z^4 \\ & + \frac{16ac^3}{\pi^3}(2\gamma+1)\sum_1^\infty \frac{(-1)^n}{n^3 I_1(\alpha)}\left(\frac{I_1(\rho)}{\rho} - I_0(\rho)\right)\cos\frac{n\pi z}{c} \\ & - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\,\frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma a^2(I_0^2 - I_1^2) - I_1^2} \\ & \qquad \left[\left(\frac{\alpha I_0}{I_1}+1\right)I_0(\rho) - \left(\frac{\alpha I_0}{I_1}+\frac{1}{\gamma}\right)\frac{I_1(\rho)}{\rho} - \rho I_1(\rho)\right]\cos\frac{n\pi z}{c} \end{aligned} \right) \quad (101).$$

$$\begin{aligned} \widehat{zz} = & -Q \\ & + \mu E \left\{ \begin{aligned} & -\left\{\frac{2\gamma-1}{6}a^4 + \tfrac{1}{3}a^2c^2 - \tfrac{14}{45}(2\gamma+1)c^4\right\} + r^2\left\{\frac{(2\gamma-1)}{2}a^2 + \frac{2c^2}{3}\right\} \\ & - z^2\left(\tfrac{4}{3}(2\gamma+1)c^2 - a^2\right) - r^4\frac{2\gamma-1}{4} - 2r^2z^2 + \tfrac{2}{3}(2\gamma+1)z^4 \\ & + \frac{16ac^3}{\pi^3}(2\gamma+1)\sum_1^\infty \frac{(-1)^n}{n^3}\,\frac{I_0(\rho)}{I_1(\alpha)}\cos\frac{n\pi z}{c} \\ & - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\,\frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2} \\ & \qquad \left[\rho I_1(\rho) - I_0(\rho)\left\{\frac{\alpha I_0}{I_1} - 2\right\}\right]\cos\frac{n\pi z}{c} \end{aligned} \right\} \end{aligned} \quad . \quad (102).$$

$$\widehat{\phi\phi} = \mu E \left\{ \begin{aligned} & -\left\{\frac{2\gamma+1}{12}a^4 - \frac{2\gamma}{3}a^2c^2 + \tfrac{14}{15}\gamma c^4\right\} + r^2\left\{(3\gamma-1)\frac{a^2}{4}\right. \\ & \qquad \left. - (4\gamma-1)\frac{c^2}{3}\right\} + 2\gamma z^2\{2c^2 - a^2\} \\ & - r^4\left(\frac{5\gamma-2}{12}\right) + r^2z^2(4\gamma-1) - 2\gamma z^4 \\ & \qquad - \frac{16ac^3}{\pi^3}(2\gamma+1)\sum_1^\infty \frac{(-1)^n}{n^3}\,\frac{I_1(\rho)}{\rho I_1(\alpha)}\cos\frac{n\pi z}{c} \\ & - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\,\frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2} \\ & \qquad \left[\frac{I_1(\rho)}{\rho}\left(\frac{\alpha I_0}{I_1}+\frac{1}{\gamma}\right) - I_0(\rho)\left(\frac{1}{\gamma} - 1\right)\right]\cos\frac{n\pi z}{c} \end{aligned} \right\} \quad . \quad (103).$$

$$\widehat{rz} = \mu E \left\{ \begin{array}{l} \left\{\frac{4}{3}(2\gamma+1)c^2 - a^2\right\}rz + r^3 z - \frac{4}{3}(2\gamma+1)rz^3 \\ + (2\gamma+1)\frac{16ac^3}{\pi^3}\sum_1^\infty \frac{(-1)^n}{n^3}\frac{I_1(\rho)}{I_1(\alpha)}\sin\frac{n\pi z}{c} \\ - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2} \\ \qquad\qquad \left[\rho I_0(\rho) - \alpha\frac{I_0}{I_1}I_1(\rho)\right]\sin\frac{n\pi z}{c} \end{array} \right\} \quad . \quad (104),$$

where, putting in for f, g, h in (98) their values,

$$E = -\frac{Q}{\mu}\frac{2\gamma - 1}{(4\gamma-1)\left(\frac{8}{45}c^4 - \frac{a^4\zeta}{6}\right) - \frac{a^4}{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105).$$

§ 17. *Numerical Example.*

Let us now consider a concrete example. Take a cylinder whose diameter is nearly equal to its length. This corresponds about to the dimensions used in practice for test pieces under pressure. We will take $\pi a/c = 3$ in order to simplify the calculation of the I-functions.

Further, we shall assume uniconstant isotropy, so that $\gamma = 2/3$.

We then find :

$$\left. \begin{array}{lll} u_0 = -\frac{Q}{\mu}(\cdot 10695), & u_1 = -\frac{Q}{\mu a^2}(\cdot 481906), & u_2 = -\frac{Q}{\mu a^4}(\cdot 079057), \\ w_0 = -\frac{Q}{\mu}(\cdot 090552), & w_1 = -\frac{Q}{\mu a^2}(1\cdot 46046), & w_2 = \frac{Q}{\mu a^4}(1\cdot 01193), \\ E = \frac{Q}{\mu a^4}(1\cdot 13842), & E' = -\frac{Q}{\mu a^4}(1\cdot 01193), & F = -\frac{Q}{\mu a^4}(2\cdot 52982), \\ & D = \frac{Q}{\mu a^2}(1\cdot 63584), & D' = \frac{Q}{\mu a^2}(1\cdot 10970), \\ & \zeta = 2\cdot 036847, & \\ f = -a^4(\cdot 451889), & g = a^4(\cdot 093947), & h = -a^4(\cdot 455329) \end{array} \right\} \quad . \quad (106),$$

and the stresses become :

$$\left.\begin{aligned}
\frac{\widehat{rr}}{\mu\mathrm{E}} &= -\cdot455329a^4 - \cdot923650r^2a^2 + 1\cdot590994z^2a^2 \\
&\quad - \tfrac{2}{9}r^4 + \tfrac{11}{3}r^2z^2 - \tfrac{4}{3}z^4 + \tfrac{112}{81}a^4\sum_1^\infty\frac{(-1)^n}{n^3}c_n\cos\frac{n\pi z}{c} \\
&\quad - \frac{8a^4}{27}\sum_1^\infty\frac{(-1)^n}{n^2}e_n\cos\frac{n\pi z}{c} \\
\frac{\widehat{zz}+\mathrm{Q}}{\mu\mathrm{E}} &= \cdot451889a^4 + \cdot897748r^2a^2 - 2\cdot411716z^2a^2 \\
&\quad - \frac{r^4}{12} - 2r^2z^2 + \tfrac{14}{9}z^4 + \tfrac{112}{81}a^4\sum_1^\infty\frac{(-1)^n}{n^3}f_n\cos\frac{n\pi z}{c} \\
&\quad - \frac{8a^4}{27}\sum_1^\infty\frac{(-1)^n}{n^2}g_n\cos\frac{n\pi z}{c} \\
\frac{\widehat{\phi\phi}}{\mu\mathrm{E}} &= -\cdot455329a^4 - \cdot359235r^2a^2 + 1\cdot590994z^2a^2 - \frac{r^4}{9} \\
&\quad + \frac{5r^2z^2}{3} - \tfrac{4}{3}z^4 - \tfrac{112}{81}a^4\sum_1^\infty\frac{(-1)^n}{n^3}h_n\cos\frac{n\pi z}{c} - \frac{8a^4}{27}\sum_1^\infty\frac{(-1)^n}{n^2}l_n\cos\frac{n\pi z}{c} \\
\frac{\widehat{rz}}{\mu\mathrm{E}} &= 2\cdot411716a^2rz + r^3z - \tfrac{28}{9}rz^3 + \tfrac{112}{81}a^4\sum_1^\infty\frac{(-1)^n}{n^3}p_n\sin\frac{n\pi z}{c} \\
&\quad - \tfrac{8}{27}a^4\sum_1^\infty\frac{(-1)^n}{n^2}q_n\sin\frac{n\pi z}{c}
\end{aligned}\right\} \quad . \ . \ (106a),$$

where

$$\left.\begin{aligned}
c_n &= \left\{\frac{\mathrm{I}_1(\rho)}{\rho} - \mathrm{I}_0(\rho)\right\}/\,\mathrm{I}_1(\alpha) \\
e_n &= \frac{\left[\left(7n+\dfrac{76}{9n}\right)\mathrm{I}_1(\alpha) - \tfrac{28}{3}\mathrm{I}_0(\alpha)\right]\left[\left(\dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1}+1\right)\mathrm{I}_0(\rho) - \left(\dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1}+\dfrac{1}{\gamma}\right)\dfrac{\mathrm{I}_1(\rho)}{\rho} - \mathrm{I}_1(\rho)\right]}{6n^2(\mathrm{I}_0^2-\mathrm{I}_1^2)-\mathrm{I}_1^2} \\
f_n &= \mathrm{I}_0(\rho)\,/\,\mathrm{I}_1(\alpha) \\
g_n &= \frac{\left[\left(7n+\dfrac{76}{9n}\right)\mathrm{I}_1 - \tfrac{28}{3}\mathrm{I}_0\right]\left[\rho\mathrm{I}_1(\rho) - \mathrm{I}_0(\rho)\left\{\dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1}-2\right\}\right]}{6n^2(\mathrm{I}_0^2-\mathrm{I}_1^2)-\mathrm{I}_1^2} \\
h_n &= \mathrm{I}_1(\rho)\,/\,\rho\mathrm{I}_1(\alpha) \\
l_n &= \frac{\left[\left(7n+\dfrac{76}{9n}\right)\mathrm{I}_1 - \tfrac{28}{3}\mathrm{I}_0\right]\left[\dfrac{\mathrm{I}_1(\rho)}{\rho}\left(\dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1}+\dfrac{1}{\gamma}\right) - \left(\dfrac{1}{\gamma}-1\right)\mathrm{I}_0(\rho)\right]}{6n^2(\mathrm{I}_0^2-\mathrm{I}_1^2)-\mathrm{I}_1^2} \\
p_n &= \mathrm{I}_1(\rho)/\mathrm{I}_1(\alpha) \\
q_n &= \frac{\left[\left(7n+\dfrac{76}{9n}\right)\mathrm{I}_1 - \tfrac{28}{3}\mathrm{I}_0\right]\left[\rho\mathrm{I}_0(\rho) - \dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1}\mathrm{I}_1(\rho)\right]}{6n^2(\mathrm{I}_0^2-\mathrm{I}_1^2)-\mathrm{I}_1^2}
\end{aligned}\right\} \quad . \ (106b)$$

§ 18. *Tables of the Constants for the special case taken.*

The values of these constants I have calculated for the values 1 to 6 of n and the values $\rho = 0$, $\alpha/3$, $2\alpha/3$, α, *i.e.*, remembering that in our case $\alpha = 3n$, for $\rho = 0$, n, $2n$, $3n$. These values are given in the following tables:—

TABLE of Constants.

c_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$-\cdot 126474$	$-\cdot 177294$	$-\cdot 375444$	$-\cdot 901257$
2	$-\cdot 815103 \times 10^{-2}$	$-\cdot 241966 \times 10^{-1}$	$-\cdot 144470$	$-\cdot 929393$
3	$-\cdot 485003 \times 10^{-3}$	$-\cdot 345613 \times 10^{-2}$	$-\cdot 553007 \times 10^{-1}$	$-\cdot 949670$
4	$-\cdot 275613 \times 10^{-4}$	$-\cdot 488500 \times 10^{-3}$	$-\cdot 208129 \times 10^{-1}$	$-\cdot 961180$
5	$-\cdot 152381 \times 10^{-5}$	$-\cdot 681835 \times 10^{-4}$	$-\cdot 776521 \times 10^{-2}$	$-\cdot 968455$
6	$-\cdot 825385 \times 10^{-7}$	$-\cdot 943397 \times 10^{-5}$	$-\cdot 288544 \times 10^{-2}$	$-\cdot 973449$

e_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$\cdot 971897$	$1\cdot 132478$	$1\cdot 573394$	$1\cdot 960798$
2	$\cdot 120292$	$\cdot 260646$	$\cdot 914018$	$1\cdot 998081$
3	$\cdot 117631 \times 10^{-1}$	$\cdot 586808 \times 10^{-1}$	$\cdot 534190$	$2\cdot 319025$
4	$\cdot 945472 \times 10^{-3}$	$\cdot 115601 \times 10^{-1}$	$\cdot 273635$	$2\cdot 545291$
5	$\cdot 678980 \times 10^{-4}$	$\cdot 208131 \times 10^{-2}$	$\cdot 129351$	$2\cdot 702777$
6	$\cdot 454266 \times 10^{-5}$	$\cdot 353345 \times 10^{-3}$	$\cdot 582489 \times 10^{-1}$	$2\cdot 816948$

f_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$\cdot 252949$	$\cdot 320250$	$\cdot 576618$	$1\cdot 234590$
2	$\cdot 163021 \times 10^{-1}$	$\cdot 371619 \times 10^{-1}$	$\cdot 184245$	$1\cdot 096059$
3	$\cdot 970005 \times 10^{-3}$	$\cdot 473439 \times 10^{-2}$	$\cdot 652177 \times 10^{-1}$	$1\cdot 060779$
4	$\cdot 551226 \times 10^{-4}$	$\cdot 622991 \times 10^{-3}$	$\cdot 235682 \times 10^{-1}$	$1\cdot 044513$
5	$\cdot 304762 \times 10^{-5}$	$\cdot 830167 \times 10^{-4}$	$\cdot 857923 \times 10^{-2}$	$1\cdot 035120$
6	$\cdot 165477 \times 10^{-6}$	$\cdot 111258 \times 10^{-4}$	$\cdot 313561 \times 10^{-2}$	$1\cdot 029005$

g_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	−·787811	−·736099	−·324884	1·638888
2	−·155588	−·246518	−·431227	2·052233
3	−·176721 × 10^{-1}	−·584824 × 10^{-1}	−·326343	2·400197
4	−·152827 × 10^{-2}	−·116088 × 10^{-1}	−·189318	2·623837
5	−·114613 × 10^{-3}	−·209107 × 10^{-2}	−·963285 × 10^{-1}	2·774995
6	−·789128 × 10^{-5}	−·354776 × 10^{-3}	−·455547 × 10^{-1}	2·882627

h_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	·126474	·142956	·201175	·333333
2	·815103 × 10^{-2}	·129653 × 10^{-1}	·397748 × 10^{-1}	·166667
3	·485003 × 10^{-3}	·127826 × 10^{-2}	·991703 × 10^{-2}	·111111
4	·275613 × 10^{-4}	·134492 × 10^{-3}	·275530 × 10^{-2}	·853333 × 10^{-1}
5	·152381 × 10^{-5}	·148331 × 10^{-4}	·814018 × 10^{-3}	·666667 × 10^{-1}
6	·827385 × 10^{-7}	·169178 × 10^{-5}	·250165 × 10^{-3}	·555556 × 10^{-1}

l_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	·971897	1·067168	1·386651	2·042427
2	·120292	·179629	·477822	1·664308
3	·117631 × 10^{-1}	·283739 × 10^{-1}	·185745	1·682707
4	·945472 × 10^{-3}	·414783 × 10^{-2}	·707563 × 10^{-1}	1·703514
5	·678980 × 10^{-4}	·586774 × 10^{-3}	·266082 × 10^{-1}	1·716955
6	·454266 × 10^{-5}	·817120 × 10^{-4}	·993188 × 10^{-2}	1·725546

p_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	·000000	·142956	·402350	1·000000
2	·000000	·259307 × 10^{-1}	·159100	1·000000
3	·000000	·383479 × 10^{-2}	·595020 × 10^{-1}	1·000000
4	·000000	·537967 × 10^{-3}	·220423 × 10^{-1}	1·000000
5	·000000	·741658 × 10^{-4}	·814019 × 10^{-2}	1·000000
6	·000000	·101507 × 10^{-4}	·300198 × 10^{-2}	1·000000

q_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	·000000	$-\cdot382470$	$-\cdot615993$	·000000
2	·000000	$-\cdot200638$	$-\cdot645084$	·000000
3	·000000	$-\cdot540924 \times 10^{-1}$	$-\cdot426702$	·000000
4	·000000	$-\cdot111882 \times 10^{-1}$	$-\cdot230912$	·000000
5	·000000	$-\cdot204756 \times 10^{-2}$	$-\cdot112874$	·000000
6	·000000	$-\cdot349986 \times 10^{-3}$	$-\cdot518839 \times 10^{-1}$	·000000

The above, when substituted in the formulæ, give quite fairly rapid convergence when $r<a$, the convergency ratio being in this case less than unity by a finite amount. But when $r = a$, the series becomes comparable with the series $\Sigma\frac{(-1)^n}{n^i}\cos\frac{n\pi z}{c}$, where i is a positive integer, and, as in the first problem, a special procedure has to be adopted.

§ 19. *Methods of Evaluation at the Curved Boundary.*

When $r = a$, $\widehat{rr}$ and $\widehat{rz}$ are of course zero; but the stresses $\widehat{zz}$ and $\widehat{\phi\phi}$ require separate evalution.

Now, if we use the series for I_0 and I_1 in descending powers of the argument, the first few terms of these give a very good representation of the function when α is at all large, and will be quite sufficient for $\alpha > 18$, at which point the tables of the last paragraph stop.

Replacing I_0, I_1 by these series, we find

$$\left.\begin{aligned} f_n(\alpha) &= 1 + \frac{1}{2\alpha} + \frac{3}{8\alpha^2} + \frac{3}{8\alpha^3} + \ldots \\ c_n(\alpha) &= -\left(1 - \frac{1}{2\alpha} + \frac{3}{8\alpha^2} + \frac{3}{8\alpha^3} + \ldots\right) \\ l_n(\alpha) &= \tfrac{7}{4} - \frac{165}{16\alpha^2} + \frac{335}{8\alpha^3} + \ldots \\ e_n(\alpha) &= \tfrac{7}{2} - \frac{14}{\alpha} + \frac{31}{\alpha^2} - \frac{21}{4\alpha^3} - \frac{21}{4\alpha^4} + \ldots \\ g_n(\alpha) &= \tfrac{7}{2} - \frac{49}{4\alpha} + \frac{157}{8\alpha^2} + \frac{503}{32\alpha^3} + \ldots \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (107),$$

and

$$h\;(\alpha) = \frac{1}{\alpha}, \quad p_n(\alpha) = 1, \quad q_n(\alpha) = 0.$$

Now write

$$
\begin{aligned}
l_n'(\alpha) &= -\,l_n^{(4)}(\alpha) + l_n(\alpha),\\
g_n'(\alpha) &= -\,g_n^{(4)}(\alpha) + g_n(\alpha),\\
f_n'(\alpha) &= -f_n^{(4)}(\alpha) + f_n(\alpha),
\end{aligned}
$$

where $l_n^{(4)}(\alpha)$ denotes the first four terms of the above series for $l_n(\alpha)$, with a similar meaning for $g_n^{4}(\alpha)$ and $f_n^{(4)}(\alpha)$.

Then if we substitute in the equations for $\widehat{zz} + Q$, $\widehat{\phi\phi}$ we find:

$$
\frac{(\widehat{zz} + Q)_{r=a}}{\mu E} = 1{\cdot}266304\, a^4 - 4{\cdot}411716\, a^2 z^2 + \tfrac{14}{9} z^4
$$

$$
+ a^4 \times \tfrac{8}{27} \left\{ \begin{array}{l} -\tfrac{7}{2} \sum\limits_1^{\infty} \dfrac{(-1)^n}{n^2} \cos \dfrac{n\pi z}{c} - \tfrac{101}{72} \sum\limits_1^{\infty} \dfrac{(-1)^n}{n^4} \cos \dfrac{n\pi z}{c} \\ \quad + \tfrac{7}{108} \sum\limits_1^{\infty} \dfrac{(-1)^n}{n^6} \cos \dfrac{n\pi z}{c} \end{array} \right\}
$$

$$
+ \frac{8a^4}{27} \sum_1^{\infty} (-1)^n \cos \frac{n\pi z}{c} \left\{ \tfrac{14}{3} \frac{f_n'}{n^3} - \frac{g_n'}{n^2} \right\} + \frac{8a^4}{27} \sum_1^{\infty} \; - 1)^n \cos \frac{n\pi z}{c} \left\{ \tfrac{35}{4} \frac{1}{n^3} - \tfrac{335}{864} \frac{1}{n^5} \right\}.
$$

Now remembering that

$$
\begin{aligned}
\sum_1^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi z}{c} &= \frac{\pi^2}{4} \frac{z^2}{c^2} - \frac{\pi^2}{12}\\
\sum_1^{\infty} \frac{(-1)^n}{n^4} \cos \frac{n\pi z}{c} &= -\frac{z^4}{c^4} \frac{\pi^4}{48} + \frac{z^2}{c^2} \frac{\pi^4}{24} - \frac{7\pi^4}{720}\\
\sum_1^{\infty} \frac{(-1)^n}{n^6} \cos \frac{n\pi z}{c} &= \frac{z^6}{c^6} \frac{\pi^6}{1440} - \frac{z^4}{c^4} \frac{\pi^6}{288} + \frac{z^2}{c^2} \frac{7\pi^6}{1440} - \frac{31\pi^6}{30240},
\end{aligned}
$$

we find, using $\pi a/c = 3$,

$$
\begin{aligned}
\frac{(\widehat{zz} + Q)_{r=a}}{\mu E a^4} = 2{\cdot}493930 & \\
- 8{\cdot}201525 \frac{z^2}{a^2} &+ 2{\cdot}251614 \frac{z^4}{a^4} + {\cdot}00972222 \frac{z^6}{a^6}\\
+ \sum_1^{\infty} (-1)^n \cos \frac{n\pi z}{c} &\left\{ \tfrac{112}{81} \frac{f_n'}{n^3} - \tfrac{8}{27} \frac{g_n'}{n^2} + \tfrac{70}{27} \frac{1}{n^3} - \tfrac{335}{2916} \frac{1}{n^5} \right\} \quad . \quad . \quad . \quad (108).
\end{aligned}
$$

And, in a similar manner,

$$
\begin{aligned}
\frac{(\widehat{\phi\phi})_{r=a}}{\mu E a^4} &= -\,{\cdot}3842415 + 1{\cdot}6416832 \frac{z^2}{a^2} - 1{\cdot}1284672 \frac{z^4}{a^4}\\
&\quad - \sum_1^{\infty} (-1)^n \cos \frac{n\pi z}{c} \left\{ \tfrac{8}{27} \frac{l_n'}{n^2} + \tfrac{335}{729} \frac{1}{n^5} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathbf{(109)}.
\end{aligned}
$$

§ 20. *Calculation of the Series in the preceding Section.*

If we work out the values of f_n', g_n', l_n' we find they are as tabulated below :

n.	f_n'.	g_n'.	l_n'.
1	+ ·012367	− ·540511	− ·112667
2	+ ·000573	− ·024011	+ ·006901
3	+ ·000078	− ·002538	+ ·002580
4	+ ·000025	− ·001259	+ ·000896
5	+ ·000009	− ·000217	+ ·000381
6	+ ·000005	− ·000083	+ ·000195

Hence the parts of the Σ which depend upon $l_n'/n^2, f_n'/n^3, g_n'/n^2$ converge quite rapidly enough to allow us to stop after the sixth term. It therefore merely remains to evaluate the series

$$\sum_1^\infty \frac{(-1)^n}{n^3} \cos \frac{n\pi z}{c} \quad \text{and} \quad \sum_1^\infty \frac{(-1)^n}{n^5} \cos \frac{n\pi z}{c}.$$

These cannot be expressed in finite terms, and although we may apply the EULER-MACLAURIN sum-formula to these series directly, though in a slightly modified form, this sum-formula is not really of very great advantage, as its rapidity of convergence depends on z, and is such, for certain values of this variable, as to render the formula useless as an approximation to the remainder. As a matter of fact, however, the series were to be calculated only for values of $z = ic/6$, i being any integer from 0 to 6. But for such values of z, the cosine terms repeat themselves after $n = 6$.

Thus,

$$\sum_1^\infty \frac{(-1)^{n-1}}{n^3} \cos\left(\frac{n\pi i}{6}\right)$$

$$= \cos i\pi/6 \left(\sum_{m=0}^{m=\infty} \frac{1}{(1+12m)^3} + (-1)^i \sum_{m=0}^{m=\infty} \frac{1}{(7+12m)^3}\right)$$

$$- \cos 2i\pi/6 \left(\sum_{m=0}^{m=\infty} \frac{1}{(2+12m)^3} + (-1)^i \sum_{m=0}^{m=\infty} \frac{1}{(8+12m)^3}\right)$$

$$+ \cdot \cdot \cdot \cdot \cdot$$

$$- \cos i\pi \left(\sum_{m=0}^{m=\infty} \frac{1}{(6+12m)^3} + (-1)^i \sum_{m=0}^{m=\infty} \frac{1}{(12+12m)^3}\right).$$

A precisely similar formula holds for

$$\sum_1^\infty \frac{(-1)^{n-1}}{n^5} \cos\left(\frac{n\pi i}{6}\right).$$

Thus we see we need only work out the series

$$\sum_{m=0}^{m=\infty} \frac{1}{(s+12m)^3} \quad \text{and} \quad \sum_{m=0}^{m=\infty} \frac{1}{(s+12m)^5},$$

where s has integral values ranging from 1 to 12.

These series are easily calculated, and, to them, the sum-formula is quite applicable.

By this means it was found that

$$\begin{aligned}
&\sum_1^\infty \frac{(-1)^{n-1}}{n^5} \cos\left(\frac{n\pi i}{6}\right) \\
&\quad = \cos\frac{i\pi}{6} \quad (1{\cdot}000{,}0027 + (-1)^i \cdot 000{,}0598) \\
&\quad - \cos 2i\pi/6\,({\cdot}031{,}2519 + (-1)^i \cdot 000{,}0308) \\
&\quad + \cos 3i\pi/6\,({\cdot}004{,}1165 + (-1)^i \cdot 000{,}0171) \\
&\quad - \cos 4i\pi/6\,({\cdot}000{,}9776 + (-1)^i \cdot 000{,}0102) \\
&\quad + \cos 5i\pi/6\,({\cdot}000{,}3207 + (-1)^i \cdot 000{,}0064) \\
&\quad - \cos i\pi \quad ({\cdot}000{,}1291 + (-1)^i \cdot 000{,}0041)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_1^\infty \frac{(-1)^{n-1}}{n^3} \cos\left(\frac{n\pi i}{6}\right) \\
&\quad = \cos i\pi/6\,(1{\cdot}000{,}5611 + (-1)^i \cdot 003{,}1246) \\
&\quad - \cos 2i\pi/6\,({\cdot}125{,}4607 + (-1)^i \cdot 002{,}1368) \\
&\quad + \cos 3i\pi/6\,({\cdot}037{,}4212 + (-1)^i \cdot 001{,}5343) \\
&\quad - \cos 4i\pi/6\,({\cdot}015{,}9496 + (-1)^i \cdot 001{,}1448) \\
&\quad + \cos 5i\pi/6\,({\cdot}008{,}2777 + (-1)^i \cdot 000{,}8811) \\
&\quad - \cos i\pi \quad ({\cdot}004{,}8694 + (-1)^i \cdot 000{,}6956),
\end{aligned}$$

and the calculation of the stresses on the boundary then became a simple matter.

§ 21. *Numerical Values of the Stresses.*

The values of the stresses, referred to the mean pressure as unit, are tabulated below:

TABLE of Stresses.

$\widehat{zz}/Q$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	1·13382	-1·13436	-1·13322	-1·12145	-1·08000	-1·03372	-·68576
$a/3$	-1·09971	-1·10053	-1·10017	-1·08998	-1·05441	-·96416	-·74952
$2a/3$	-1·00724	-1·00896	-1·01314	-1·01553	-1·00557	-·97136	-·92845
$5a/6$	-·94841	-·94843	-·95056	-·96037	-·98465	-1·01726	-1·08211
a	-·89430	-·88809	-·87216	-·85845	-·88177	-1·04077	-1·68635

$\widehat{rr}/Q$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	-·00274	-·01414	-·05134	-·12416	-·25325	-·48151	-·89668
$a/3$	-·00181	-·01118	-·04201	-·10220	-·20471	-·37182	-·65935
$2a/3$	-·00110	-·00459	-·01759	-·04481	-·09244	-·14271	-·09977
a	·00000	·00000	·00000	·00000	·00000	·00000	·00000

$\widehat{\phi\phi}/Q$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	-·00274	-·01414	-·05134	-·12416	-·25325	-·48151	-·89668
$a/3$	·00299	-·00737	-·04126	-·10748	-·22288	-·42120	-·77728
$2a/3$	·01744	·01017	-·01461	-·06480	-·15199	-·28607	-·48160
a	·03265	·03013	·02030	-·00516	-·06344	-·18728	-·43801

$\widehat{rz}/Q$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	·00000	·00000	·00000	·00000	·00000	·00000	·00000
$a/3$	·00000	·00705	-·00388	-·02020	-·05902	-·14798	-·35357
$2a/3$	·00000	·00425	-·00171	-·02712	-·08645	-·19750	-·44154
a	·00000	·00000	·00000	·00000	·00000	·00000	·00000

In the above the values of $\widehat{zz}$ for an additional value of r (viz., $r = 5a/6$) have been computed in order to exhibit more clearly the variation in the pressure along the radius.

The numerical results here given are shown graphically in Diagrams 7–10. We see at once that, save near the ends, the stresses $\widehat{rr}$, $\widehat{\phi\phi}$, and $\widehat{rz}$ do not differ very much from zero, which is the value they should have on the uniform pressure hypothesis. On the other hand, the axial pressure deviates throughout from uniformity over the cross-section, the total variation in any section remaining tolerably constant over nearly two-thirds of the length of the cylinder, and equal to about 25 per cent. of the mean pressure.

Diagram 7.—Showing Stress $\widehat{zz}$ for Cylinder compressed between Rough Planes (second example).

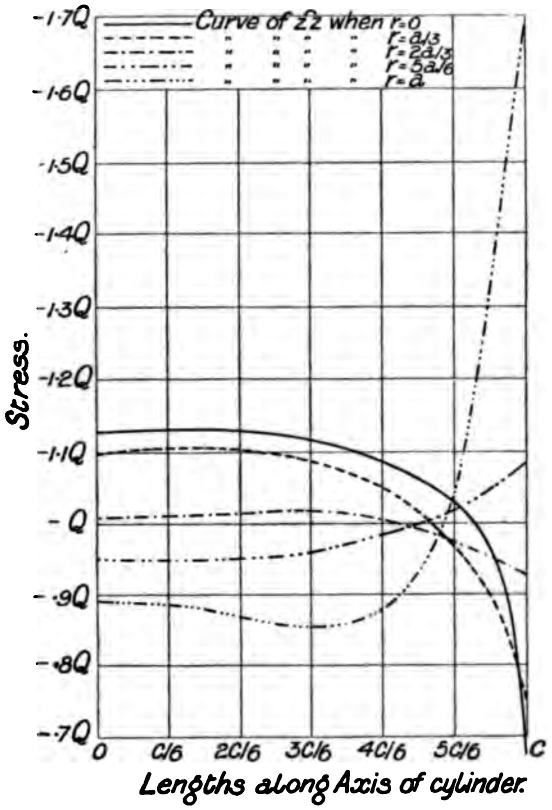
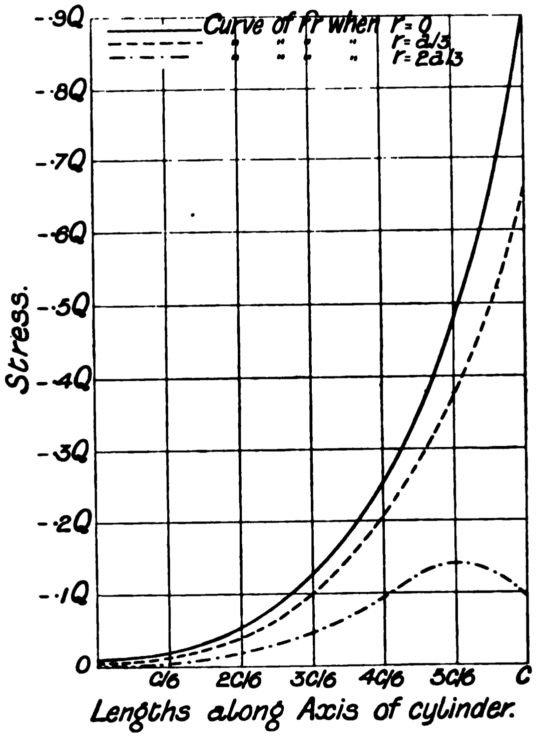


Diagram 8.—Showing Stress $\widehat{rr}$ for Cylinder compressed between Rough Planes (second example).



We notice also, that, near the centre of the cylinder, the greatest pressures occur at the centre of the cross-section; whereas the reverse takes place at the ends, the pressure at the perimeter of the ends amounting to about $1\frac{2}{3}$ times the mean pressure.

If we bear in mind the suggestion of the first problem of the present paper, that surface shear depresses those parts of the material towards which it acts, it is easy to see, physically, why such a distribution of pressure should be expected in practice.

The system of frictional shears required to prevent the ends expanding will be towards the centre: the parts of the material round the centre will therefore be depressed, and the compressing planes (supposed rigid) will have to compress the outer portions more than the inner, if the cross-section is to retain its original plane form, *i.e.*, remain in contact with the compressing planes throughout. It is thus not surprising that the greatest pressure should be at the perimeter, being in fact nearly $2\frac{1}{2}$ times the pressure at the centre.

Diagram 9.—Showing Stress $\widehat{\phi\phi}$ for Cylinder compressed between Rough Planes (second example).

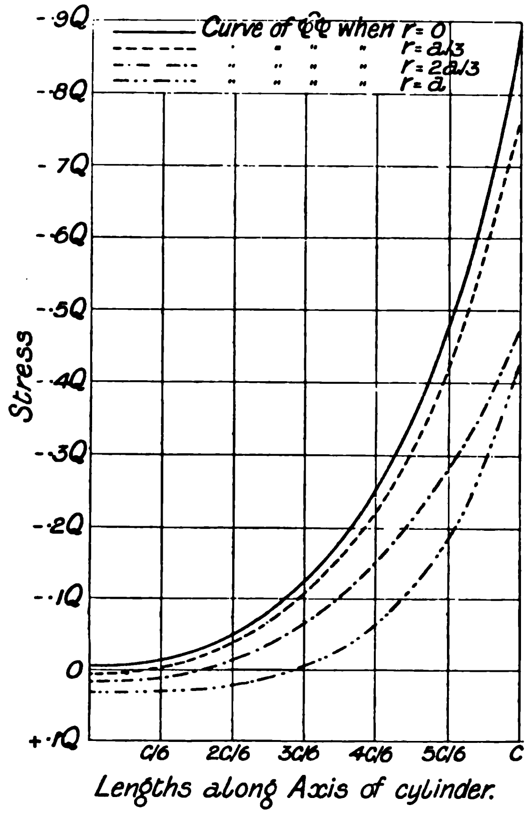
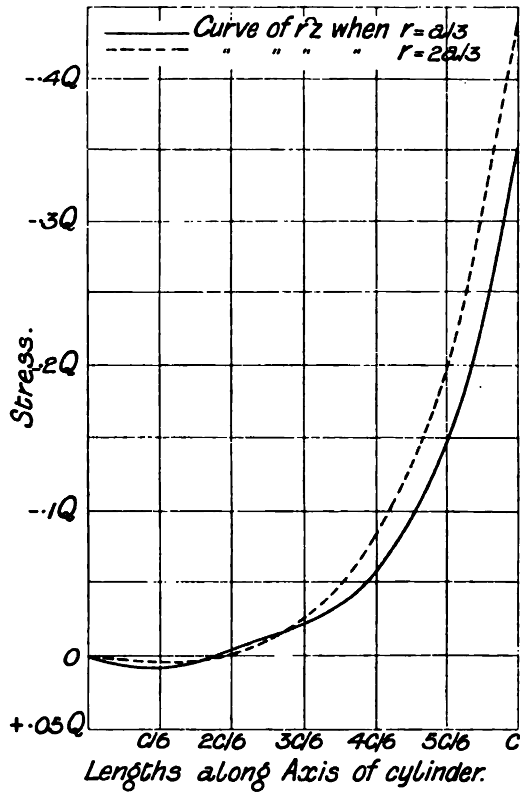


Diagram 10.—Showing Stress $\widehat{rz}$ for Cylinder compressed between Rough Planes (second example).



We see also that, near the mid-sections, the cross-radial traction changes from a pressure to a tension as we go towards the circumference, so that the outer parts of the material are subject to two pressures, parallel to the axis and the radius respectively and to a tension, at right angles to these. It should be noticed here that, in the diagrams, the ordinates representing the stresses increase negatively upwards. This has been found convenient in this case, where, owing to the general predominance of pressures, the greater part of the stresses have the negative sign.

§ 22. *Principal Stresses at each Point. Lines of Principal Stress.*

Now, when we have a distribution of stress

$$\widehat{rr},\ \widehat{\phi\phi},\ \widehat{zz},\ \widehat{rz},$$

which is symmetrical about an axis, then the principal stresses are

$$\widehat{\mathrm{RR}},\ \widehat{\phi\phi},\ \widehat{\mathrm{ZZ}};$$

where $\widehat{\phi\phi}$ is the same as before, and $\widehat{\mathrm{RR}}$, $\widehat{\mathrm{ZZ}}$ are two tractions in the meridian plane, $\widehat{\mathrm{RR}}$ making an angle θ with $\widehat{rr}$, where

$$\tan 2\theta = \frac{2\widehat{zr}}{\widehat{rr} - \widehat{zz}} \quad \text{. . . . . (110)},$$

and the values of $\widehat{\mathrm{RR}}$ and $\widehat{\mathrm{ZZ}}$ are given by

$$\left.\begin{aligned} \widehat{\mathrm{RR}} &= \frac{\widehat{rr} + \widehat{zz}}{2} + \tfrac{1}{2}\sqrt{(\widehat{rr} - \widehat{zz})^2 + 4(\widehat{zr})^2} \\ \widehat{\mathrm{ZZ}} &= \frac{\widehat{rr} + \widehat{zz}}{2} - \tfrac{1}{2}\sqrt{(\widehat{rr} - \widehat{zz})^2 + 4(\widehat{zr})^2} \end{aligned}\right\} \quad \text{. . . . . (111)}.$$

Whence, using the tables in § 21, we find the following values for $\widehat{\mathrm{RR}}$ and $\widehat{\mathrm{ZZ}}$, compared with the mean pressure over the ends.

TABLE of Principal Stresses.

$\widehat{\mathrm{RR}}/\mathrm{Q}$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	- ·00274	- ·01414	- ·05134	- ·12416	- ·25325	- ·48151	- ·89668
$a/3$	- ·00181	- ·01113	- ·04199	- ·10178	- ·20063	- ·33690	- ·34800
$2a/3$	- ·00110	- ·00457	- ·01758	- ·04405	- ·08433	- ·09804	·09139
a	·00000	·00000	·00000	·00000	·00000	·00000	·00000

$\widehat{\mathrm{ZZ}}/\mathrm{Q}$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	- 1·13382	- 1·13436	- 1·13322	- 1·12146	- 1·08000	- 1·03372	- 0·68576
$a/3$	- 1·09971	- 1·10057	- 1·10018	- 1·09039	- 1·05849	- 0·99907	- 1·06087
$2a/3$	- 1·00724	- 1·00898	- 1·01314	- 1·01628	- 1·01467	- 1·01602	- 1·11961
a	- 0·89430	- 0·88809	- 0·87216	- 0·85845	- 0·88177	- 1·04077	- 1·68635

The values of tan θ for the same points are given in the following table :—

TABLE of Slope of Lines of Principal Stress (tan θ).

r.	$z = 0$.	$z = c/6$.	$z = 2z/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	0	0	0	0	0	0	0
$a/3$	0	·0064	− ·0038	− ·0204	− ·0690	− ·2358	− ·8806
$2a/3$	0	·0044	− ·0017	− ·0279	− ·0936	− ·2303	− ·4331
a	0	0	0	0	0	0	0

By the aid of the above, we may draw the lines of principal stress, which is done in Diagram 11, the slope being exaggerated in the ratio 10 : 1. In order to do so, we suppose the line of principal stress to remain always in the neighbourhood of the same generator, so that, in the above table, the values in any row apply to the same line of principal stress. This of course is not correct near the ends, but it is sufficient for our purpose, the diagram being merely intended to show the general course of the lines of stress. These are sensibly parallel to the generators throughout the middle part of the cylinder, but slope outwards near the ends.

Diagram 11.—Lines of principal Stress for Cylinder compressed between Rough Planes (second example)

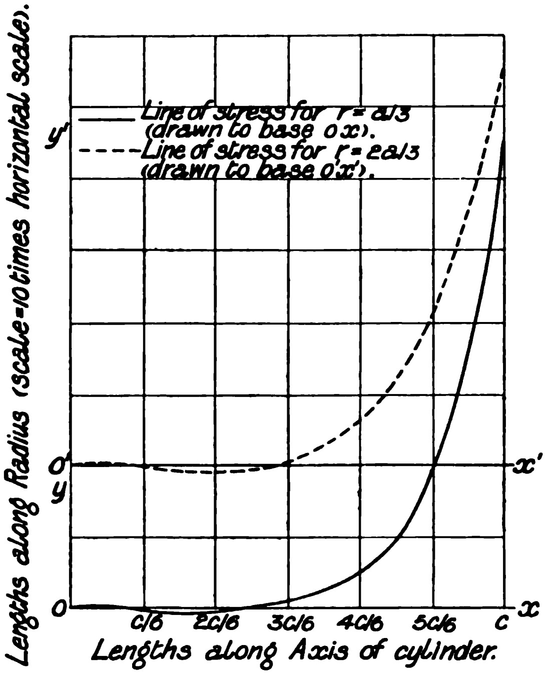


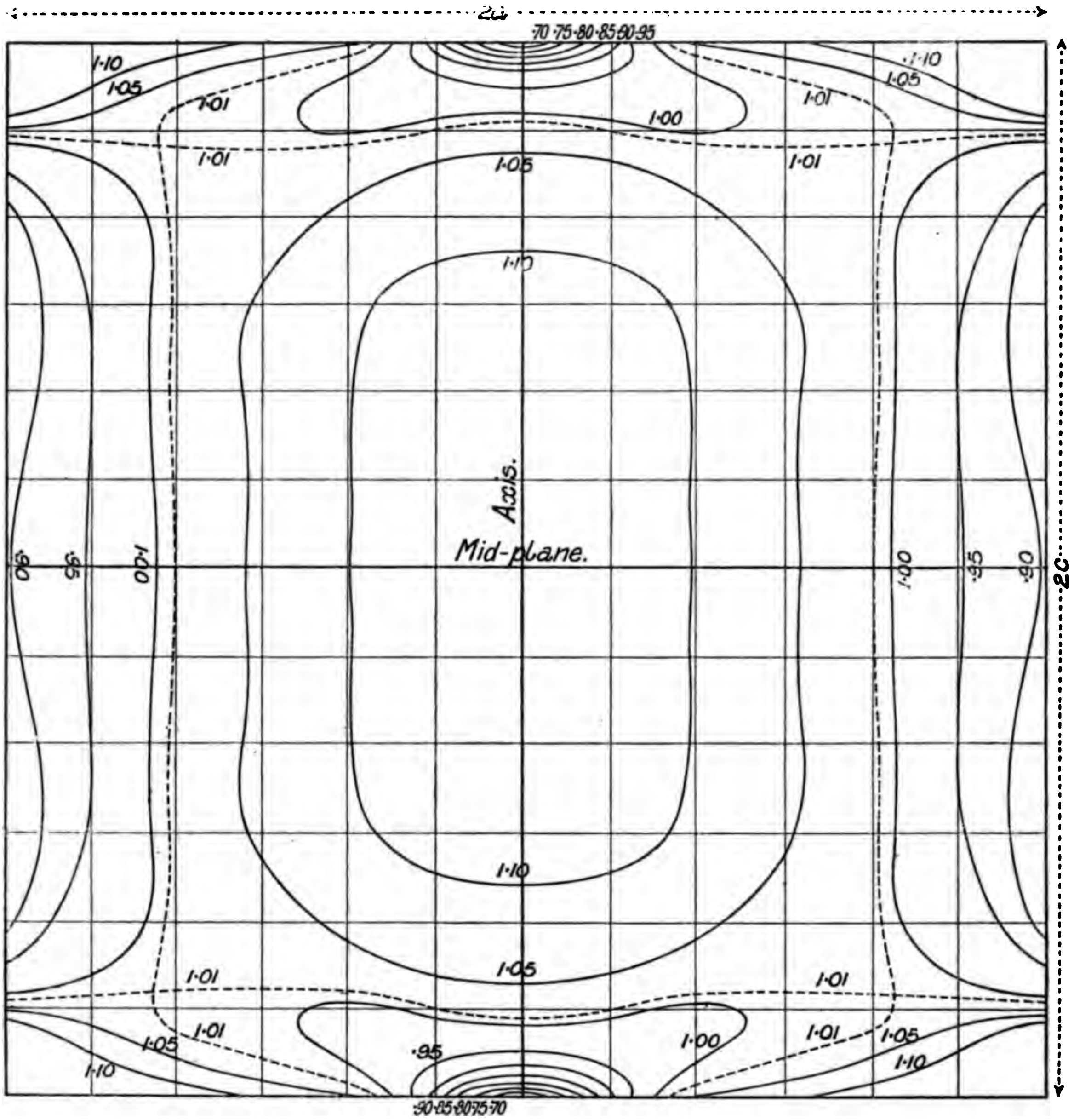
Diagram 12* shows the distribution of the lines of equal principal stress ZZ

inside the cylinder. I have chosen the stress $\widehat{ZZ}$ in preference to the others, as, except in the neighbourhood of the centres of the plane ends, it is everywhere the

---

* In Diagram 12, as in several others, a is represented as having the same value as c, so that the horizontal and vertical scales are not strictly the same, but differ in the ratio $\pi/3$. This has been done for convenience in plotting.

Diagram 12.—Distribution of Principal Stress $\widehat{ZZ}$ inside the Cylinder (case of compression between Rough Rigid Planes).



The lines are drawn at intervals of ·05 Q of the stress. The critical line corresponds to $\widehat{ZZ}$ = 1·01 Q nearly

greatest of the three principal stresses, and therefore $\widehat{ZZ}$ will be the important quantity when we come to discuss the maximum stress.

The diagram has been constructed by careful interpolation from the values tabulated above, and from it we see that the surfaces of equal principal stress in the cylinder are in general made up of three sheets, and they fall into two classes: (a) those for which $\widehat{ZZ}$ has a value less than a certain critical value, which, as nearly as I can find out from graphical methods, is about 1·01 Q, and (b) those for which $\widehat{ZZ}$ has a value greater than 1·01 Q.

The surfaces (a) consist of two solid caps or buttons, round the centres of the end sections, together with a hollow cylindrical shell surrounding the middle of the cylinder. For values sensibly $<$ ·9 Q the latter sheet disappears, and only the caps remain, their volume gradually dwindling down to zero as $\widehat{ZZ}$ falls to ·686 Q.

The surfaces (b) consist of an elongated core, resembling a cylinder closed by curved ends, surrounding the centre of the compressed block, together with two annuli at the ends, as shown in the figure referred to.

The critical surface $\widehat{ZZ}$ 1·01 Q consists of two nearly plane sheets, roughly coinciding with the cross-sections $Z = \pm 5c/6$, and one cylindrical sheet, which bends inwards towards the end, though without completely closing in, and which roughly coincides with the cylinder $r = 2a/3$ over the greater part of its surface.

§ 23. *Application to Rupture. Distribution of Maximum Stress, Strain, and Stress Difference.*

In considering what happens when a material breaks, we have to ask, first of all, whether it be brittle or ductile. In the first case, the law of stress to strain will be approximately linear up to the point where rupture takes place; in the second case, the stress-strain relation remains approximately linear until a point is reached (called the yield-point) at which a large and sudden change occurs in the stress-strain curve, after which the material becomes sensibly plastic, so that rupture finally takes place after a large permanent deformation.

In applying an elastic theory to practice, we can, in strictness, treat of rupture only in the case of a brittle solid. Even then it has to be borne in mind that the mathematical theory of strains—upon which the equations of elasticity depend—requires the strains to be so small that their squares are negligible. It is possible that, even in the case of the most brittle solids known, this condition may cease to hold before rupture occurs, although the stress-strain relation may continue to be linear. Nevertheless, the calculated values of even the breaking strains in a material like cast iron, for instance, are so small as to render this unlikely.

For a ductile metal, such as mild steel, the elastic results only tell you where the material will begin to take permanent set.

In the case of stone or cement, however, to which the present results would be applied, there seems to be no definite yield-point or elastic limit, the material being, in fact, only imperfectly elastic throughout. Still, we may consider that the results of elastic theory give in such a case an indication of the state of stress when the specimen breaks.

There are three distinct theories, both of rupture and of failure of elasticity. According to Lamé and Navier, failure occurs when the greatest stress at any point exceeds a certain limiting value. This is also often taken as the criterion of absolute rupture. According to Saint-Venant, the maximum strain, and not the maximum stress, is that which determines failure. Finally, a theory has lately been put forward by various elasticians to the effect that failure occurs when the greatest shear at any point, that is, the greatest principal stress-difference, exceeds a definite amount.

Professor Perry has proposed another criterion, suggested by the angle at which compressed cylinders shear (see 'Applied Mechanics,' pp. 344–345), namely, that rupture takes place when $s - \mu p$ exceeds a certain value, where s is the shear across any element of area at a point, p is the normal pressure on this element of area, and μ is a constant depending on the material. This theory, however, need not concern us so much, as it appears more specially applicable to the final breakdown of ductile materials long after they have become plastic. On the other hand, it has been shown by Mr. J. J. Guest ('Phil. Mag.,' July, 1900) that the beginning of plasticity was very probably determined by the maximum stress-difference.

Let us now proceed to apply these three criteria, namely, those of the maximum stress, maximum strain, and maximum stress-difference to the cylinder in the present example, and see what results they give us, on the hypothesis that for materials like stone and cement, plastic yielding and rupture are simultaneous.

Consider first the greatest stress theory. This would make failure of elasticity first begin to occur round the perimeter of the plane ends, and that as soon as $1{\cdot}68635\,Q > $ a certain limiting value S_0. If the pressure be uniformly applied, and the ends expand, we get failure of elasticity when

$$Q > S_0.$$

Hence the apparent strength of a cylinder tested in this way would be about ·593 of the strength of a cylinder tested under a distribution such as is usually assumed.

Further, if we consider the regions where the stress is greater than a given value S, we find that they consist of separate spaces, which join on to each other as S diminishes, the critical value for which this occurs being given by $S = 1{\cdot}01\,Q$. The regions of greatest stress consist therefore of a central core, which spreads out into a sort of hollow cone near the ends. If then we suppose fracture to occur over regions of greatest stress, we see why it is that the material breaks off in conical pieces at the ends.

Consider now the greatest strain theory. Let T_1, T_2, T_3 be the three principal stresses, and s_1, s_2, s_3 the corresponding stretches.

Then
$$s_1 = \frac{1}{2\mu}\left(T_1 - \frac{\lambda}{3\lambda + 2\mu}(T_1 + T_2 + T_3)\right),$$

so that the greatest s will correspond to the greatest T, if T_1, T_2, T_3 have the same sign. This is our case everywhere, except in cases where $\widehat{\phi\phi} > 0$, and then $\widehat{\phi\phi}$ is so small that it still leaves the strain corresponding to $\widehat{ZZ}$ numerically the greatest.

We have then, remembering we have assumed $\lambda = \mu$, to investigate the values of

$$\widehat{ZZ} - \tfrac{1}{5}(\widehat{ZZ} + \widehat{RR} + \widehat{\phi\phi}) = 2\mu s_z.$$

This will be proportional to the greatest strain, except near $z = \pm c$, $r = 0$, where

$$\widehat{RR} - \tfrac{1}{5}(\widehat{ZZ} + \widehat{RR} + \widehat{\phi\phi}) = 2\mu s_r$$

should be taken. It is found, however, that at this point the strain is comparatively small, and the maximum strain there is a matter of indifference.

TABLE of s_z/s, where s = maximum stretch under the same uniform pressure.

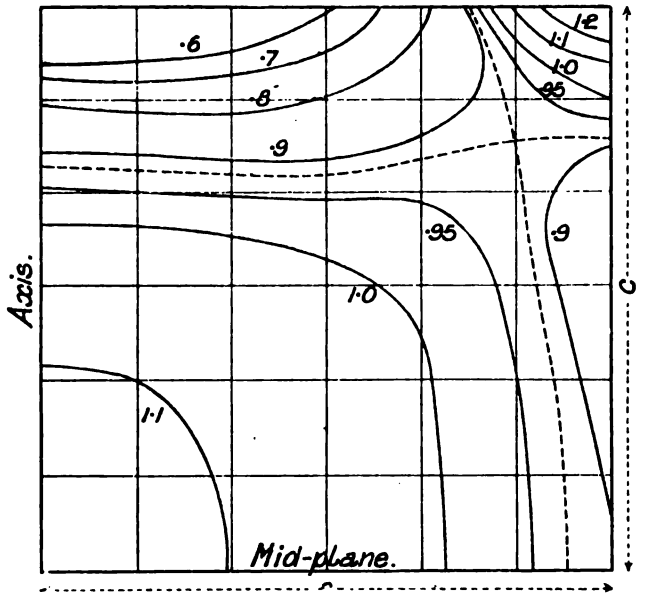
r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	1·13245	1·12729	1·10755	1·05938	·95338	·77296	·23743
$a/3$	1·10000	1·09589	1·07935	1·03756	·94751	·76590	·39034
$2a/3$	1·01133	1·01035	1·00509	·98813	·94446	·86416	·78311
a	·90246	·89563	·87724	·85716	·86591	·99395	1·57685

It we take therefore the "greatest stretch" theory, failure of elasticity still occurs at the perimeter of the ends, but this time only when the stress is $\frac{1}{1\cdot577}$ (limiting stress in the case of uniformly compressed cylinder), so that although the apparent strength is less than in the uniform case, it is greater than if we adopt the "greatest stress" theory.

The lines of equal principal stretch s_z/s are shown in Diagram 13. They are drawn for only one quarter of the meridian plane, the rest being symmetrical. They present the same general characteristics as the curves of equal stress $\widehat{ZZ}$, with this difference, that the critical line corresponds to $s_z = \cdot 915\, s$. Again, the caps or buttons at the ends are far larger; so that if pieces are cut out, they will be considerably larger than on the "greatest stress" theory. Also, looking at the inclination of the lines joining the corners to the critical points, *i.e.*, the points where the two branches of the critical line intersect, we see that the fragments, if approximately conical near their base, will probably be cut off at a much higher angle than in the previous case.

Let us now proceed to consider what happens if we adopt the third or "greatest stress-difference" theory of rupture. It is easy to see from the tables of $\widehat{RR}$, $\widehat{ZZ}$, and $\widehat{\phi\phi}$ that the greatest stress-difference is either $\widehat{RR}$-$\widehat{ZZ}$ or $\widehat{\phi\phi}$-$\widehat{ZZ}$. In the sixteen cases tabulated, for which $z > c/3$, the first of these is the greatest stress-difference, and in the twelve remaining cases the second is the greatest, although, as a matter of fact, the two stress-differences, for these twelve cases, do not diverge very much.

Diagram 13.—Distribution of Principal Stretch, s_z, inside the Cylinder (case of Compression between Rough Rigid Planes).



The number corresponding to each line = the value of s_z/s for that line.
----------, critical line. s_z/s = ·915.

The actual greatest stress-difference is given in the following table :—

TABLE of (Maximum Stress-difference)/Q.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	1·13108	1·12022	1·08188	·99730	·82675	·55221	– ·21092
$a/3$	1·10270	1·09320	1·05892	·98861	·85786	·66217	·71287
$2a/3$	1·02486	1·01915	·99853	·97223	·93034	·91798	1·21101
a	·92695	·91821	·89246	·85845	·88177	1·04077	1·68635

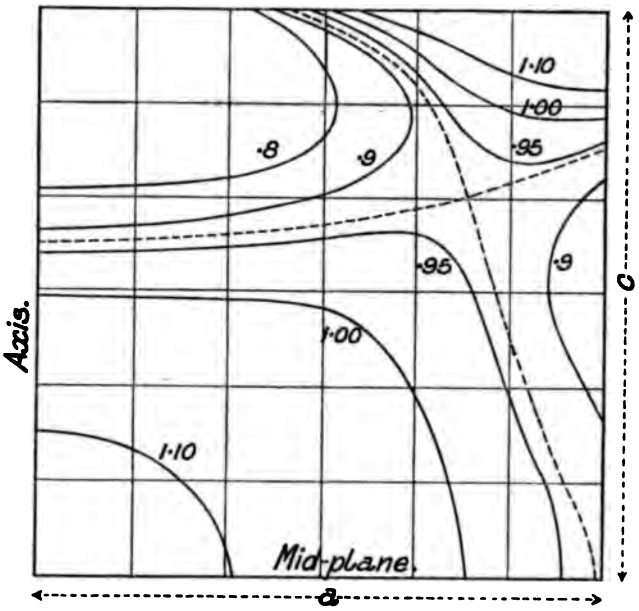
Here again plastic deformation will first occur round the perimeter of the ends when $Q = \left(\frac{1}{1 \cdot 68635}\right)$ of the value it should have, on the same theory of rupture, in order to produce failure of elasticity in a uniformly compressed cylinder.

So far, then, this theory leads to the same results as the maximum stress theory.

Diagram 14 shows the distribution of maximum stress-difference. The lines of equal maximum stress-difference present very similar characteristics to those of equal maximum stretch. The critical line corresponds to a maximum stress-difference = ·933 Q.

Remarks similar to those used in the last case apply in this.

Diagram 14.—Distribution of Principal Stress-difference inside the Cylinder (case of Compression between Rough Rigid Planes).



The number corresponding to each line denotes the value of the ratio (maximum stress-difference : Q) for that line.

- - - - - - - - - - -, critical line. Stress-difference = ·933 Q.

§ 24. *Distorted Shape of the Curved Surface.*

If we work out the values of u when $r = a$, we find, after some reductions,

$$u_{r=a} = \mathrm{E}a^5\left\{\frac{\zeta}{12} - \frac{1}{6}\frac{(z^2-c^2)^2}{a^4} - \sum_1^\infty \frac{2r_n}{\alpha^3}(-1)^n \cos\frac{n\pi z}{c}\right\} \quad . \quad . \quad . \quad . \quad (112),$$

where

$$r_n = \frac{\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}\mathrm{I}_1^2 - 4\mathrm{I}_0\mathrm{I}_1(2\gamma+1)}{\gamma\alpha^2\mathrm{I}_0^2 - (1+\gamma\alpha^2)\mathrm{I}_1^2} \quad . \quad . \quad . \quad . \quad (113).$$

Now in the particular case we are dealing with, if α be large, $r_n(\alpha)$ approximates to

$$\frac{7}{2} - \frac{49}{4\alpha} + \frac{335}{16\alpha^2} + \frac{55}{4\alpha^3} + \ldots$$

Write then

$$r_n(\alpha) = \frac{7}{2} - \frac{49}{4\alpha} + \frac{335}{16\alpha^2} + \frac{55}{4\alpha^3} + r_n'(\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad (114),$$

and substitute in (112). We find, putting in for $\Sigma\frac{1}{n^2}\cos\frac{n\pi z}{c}$ and $\Sigma\frac{1}{n^4}\cos\frac{n\pi z}{c}$ their known values,

$$u_{r=a} = \mathrm{E}a^5 \left\{ \begin{array}{l} \dfrac{\zeta}{12} - \dfrac{11}{378}\dfrac{\pi^6}{729} + \dfrac{49}{180}\dfrac{\pi^4}{81} - \left(\dfrac{z^2 - c^2}{a^2}\right)^2 \left\{\dfrac{65}{96} + \dfrac{55}{2880}\left(\dfrac{z^2}{a^2} - \dfrac{3c^2}{a^2}\right)\right\} \\ - \dfrac{7c^3}{\pi^3 a^3} \sum_1^\infty \dfrac{(-1)^n}{n^3} \cos \dfrac{n\pi z}{c} - \dfrac{335}{8\pi^5}\dfrac{c^5}{a^5} \sum_1^\infty \dfrac{(-1)^n}{n^5} \cos \dfrac{n\pi z}{c} \\ - 2\sum_1^\infty (-1)^n \dfrac{r_n'}{\alpha^3} \cos \dfrac{n\pi z}{c} \end{array} \right\} \quad . \quad (115),$$

where, putting in now $\pi a = 3c$, we have the following values for r_n' :—

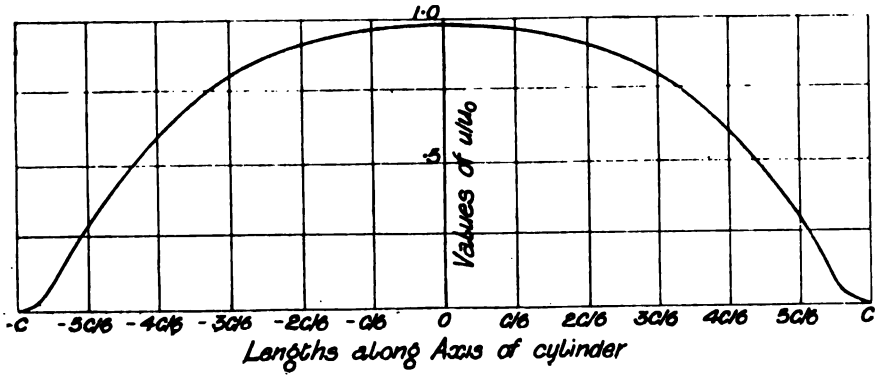
n.	$r_n'(\alpha)$.
1 . . .	—·42430
2 . . .	—·01807
3 . . .	—·00223
4 . . .	—·00063
5 . . .	—·00024
6 . . .	—·00010

Whence, using the methods of § 20, we find the following values for u/u_0, where u_0 = lateral expansion of a cylinder of the same dimensions under the same total pressure uniformly distributed :—

z.	u/u_0.
0	·97861
$c/6$	·96325
$2c/6$	·91118
$3c/6$	·80402
$4c/6$	·61209
$5c/6$	·30863
c	·00000

These results are exhibited in Diagram 15. We see that the cylinder has a single bulge in the centre, and indeed it is easy to verify that $(d^2u/dz^2)_0 < 0$ by differentiating equation (115) and putting in the numerical values.

Diagram 15.—Showing Distortion of Curved Surface (second example).



This agrees with the figures shown by **Bach** in his 'Elasticität und Festigkeit' (figs. 2 and 3, Par. 11) of the distorted shapes of such cylinders. There are, however, in the possession of Professor **Karl Pearson**, at University College, London, specimens of iron which have been strongly compressed, so that the strain has been large and permanent and the meridian section of the distorted curved surface is an obviously wavy curve, with *two* maxima, one on either side of the mid-section. With regard to the apparent disagreement here between theory and practice, I would observe that these specimens have been subjected to enormous stresses, for which the equations of elasticity certainly do not apply, and probably are not even an approximation; in the second place, the specimens I have seen are longer, compared with their diameter, than the cylinder of the present numerical example, so that it is easy to see why the conclusions above need not apply to these specimens.

§ 25. *Apparent Young's Modulus and Poisson's Ratio.*

We find that the total shortening of the bar

$$= -2\,(w)_{z=c} = 2\left[\mathrm{Q}/\mu\,\frac{\gamma\left\{\frac{8}{45}c^5 - \frac{ca^4\zeta}{6}\right\} - \frac{ca^4}{12}(1-\gamma)}{(4\gamma-1)\left\{\frac{8}{45}c^4 - \frac{a^4\zeta}{6}\right\} - \frac{a^4}{12}}\right] \quad . \quad . \quad . \quad . \quad (116),$$

where

$$\zeta = 24\sum_1^\infty \frac{1}{\alpha^3}\,\frac{\left[(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right] - \frac{4\mathrm{I}_0}{\mathrm{I}_1}(2\gamma+1)}{\left(\frac{\mathrm{I}_0}{{}_1\mathrm{I}}\right)^2\gamma\alpha^2 - (1+\gamma\alpha^2)}.$$

Hence the apparent **Young's** modulus $\mathrm{E_Y}' = \mathrm{Q}c/(w)_{z=c}$,

$$\mathrm{E_Y}' = \mu\,\frac{\left[(4\gamma-1)\left\{\frac{8}{45}c^4 - \frac{a^4\zeta}{6}\right\} - \frac{a^4}{12}\right]}{\gamma\left\{\frac{8}{45} \quad - \frac{a^4\zeta}{6}\right\} - \frac{a^4}{12}(1-\gamma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117).$$

Now if α be small, *i.e.*, if the bar be very long

$$\mathrm{I}_0/\mathrm{I}_1 = \frac{2}{\alpha}\left(1 + \frac{\alpha^2}{8} - \frac{7\alpha^4}{192} + \ldots\right)$$

and

$$\zeta = 96\,\Sigma\,\frac{1}{\alpha^4}\left(1 + \frac{36\gamma+7}{4\gamma-1}\,\frac{\alpha^4}{96}\right),$$

retaining only terms of 4th order,

$$= \frac{96}{90}\,\frac{c^4}{a^4} + \nu\,\frac{36\gamma+7}{4\gamma-1},$$

where ν is the greatest integral value for which $\alpha = \nu\pi a/c$ makes the above approximation sufficient.

Hence if a/c is very small, ν may be large, and thus although the first term in $a^4\zeta/6$ cancels $\frac{8}{45}c^4$, yet the terms in curled brackets will become indefinitely large compared with the other terms. Thus, for a very long cylinder,

$$E_Y' = \frac{\mu(4\gamma - 1)}{\gamma} = E_Y,$$

where E_Y is the true YOUNG's modulus.

On the other hand, for a very short cylinder, c/a is very small, and ζ being of the order $\frac{24}{\pi^3}\frac{c^3}{a^3} \times \frac{2\gamma+1}{\gamma}\sum_1^\infty \frac{1}{n^3}$, the two leading terms in the numerator and denominator of the right-hand side of (117) are negligible and

$$E_Y' = \frac{\mu}{1-\gamma} = \frac{\gamma}{(1-\gamma)(4\gamma-1)} E_Y.$$

This is identical with the modulus of compression for a cylinder which is prevented from expanding laterally by a constant pressure applied to the sides. So that we see that for a very flat disc the effect on the modulus of compression is the same, whether the lateral expansion be prevented by means of shearing-stress over the flat ends, or by hydrostatic pressure over the curved surface.

The apparent YOUNG's modulus for intermediate cases will be between those two values (these, for uniconstant isotropy, being E_Y and $6/5\,E_Y$). Thus, in the given example, where $\gamma = 2/3$,

$$E_Y' = 1{\cdot}0498 E_Y.$$

POISSON's ratio comes out to be apparently ·2690 instead of ·2500.

Thus the errors in the values of YOUNG's modulus and POISSON's ratio, as deduced from an experiment with cylinders under the given conditions, will be 5 per cent. and 7·6 per cent. respectively.

§ 26. *Solution involving Discontinuities at the Perimeter of the Plane Ends.*

In § 13 it was stated that a solution, obtained by methods strictly analogous to those used in that section, but which neglected the condition that the shear $\widehat{rz}$ should be continuous at the perimeter of the plane ends, could be found.

It seems of interest to give, for purposes of comparison, the expressions for the displacements and stresses, deduced from this solution. They are :

$$\left.\begin{aligned} u &= u_0 r + u_1 r^3/3 + Drz^2/2 + \sum_1^\infty \left\{ -\frac{A_1}{k} I_1(kr) - \frac{C}{k} r I_0(kr) \right\} \cos kz \\ w &= w_0 z + w_1 z^3/3 + \sum_1^\infty \left\{ \frac{A_2}{k} I_0(kr) + \frac{C}{k} r I_1(kr) \right\} \sin kz \end{aligned}\right\} . \quad (118)$$

where

$$\left.\begin{aligned} u_1 &= -\tfrac{3}{8}(1-\gamma)\mathrm{D} \\ w_1 &= -\gamma\mathrm{D} \end{aligned}\right\} \tag{119}$$

$$\left.\begin{aligned} \mathrm{A}_2 - \mathrm{A}_1 &= 2\mathrm{C}/k\gamma \\ \mathrm{A}_2 + \mathrm{A}_1 &= -\frac{2\mathrm{C}}{k}\frac{\alpha \mathrm{I}_0(\alpha)}{\mathrm{I}_1(\alpha)} + \frac{2\mathrm{D}ac}{n\pi}(-1)^n \end{aligned}\right\} \tag{120}$$

where

$$k = n\pi/c.$$

Also

$$\mathrm{C} = -\frac{\mathrm{D}}{k}(-1)^n\frac{\gamma\alpha \mathrm{I}_1(\alpha)(4\gamma+1) - \gamma\alpha^2 \mathrm{I}_0(\alpha)}{\gamma\alpha^2(\mathrm{I}_0^2(\alpha) - \mathrm{I}_1^2(\alpha)) - \mathrm{I}_1^2(\alpha)} \tag{121}$$

and

$$u_0 = \mathrm{D}\left[\tfrac{1}{8}(1-\gamma)a^2 - \frac{c^2\zeta}{3}\right]$$

$$w_0 = \mathrm{D}\left(\frac{1-\gamma}{2\gamma-1}\right)\left[\tfrac{1}{4}a^2 + \frac{2\gamma c^2}{3(1-\gamma)}(\zeta+\gamma-1)\right]$$

where

$$\mathrm{D} = -\frac{\mathrm{Q}}{\mu}\frac{2\gamma-1}{\frac{a^2}{4} - \frac{2}{3}c^2[\zeta(4\gamma-1) + (3\gamma-1)]}$$

and

$$\zeta = -\frac{3}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left\{\frac{\mathrm{I}_1^2(4\gamma+1) - \alpha \mathrm{I}_0 \mathrm{I}_1}{\gamma\alpha^2(\mathrm{I}_0^2 - \mathrm{I}_1^2) - \mathrm{I}_1^2}\right\} \tag{122}$$

The method by which the constants are obtained is precisely the same as that used in §§ 14, 15.

The expressions for the stresses are given below:

$$\frac{\widehat{rr}}{\mu} = \mathrm{D}\left[\tfrac{1}{4}(1+\gamma)(a^2-r^2) + 2\gamma\left(z^2 - \frac{c^2}{3}\right) + \frac{2ac}{\pi}\sum_1^\infty\frac{(-1)^n}{n}\left[\frac{\mathrm{I}_1(\rho)}{\rho \mathrm{I}_1(\alpha)} - \frac{\mathrm{I}_0(\rho)}{\mathrm{I}_1(\alpha)}\right]\cos\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2\mathrm{C}}{k}\left[\left\{1 + \frac{\alpha \mathrm{I}_0}{\mathrm{I}_1}\right\}\mathrm{I}_0(\rho) - \left\{\frac{\alpha \mathrm{I}_0}{\mathrm{I}_1} + \frac{1}{\gamma}\right\}\frac{\mathrm{I}_1(\rho)}{\rho} - \rho \mathrm{I}_1(\rho)\right]\cos\frac{n\pi z}{c} \tag{123}$$

$$\frac{\widehat{\phi\phi}}{\mu} = \mathrm{D}\left[\tfrac{1}{4}(1+\gamma)a^2 + 2\gamma\left(z^2 - \frac{c^2}{3}\right) - \tfrac{1}{4}(3\gamma-1)r^2 - \frac{2ac}{\pi}\sum_1^\infty\frac{(-1)^n}{n}\frac{\mathrm{I}_1(\rho)}{\rho \mathrm{I}_1(\alpha)}\cos\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2\mathrm{C}}{k}\left[\left(1 - \frac{1}{\gamma}\right)\mathrm{I}_0(\rho) + \left(\frac{1}{\gamma} + \frac{\alpha \mathrm{I}_0}{\mathrm{I}_1}\right)\frac{\mathrm{I}_1(\rho)}{\rho}\right]\cos\frac{n\pi z}{c} \tag{124}$$

$$\frac{\widehat{zz} + \mathrm{Q}}{\mu} = \mathrm{D}\left[\frac{2c^2}{3} + \frac{\widehat{2\gamma-1}a^2}{4} - z^2 - \frac{\widehat{2\gamma-1}}{2}r^2 + \frac{2ac}{\pi}\sum_1^\infty\frac{(-1)^n}{n}\frac{\mathrm{I}_0(\rho)}{\mathrm{I}_1(\alpha)}\cos\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2\mathrm{C}}{k}\left[\left\{2 - \frac{\alpha \mathrm{I}_0}{\mathrm{I}_1}\right\}\mathrm{I}_0(\rho) + \rho \mathrm{I}_1(\rho)\right]\cos\frac{n\pi z}{c} \tag{125}$$

$$\frac{\widehat{rz}}{\mu} = \mathrm{D}\left[rz + \frac{2ac}{\alpha}\sum_1^\infty\frac{(-1)^n}{n}\frac{\mathrm{I}_1(\rho)}{\mathrm{I}_1(\alpha)}\sin\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2\mathrm{C}}{k}\left(\rho \mathrm{I}_0(\rho) - \mathrm{I}_1(\rho)\frac{\alpha \mathrm{I}_0}{\mathrm{I}_1}\right)\sin\frac{n\pi z}{c} \tag{126}$$

It is now easy to see, if we bear in mind that when n and therefore α is large, CI_0 remains finite, as appears on examination of (121), how it is that the stresses at the boundary become infinite.

For in both $\widehat{zz} + Q$ and $\widehat{\phi\phi}$ we have, when $r = a$, terms of order $1/n$, when n is large. These are of alternate sign, C containing $(-1)^n$. But if $z = \pm c$ they become all of the same sign, and the series become logarithmically infinite.

§ 27. *Summary of Results.*

Looking back upon the results obtained, we notice :

(*a.*) That the three solutions we have been considering successively are only the simplest of an infinite series of solutions, which are continually growing more complicated ; for we need not necessarily stop, as has been done, at terms of the fifth degree, but might go on to terms of any degree in r and z and thus construct, as it were, solutions of successive orders. We should then have an infinite number of free constants, which might be determined by introducing further limitations at the plane ends, such as, for instance, restricting n to be zero at every point and not merely along the perimeter.

The analytical complexity of such a complete solution would, however, be very great, and would render it quite beyond the reach of arithmetical expression, and consequently valueless for the purposes of the engineer and the physicist. No attempt has therefore been made to develop this solution, although, as an analytical possibility, it appears interesting.

(*b.*) That the different solutions all agree in giving the perimeter of the plane ends as the locus of the points where the elastic limit will first be passed, one of these solutions actually making the stress infinite at this perimeter.

In the more important solution, however, where continuity and finiteness are preserved, the conclusion still holds, and, further, is independent of whatever theory of tendency to rupture we adopt, whether we suppose it due to maximum stress, to maximum stretch or squeeze, or to maximum shear or stress-difference.

(*c.*) That in the numerical example considered, plastic deformation begins to occur round the perimeter for a stress between 2/3 and 1/2 of that which is required to cause a cylinder under uniform pressure to pass the elastic limit.

This is apparently in contradiction with the results of engineering experience, both UNWIN and PERRY stating that blocks of stone or cement, pressed between millboard, which hinders the expansion of the ends, show greater strength than the same blocks when the ends are allowed to expand.

The key to this appears to be found in a remark of UNWIN, which Professor EWING confirms, that the lead sheets do not merely *allow* the expansion of the block, they *force* it, *i.e.*, lead in its plastic state will expand more than the stone or cement would do laterally under a uniform axial pressure.

But the solution, when the ends are compelled to expand by a given quantity α_0, is easily deducible from that given for non-expanding ends. Thus, let u_1, w_1 be the values of u and w in the case worked out, when $Q = 1$, and let us write

$$u = P\left(\frac{\lambda r}{2\mu(3\lambda + 2\mu)}\right) + Ru_1,$$

$$w = P\left(\frac{-(\lambda + \mu)z}{\mu(3\lambda + 2\mu)}\right) + Rw_1.$$

Then this will satisfy all the conditions, provided

$$\frac{P}{\mu}\frac{\lambda}{6\lambda + 4\mu} \times a = \alpha_0$$

and

$$P + R = Q.$$

Therefore

$$u = \frac{\alpha_0 r}{a} + \left(Q - \frac{\alpha_0}{a}\frac{\mu}{\lambda}(6\lambda + 4\mu)\right)u_1.$$

$$w = \frac{-2(\lambda + \mu)}{\lambda}\frac{\alpha_0 z}{a} + \left(Q - \frac{\alpha_0}{a}\frac{\mu}{\lambda}(6\lambda + 4\mu)\right)w_1,$$

giving the solution under a given mean pressure Q, which produces a flow α_0 of a lead plate, and thereby constrains the ends of the test piece to expand by that amount.

The principal stress at the perimeter of the section is now

$$P + (1{\cdot}686)R = 1{\cdot}686\,Q - {\cdot}686\,P,$$

and if P, *i.e.*, α_0, be made large enough, this can be made much smaller than Q. It begins to be smaller than Q as soon as the expansion induced by the flow of the lead is greater than the natural expansion of the stone under uniform pressure.

On the other hand, the principal stress at the centre of the plane ends is

$$P + {\cdot}686\,R = {\cdot}686\,Q + {\cdot}314\,P,$$

and this again may be made great by making α_0 large.

The principal stress-differences are :

at the perimeter $\quad 1{\cdot}686\,Q - {\cdot}686\,P$
at the centre $\quad P - {\cdot}211\,R = -{\cdot}211\,Q + 1{\cdot}211\,P.$

Hence we see that whatever theory of failure we adopt, if the ends are forced to expand, so that $P > Q$, the material first becomes plastic (or else breaks) at the centre of the cross-section, the strength of the test-piece diminishing as P increases, but having no definite value. That some such thing as this does really occur in practice is very well shown by the results published by UNWIN ('The Testing of Materials of

Construction'), where blocks show less strength when three sheets of lead are introduced between the compressing planes and the test piece than when one sheet only is introduced, the lateral flow being greater in the first case owing to the larger amount of lead.

It would seem, therefore, as if the true strength of a cylinder were really greater than its strength as tested either between millboards or between lead sheets, and not, as Professor PERRY states in his 'Applied Mechanics,' equal to the strength shown in the lead test—this test, as we see, leading to results that are not definite, but vary with the expansion of the lead. The millboard test, however, which is advocated by UNWIN, should give a constant value, although it is not the value which would hold for a cylinder under uniform pressure.

(*d.*) Diagrams 12–14 suggest an explanation of the fact that, when short cylinders are strongly compressed between very hard surfaces, pieces are sometimes cut out at the ends of an approximately conical shape. The same occurs when spherical pieces of metal, such as ball-bearings, are compressed between parallel plates. This is usually explained by saying that the material breaks along the planes of principal shear. On the other hand, it may be argued simply that rupture should take place over the regions of greatest stress. These are near the perimeter at the ends, and gradually close in upon the centre, forming hollow caps.

Further, in the case of the lead tests, where $P > Q$, this state of things is reversed, and the material should give way from the inside, so that we should expect it to split axially, and possibly along meridian planes as well. That this is what really occurs can be verified by referring to the figures in the chapter on testing of stone in UNWIN'S 'Testing of Materials of Construction.'

(*e.*) The results both of this and of the first problem show us how unreliable any experiments on short cylinders must be, which have in view the determination, by tensile strain, of either YOUNG'S modulus or POISSON'S ratio. Thus any results obtained in such a case without the dimensions and the mode of application of the stress being exactly specified, would not justify us in general in drawing any conclusions as to whether a given material possesses or not uniconstant isotropy.

§ 28. *The Third Problem. Case of Torsion. Expressions for the Displacement and Stresses.*

I now proceed to consider a case where u and w are zero, that is, where we have to deal with the solution in v, which we have seen is independent of the others.

We have in the notation of § 3

$$(\vartheta^2 + D^2) v = 0 .$$

Hence, excluding K-functions, since the solution must be finite and continuous at the origin, we have

$$v = \Sigma (A_n \sin kz + B_n \cos kz) I_1 (kr) .$$

Now, if the torsion be symmetrical on either side of the mid-section, we have $v = 0$, when $z = 0$, therefore

$$B_n = 0.$$

But also

$$\widehat{\phi z} = \mu \frac{dv}{dz} = 0 \text{ at the ends.}$$

Therefore

$$\cos kc = 0 \text{ or } kc = \widehat{2n+1}\pi/2.$$

Also

$$\widehat{r\phi} = \mu r \frac{d}{dr}\left(\frac{v}{r}\right) = \Sigma \mu A_n \sin kz \,.\, r \frac{d}{dr} \frac{I_1(kr)}{r}$$

$$= \Sigma \mu A_n \sin kz \,.\, k I_2(kr).$$

By the well-known property of the I-functions

$$\frac{d}{dx}\left(\frac{I_n(x)}{x^n}\right) = \frac{I_{n+1}(x)}{x^n}.$$

Now suppose that the cylinder is subjected to a certain system of transverse surface shears, so that $\widehat{r\phi}$ can be expanded in a FOURIER'S series in the form

$$(\widehat{r\phi})_a = \sum_0^\infty c_n \sin \frac{\widehat{2n+1}\pi z}{2c}.$$

Then

$$\mu k A_n I_2(\alpha) = c_n$$

or

$$A_n = \frac{c_n}{\mu k} \frac{1}{I_2(\alpha)}.$$

Hence

$$v = \sum_0^\infty \frac{c_n}{\mu k} \frac{I_1(\rho)}{I_2(\alpha)} \sin \frac{\widehat{2n+1}\pi z}{2c} \quad \ldots\ldots\ldots \quad (127),$$

$$\widehat{r\phi} = \sum_0^\infty c_n \frac{I_2(\rho)}{I_2(\alpha)} \sin \frac{\widehat{2n+1}\pi z}{2c} \quad \ldots\ldots\ldots \quad (128),$$

$$\widehat{\phi z} = \sum_0^\infty c_n \frac{I_1(\rho)}{I_2(\alpha)} \cos \frac{\widehat{2n+1}\pi z}{2c} \quad \ldots\ldots\ldots \quad (129),$$

where ρ α have the same meanings as before, viz., kr, ka.

In the case where a/c is very small, or the cylinder is very long relatively to its diameter, we may obtain a first approximation by retaining only the first terms in the expression for the I's. Proceeding as in § 6, we find

$$\left.\begin{aligned} v &= \frac{4r}{\mu a^2}\sum_0^\infty \frac{c_n}{k^2}\sin kz \\ \widehat{r\phi} &= \frac{r^2}{a^2}\sum_0^\infty c_n \sin kz \\ \widehat{\phi z} &= \frac{4r}{a^2}\sum_0^\infty \frac{c_n}{k}\cos kz \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130).$$

Now if $$(\widehat{r\phi})_a = \psi(z),$$
we see that
$$\widehat{r\phi} = r^2\psi(z)/a^2,$$

$$\widehat{\phi z} = (4r/a)\int_z^c \psi(z)\,dz,$$

$$v = \frac{4r}{\mu a^2}\int_0^z dz\int_z^c \psi(z)\,dz.$$

Now if M be the total torsion moment up to any cross-section,

$$\mathrm{M} = 2\pi a^2\int_z^c \psi(z)\,dz,$$

$$\widehat{\phi z} = \frac{2r}{\pi a^4}\mathrm{M},$$

v/r = angle turned through by a radius = θ say.

Therefore $$\theta = \frac{2}{\pi\mu a^4}\int_0^z \mathrm{M}\,dz,$$

$$\frac{d\theta}{dz} = \text{torsion at the point} = \tau.$$

Therefore $$\tau = \frac{2\mathrm{M}}{\pi\mu a^4}.$$

Therefore $$\mathrm{M} = \mu \times \frac{\pi a^4}{2} \times \tau \text{ and } \widehat{\phi z} = \mu\tau r.$$

But these are the actual formulæ connecting the torsion with the applied couple and with the shear across the section for a circular cylinder.

We see, then, that the usual formulæ continue to hold, to the first approximation, when the forces applied to the surface of the cylinder vary with z, provided we define our torsion-couple at any section (much as the bending moment at any section of a beam is defined), as the couple of all the external applied forces to one side of that cross-section.

It is interesting to note also that, to this approximation, there is no distortion of the cross-sections, v/r being constant for the section. Straight radii therefore remain straight radii.

Further, $\widehat{r\phi} = (r^2/a^2)$ (its value at the boundary).

In other words, the transverse shearing-stress across cylinders coaxial with the given one is zero for sections where there is no such external applied stress, and for other sections diminishes rapidly along the radius as we go inwards, so that near the centre it is always small compared with $\widehat{\phi z}$.

There is one very important point to be noted with regard to this method of approximation : ρ and α increase with n, and therefore, however small a/c may be, so long as it remains finite, we still reach a value of n, for which it is not justifiable to take for I_1 and I_3 the first terms of their expansions in positive integral powers of the argument. If, however, we stop at the νth term, where ν is finite, then if R_ν is the remainder after ν terms of the series

$$\sum_0^\infty c_n \frac{I_1(\rho)}{I_3(\alpha)} \cos \frac{\widehat{2n+1}\pi z}{2c}, \text{ for example,}$$

and if, on the other hand, the numerical value of the difference $\frac{4r}{ka^2} - \frac{I_1(\rho)}{I_3(\alpha)} < \epsilon$ for all values of n not greater than ν, where ϵ is a quantity which depends upon c/a, and which can be made as small as we please by making c/a small enough, then the difference

$$\sum_0^\infty c_n \frac{I_1(\rho)}{I_3(\alpha)} \cos \frac{\widehat{2n+1}\pi z}{2c} - \sum_0^\infty \frac{c_n}{k} \frac{4r}{a^2} \cos \frac{\widehat{2n+1}\pi z}{2c}$$

must be numerically less than

$$|R_\nu| + |R_\nu'| + \nu\epsilon,$$

where $|x|$ denotes the numerical value or modulus of x, and R_ν' is the remainder after ν terms of the series

$$\sum_0^\infty \frac{c_n}{k} \frac{4r}{a^2} \cos \frac{\widehat{2n+1}\pi z}{2c}.$$

Now if both the original series and the approximate series are uniformly convergent, then by giving ν a certain value, $|R_\nu|$ and $|R_\nu'|$ can both be made less than a certain small quantity $\eta/3$ which tends to zero when ν tends to infinity, and that for all values of z.

Now make c/a so small that $\epsilon < \eta/3\nu$, which we can always do so long as ν is finite. Then the difference between the two series is numerically $< \eta$, and the approximation holds.

If, however, for any value of z, it becomes impossible to assign an upper limit to R_ν or R_ν', *i.e.*, if either series cease to be uniformly convergent, then we should have to increase ν indefinitely in order to make $|R_\nu| < \eta/3$, and therefore to modify ϵ, so that no limiting value of c/a (which should, of course, be independent of z) could be found, and the approximation need not necessarily hold. As a matter of fact, it is shown in § 29 to fail for particular cases. This is true *à fortiori*, if either series cease to be convergent at all.

The same remarks apply in their entirety to the process of approximation given in § 6, and further, to the approximate expressions given by Professor POCHHAMMER in his investigation on the bending of beams ('Crelle,' vol. 81).

§ 29. *Special Case of Two Discontinuous Rings of Shear.*

Suppose that we have the following system of values for $\widehat{\phi z}$:—

$$\widehat{\phi z} = \text{T if } c - e < z < c,$$

$$\widehat{\phi z} = 0 \text{ it } -c + e < z < c - e,$$

$$\widehat{\phi z} = -\text{T if } -c < z < -c + e,$$

so that we have a cylinder twisted by two equal and opposite rings of transverse shear extending over lengths e of the cylinder, near the ends. Then we find easily

$$c_n = \frac{4\text{T}}{(2n+1)\pi}(-1)^n \sin\frac{(2n+1)\pi e}{2c}$$

with the following values of the displacements and stresses :

$$\left.\begin{aligned} v &= \sum_0^\infty \frac{8\text{T}c}{\mu(2n+1)^2\pi^2}(-1)^n \frac{\text{I}_1(\rho)}{\text{I}_2(\alpha)} \sin\frac{\widehat{2n+1}\,\pi e}{2c} \sin\frac{\widehat{2n+1}\,\pi z}{2c} \\ \widehat{r\phi} &= \sum_0^\infty \frac{4\text{T}}{(2n+1)\pi}(-1)^n \frac{\text{I}_2(\rho)}{\text{I}_2(\alpha)} \sin\frac{\widehat{2n+1}\,\pi e}{2c} \sin\frac{\widehat{2n+1}\,\pi z}{2c} \\ \widehat{\phi z} &= \sum_0^\infty \frac{4\text{T}}{(2n+1)\pi}(-1)^n \frac{\text{I}_1(\rho)}{\text{I}_2(\alpha)} \sin\frac{\widehat{2n+1}\,\pi e}{2c} \cos\frac{\widehat{2n+1}\,\pi z}{2c} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (131).$$

Now it is easy to see that in this case the conditions for uniform convergency are satisfied, except at the boundary, and except with regard to the stress $\widehat{r\phi}$, whose approximate expression is not uniformly convergent, being in fact discontinuous for $z = \pm(c - e)$.

At the boundary, $\text{I}_1(\alpha)/\text{I}_2(\alpha)$ tends to unity with n, its approximate expression, when α is large, being

$$\frac{\text{I}_1(\alpha)}{\text{I}_2(\alpha)} = 1 + \frac{3}{2\alpha} + \frac{15}{8\alpha^2} + \frac{15}{8\alpha^3} + \cdot \;.\;.\;.\;.\;.\;.\;. \quad (132).$$

Hence v is always uniformly convergent and its approximate expression likewise, so for it the approximation, for sufficiently small values of c/a, holds throughout.

For $\widehat{r\phi}$ the series is non-uniformly convergent for $r = a$ in the neighbourhood of the sections $z = \pm (c - e)$ owing to the series having a finite discontinuity. For $\widehat{\phi z}$ the approximation certainly fails, for $r = a$, in the neighbourhood of $z = \pm (c - e)$ for part of the expression for $\widehat{\phi z}$ is the series $\sum_0^\infty 1/(2n + 1)$, which is divergent.

Hence, if we are in such a case to use our approximations for the stresses, we must exclude the sections where the applied stress is discontinuous and their immediate neighbourhood from consideration.

It will be found that similar remarks apply to the example of pull given in § 7, and also to the example given by Professor POCHHAMMER in his paper on bending (*loc. cit.*), in which he also deals with discontinuous systems of stress, so that his approximate expressions leave us in the dark as to what does really happen at points of support, the cross-sections in the neighbourhood of such points being, for reasons analogous to those developed above, excluded from the regions where his approximations hold.

Before proceeding to an actual numerical concrete case, we may notice that $\widehat{\phi z}$ becomes infinite at the points $z = \pm (c - e)$ for the causes stated above. Hence any discontinuity in a system of transverse shears applied to the surface of a cylinder, or any such shear transmitted by a grip applied to a portion of the material projecting at a sharp angle, will produce in the neighbourhood a very great stress across the section. It would seem, therefore, that a cylinder treated in this way would be likely to experience plastic flow, or to rupture, not in the middle, but near the points where it is seized.

§ 30. *Approximations on the Boundary when the Cylinder is short.*

When the cylinder is short, so that α becomes rapidly large, we may use the method employed in §§ 8, 9, and 19, availing ourselves of the approximation (132). We then find :—

$$\begin{aligned} v = \frac{4\mathrm{T}c}{\mu\pi^2}\Bigg\{ & \frac{\pi}{4}\left(1 - \frac{z-e}{c}\right)\log_e \left| \cot\left\{\frac{\pi}{4}\left(1 - \frac{z-e}{c}\right)\right\}\right| \\ & - \frac{\pi}{4}\left(1 - \frac{z+e}{c}\right)\log_e \left| \cot\left[\frac{\pi}{4}\left(1 - \frac{z+e}{c}\right)\right]\right| \\ & + \frac{4\mathrm{T}c}{\mu\pi^2}\left\{\tfrac{1}{2}\int_0^{\frac{\pi}{2}\left(1-\frac{z-e}{c}\right)} x \operatorname{cosec} x \, dx - \tfrac{1}{2}\int_0^{\frac{\pi}{2}\left(1-\frac{z+e}{c}\right)} x \operatorname{cosec} x \, dx\right\} \\ & + \tfrac{3}{4}\frac{\mathrm{T}\pi}{\mu c}\left[\begin{array}{l} ez \text{ (from } z = 0 \text{ to } z = c - e) \\ \text{and} \\ ez - \tfrac{1}{2}(z - c + e)^2 \text{ (from } z = c - e \text{ to } z - c) \end{array}\right] \end{aligned}$$

$$+ 15\frac{\mathrm{T}c}{\mu\pi^2}\sum_0^\infty(-1)^n\left(\frac{1}{(2n+1)^4}+\frac{1}{(2n+1)^5}\right)\sin\frac{\overparen{2n+1}\pi e}{2c}\sin\frac{\overparen{2n+1}\pi z}{2c}$$

$$+\frac{8\mathrm{T}c}{\mu\pi^2}\sum_0^\infty\frac{(-1)^n}{(2n+1)^2}\left\{\frac{\mathrm{I}_1(\alpha)}{\mathrm{I}_2(\alpha)}-1-\frac{3}{2\alpha}-\frac{15}{8\alpha^2}-\frac{15}{8\alpha^3}\right\}\sin\frac{\overparen{2n+1}\pi e}{2c}\sin\frac{\overparen{2n+1}\pi z}{2c}\quad . \quad . \quad (133),$$

$$\widehat{\phi z}=\frac{\mathrm{T}}{\pi}\log_e\left|\frac{\tan\left(\frac{\pi}{4}-\frac{\pi}{4}\frac{z-e}{c}\right)}{\tan\left(\frac{\pi}{4}-\frac{\pi}{4}\frac{z+e}{c}\right)}\right|+\tfrac{3}{4}\frac{\pi\mathrm{T}}{c}\left[\begin{array}{l}e\ (\text{from } z=0 \text{ to } z=c-e)\\ \text{and}\\ c-z\ (\text{from } z=c-e \text{ to } z=c)\end{array}\right]$$

$$+\tfrac{15}{2}\frac{\mathrm{T}}{\pi}\sum_0^\infty(-1^n\left(\frac{1}{(2n+1)^3}+\frac{1}{(2n+1)^4}\right)\sin\frac{(2n+1)\pi e}{2c}\cos\frac{\overparen{2n+1}\pi z}{2c}$$

$$+\frac{4\mathrm{T}}{\pi}\sum_0^\infty\frac{(-1)^n}{(2n+1)}\left(\frac{\mathrm{I}_1(\alpha)}{\mathrm{I}_2(\alpha)}-1-\frac{3}{2\alpha}-\frac{15}{8\alpha^2}-\frac{15}{8\alpha^3}\right)\sin\frac{\overparen{2n+1}\pi e}{2c}\cos\frac{\overparen{2n+1}\pi z}{2c}\ (134),$$

where in the last Σ in both equations only a comparatively small number of terms need be taken.

§ 31. *Numerical Example. Values of the Coefficients and of the Displacement and Stresses.*

Take a cylinder such that $\pi a/2c = 1$, and let $e = c/2$, so that the distribution ot stress is as shown in fig. 4. Then $\alpha = 2n + 1$, and we find :

$$v=\frac{4\sqrt{2}\mathrm{T}c}{\mu\pi^2}(v_0\sin\pi z/2c-v_1\sin 3\pi z/2c-v_2\sin 5\pi z/2c+v_3\sin 7\pi z/2c+v_4\sin 9\pi z/2c$$
$$-v_5\sin 11\pi z/2c-\ldots)$$

$$\widehat{r\phi}=\frac{2\sqrt{2}\mathrm{T}}{\pi}(t_0\sin\pi z/2c-t_1\sin 3\pi z/2c-\ldots)$$

$$\widehat{\phi z}=\frac{2\sqrt{2}\mathrm{T}}{\pi}(s_0\cos\pi z/2c-s_1\cos 3\pi z/2c-\ldots)$$

the law of the signs being obvious, and the v's, t's, and s's being given below :

TABLE of t_n and s_n.

n.	t_n.				s_n.			
	$r = \cdot 2a$.	$r = \cdot 4a$.	$r = \cdot 6a$.	$r = a$.	$r = \cdot 2a$.	$r = \cdot 4a$.	$r = \cdot 6a$.	$r = a$.
0	·036956	·149306	·341554	1·000000	·740348	1·502982	2·310929	4·163295
1	·006884	·030078	·078098	·333333	·046574	·106104	·195552	·586933
2	·001551	·007871	·025651	·200000	·006457	·018173	·045167	·278032
3	·000331	·002073	·009064	·142857	·001021	·003803	·013485	·179752
4	·000068	·000547	·003339	·111111	·000169	·000813	·004522	·132502
5	·000013	·000145	·001266	·090909	·000029	·000212	·001619	·104848
6	·000003	·000039	·000492	·076923	·000005	·000053	·000604	·086720
7	·000001	·000011	·000195	·066667	·000001	·000014	·000232	·073927
8	·000000	·000003	·000078	·058824	·000000	·000004	·000091	·064419
9	·000000	·000001	·000032	·052632	·000000	·000001	·000036	·057075

Fig. 4.

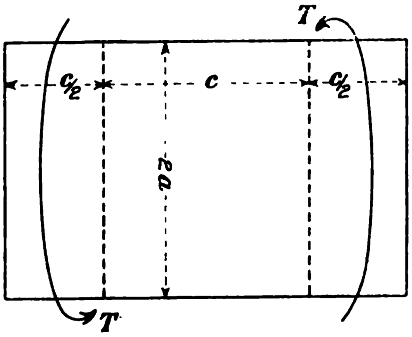


TABLE of v_n.

$r = \cdot 2a$.	$r = \cdot 4a$.	$r = \cdot 6a$.	$r = a$.
·740348	1·502982	2·310929	4·163295
·015525	·035368	·065184	·195644
·001291	·003635	·009033	·055606
·000146	·000543	·001926	·025679
·000019	·000097	·000502	·014722
·000003	·000019	·000147	·009532
·000000	·000004	·000046	·006671
·000000	·000001	·000015	·004928
·000000	·000000	·000005	·003789
·000000	·000000	·000002	·003004

From these, using the formulæ of approximation given in the last section, when $r = a$, we obtain the values of v and the two stresses. I have tabulated them in the form v/v_0 and (stress) $/T_0$ where v_0, T_0 are the greatest values of the displacement and of the shear respectively in a cylinder of the same length, subject to a uniform

Table of v/v_0.

r/a.	$z = 0$.	$z = \cdot 1c$.	$z = \cdot 2c$.	$z = \cdot 3c$.	$z = \cdot 4c$.	$z = \cdot 5c$.	$z = \cdot 6c$.	$z = \cdot 7c$.	$z = \cdot 8c$.	$z = \cdot 9c$.	$z = c$.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	·019704	·039235	·058350	·076671	·093656	·108646	·120983	·130136	·135953	·137645
·4	0	·039599	·078938	·117618	·154931	·189753	·220555	·245889	·264618	·276069	·279918
·6	0	·059762	·119340	·178387	·236094	·290682	·339273	·378990	·408066	·425706	·431612
1·0	0	·100258	·200717	·301772	·404584	·516879	·619041	·691974	·742997	·773388	·783489

Table of $\widehat{r\phi}/T_0$.

r/a.	$z = 0$.	$z = \cdot 1c$.	$z = \cdot 2c$.	$z = \cdot 3c$.	$z = \cdot 4c$.	$z = \cdot 5c$.	$z = \cdot 6c$.	$z = \cdot 7c$.	$z = \cdot 8c$.	$z = \cdot 9c$.	$z = c$.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	·000546	·001317	·002517	·004244	·006353	·008452	·010145	·011266	·011864	·012046
·4	0	·001821	·004540	·009126	·016259	·025423	·034559	·041589	·045928	·048101	·048743
·6	0	·002777	·007342	·016277	·032852	·057238	·081594	·098019	·106624	·110409	·111470
1·0	0	0	0	0	0	$\begin{cases} 0 \\ \cdot 318310 \end{cases}$	·318310	·318310	·318310	·318310	·318310

Table of $\widehat{\phi z}/T_0$.

r/a.	$z = 0$.	$z = \cdot 1c$.	$z = \cdot 2c$.	$z = \cdot 3c$.	$z = \cdot 4c$.	$z = \cdot 5c$.	$z = \cdot 6c$.	$z = \cdot 7c$.	$z = \cdot 8c$.	$z = \cdot 9c$.	$z = c$.
0	0	0	0	0	0	0	0	0	0	0	0
·2	·19730	·19650	·19373	·18795	·17757	·16102	·13766	·10819	·07429	·03774	0
·4	·39638	·39519	·39096	·38147	·36288	·33076	·28300	·22179	·15169	·07682	0
·6	·59789	·59708	·59396	·58574	·56550	·52125	·44556	·34598	·23436	·11800	0
1·0	1·00230	1·00317	1·00658	1·01614	1·04654	∞	·84647	·61593	·40614	·20217	0

torsion over its whole length, the total couple applied being the same as in the present case. We find $T_0 = \mu\tau a = \pi T$ and $v_0 = \tau c a = \pi c T/\mu$ for the given example. Tables are given on page 229.

Looking at these tables we see that, over the length free from external applied shear, the strains and stresses inside the cylinder are sensibly the same as what they would be on the hypothesis of a uniform torsion. Outside $z/c = \cdot 5$ the torsion couple diminishes, and the stresses diminish in consequence.

It is interesting to compare these results with those that we should have obtained if we had supposed the approximate results given on p. 223 to hold good in this case. Denoting by v', $\widehat{r\phi}'$, $\widehat{\phi z}'$ the values of the displacement and stresses calculated on this hypothesis, we have—

TABLE of v'/v_0.

r/a.	$z/c = 0$.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.	1.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	·020	·040	·080	·080	·100	·118	·132	·142	·148	·150
·4	0	·040	·080	·120	·160	·200	·236	·264	·284	·296	·300
·6	0	·060	·120	·180	·240	·300	·354	·396	·426	·444	·450
1·0	0	·100	·200	·300	·400	·500	·590	·660	·710	·740	·750

TABLE of $\widehat{r\phi}'/T_0$.

r/a.	$z/c = 0$.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.	1.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	0	0	0	0	0 ·012732	·012732	·012732	·012732	·012732	·012732
·4	0	0	0	0	0	0 ·050930	·050930	·050930	·050930	·050930	·050930
·6	0	0	0	0	0	0 ·114592	·114592	·114592	·114592	·114592	·114592
1·0	0	0	0	0	0	0 ·318310	·318310	·318310	·318310	·318310	·318310

TABLE of $\widehat{\phi z}'/T_0$.

r/a.	$z/c = 0$.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.	1
0	0	0	0	0	0	0	0	0	0	0	0
·2	·20	·20	·20	·20	·20	·20	·16	·12	·08	·04	0
·4	·40	·40	·40	·40	·40	·40	·32	·24	·16	·08	0
·6	·60	·60	·60	·60	·60	·60	·48	·36	·24	·12	0
1·0	1·00	1·00	1·00	1·00	1·00	1·00	·80	·60	·40	·20	0

§ 32. *Discussion of the Results.*

From the above tables we see that the radii in each cross-section do not remain straight lines, but assume distorted shapes, which are shown, on a very exaggerated scale, on Diagram 16, where, for each of the ten cross-sections, the curve of $(v - v')/v_0$, which indicates the deviation from the straight line in the distorted form of the radii, has been plotted. The variation from the straight line increases rapidly as we approach the region where the stress is applied, as can be seen from curves (1)–(4) on Diagram 16. On the other hand, towards the ends, the distortion remains fairly constant. The distorted radii meet the bounding circles at right angles when $\widehat{r\phi} = 0$ at the surface, but they meet it at a finite angle where $\widehat{r\phi} = \mathrm{T}$.

From the values of $\widehat{\phi z}$ and $\widehat{r\phi}$ we see that as soon as we get at all away from the ends the conditions that $\widehat{r\phi} = 0$, $\widehat{\phi z} = \mu\tau z$, $v = \tau rz$, which hold for uniform torsion, are very closely satisfied, and that, more generally, except where the abrupt change takes place in the shearing stress at the surface, the approximate expressions given in § 28 do not differ widely from the true expressions, the law that $\widehat{r\phi}$ varies as the square of the radius being, near the ends, tolerably well verified. It is to be noted also that, where the approximations would give a discontinuity in $\widehat{r\phi}$ *inside* the material (viz. at $z = \cdot 5c$), the true values are almost exactly the mean of the two discontinuous values obtained from the approximate formulæ assumed correct.

In like manner $\widehat{\phi z}$ is nearly the same as $\widehat{\phi z'}$, except near $z = \cdot 5c$, where, as we have seen, an infinite stress really occurs, of which the approximations give no hint.

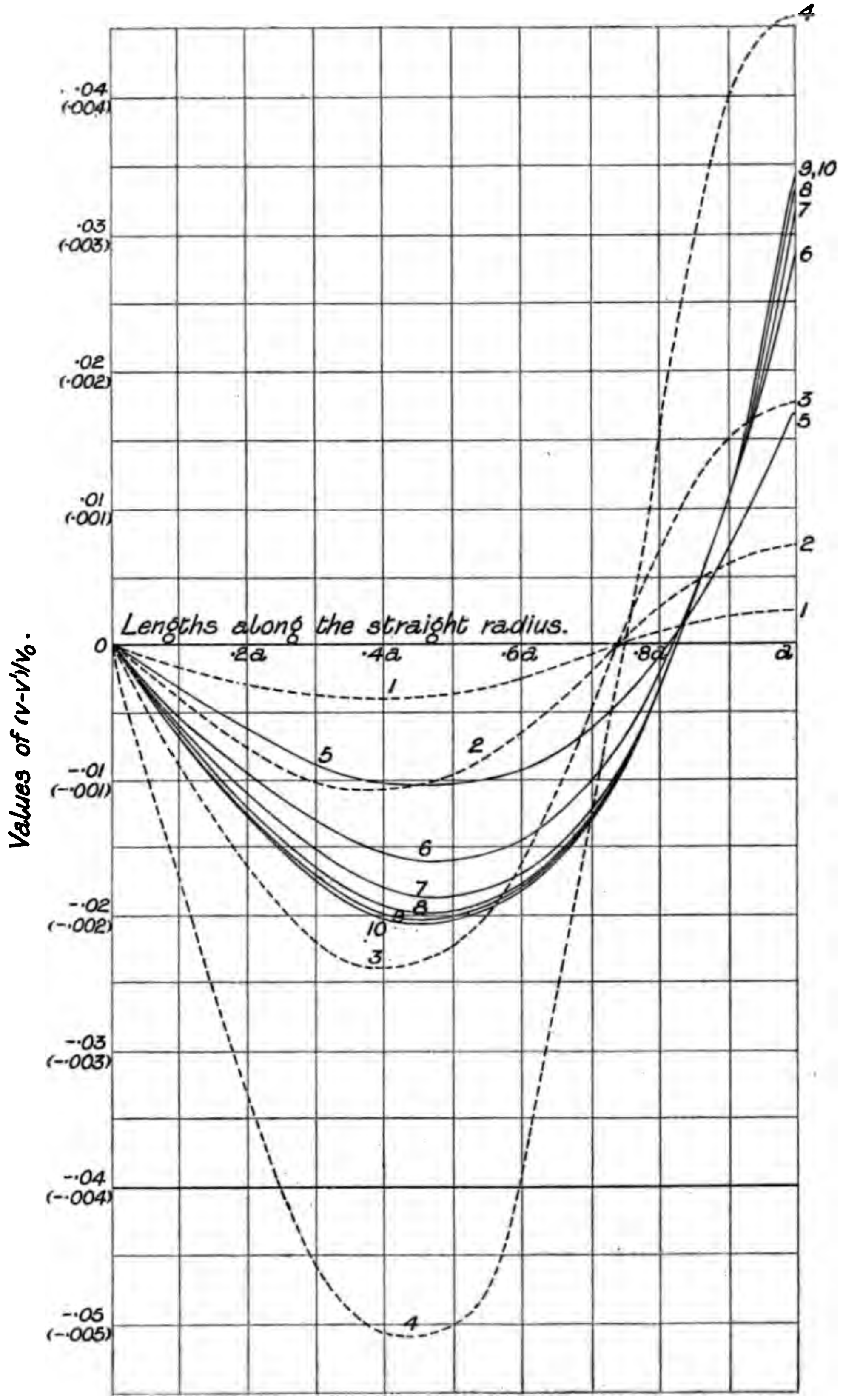
We note, however, that $\widehat{\phi z}$ does not strictly vary as r all over the section, being smaller than should be expected inside and larger at the boundary.

The theoretical result, that the stress is infinite where the transverse applied shear is discontinuous, throws much light on the case of a cylinder whose cross-section abruptly changes, as in fig. 1, with the difference that now the stress applied to the collar is transverse. We see that in such a case we should expect the material to give way at the points of sudden change. This conclusion is in accordance with practical experience, the tail ends of propeller shafts, for instance, breaking almost invariably in this manner.

§ 33. *General Conclusion.*

This example concludes the series of three which it was proposed to treat of. The object has been to obtain a clear idea of the effects of certain surface distributions of stress which come much nearer to the cases arising in practice than does the uniform distribution ordinarily taken.

Diagram 16.—Showing Distortion of a Radius originally Straight in the case of Torsion produced by applying Shearing Stress to the Curved Surface.



The curve corresponding to the section $z/c = n/10$ is numbered n. The first four curves have had the ordinates exaggerated in the ratio of 10 : 1. They are shown by the dotted lines, and to them refer the numbers in brackets.

No doubt the cases treated involve somewhat arbitrary conditions, not strictly obtained in practice, but it appeared useful to ascertain how far they gave results diverging from those which would be found on the ordinary hypothesis of uniform extension or torsion.

This furnishes us with a test of how far we may accept DE SAINT-VENANT'S principle of "equipollent" systems of load for a bar whose length is gradually made smaller compared with its diameter. The results we have here obtained indicate that, as we go away from the points of application of the stress, a "uniform" solution is reached much sooner in the case of torsion than in that of either tension or pressure.

With regard to the arithmetic of the paper, the results have been as far as possible checked. It is believed that they are correct to the number of figures given, but owing to the slow convergence of certain of the series, accumulated errors may in some cases affect the last and even the second last figure. Even this, however, would not sensibly disturb the conclusions.

For the I-functions the tables in GRAY and MATHEW'S "Bessel's Functions" were used, but the range of the tables is so limited that a large number of these functions had to be independently calculated. The semi-convergent expansions were employed, the argument being large in each case.

My very best thanks are due to Professor EWING for his unfailing kindness in coming to my aid with suggestions and advice.

V. *On the Mathematical Theory of Errors of Judgment, with Special Reference to the Personal Equation.*

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Introductory.

1. The following investigation has been in progress for some years and led to a paper, communicated to the Society on December 29, 1896.* I therein pointed out that personal judgments were frequently correlated. This correlation may be of the kind which in that paper I termed "spurious," or it may be genuine. By "spurious" correlation I understand the quantitative measure of a resemblance in judgments, which resemblance is due solely to the particular manipulation of the observations. Very customary treatment of observations will lead to the existence of a spurious correlation, which may be and generally is entirely overlooked by the observers. For example: if the quantity to be determined by judgment were the time taken by a bright point, say a star, in travelling from a position C intermediate between spider lines A and B to the line B, and the result were to be expressed by the ratio of this time to the known time from A to B, then there would be correlation in the results obtained by two observers for a number of stars, even if their absolute judgments on the

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* 'Roy. Soc. Proc.,' vol. 60, p. 489.

time from C to B were quite independent. Again, if the judgments of two observers be in both cases referred to a standard observer, then such relative judgments will be found to be correlated; and this is true, although if we could find the *absolute* errors of the two observers, we might discover that these errors were quite uncorrelated. We shall see illustrations below of the manner in which this spurious correlation almost imperceptibly creeps into any ordinary method of manipulating observations, and how very little attention has hitherto been paid to it.

But apart from this spurious correlation the experiments described in this memoir seem to show that there exists almost invariably a genuine correlation between the judgments of independent observers. This may be due to two sources: (i.) Likeness of the environment in the case of each individual observation, which leads to likeness of judgment in the individual observers. One experiment may appear to be made under precisely the same conditions as a second, but really it has a certain atmosphere of its own which influences the observers in a like manner. (ii.) Likeness in the physical or intellectual characters of the observers leading to a likeness in their judgments of what took place.

It is usual to suppose that the error made by an individual observer depends upon a great variety of small causes largely peculiar to that individual; or, if peculiar to the individual experiment, that they will affect different observers in different ways. Our experiments show such considerable correlation between the judgments of individual observers, that I have been compelled to discard this view; I consider that very slight variations of the environment (for example, similar observations on stars of different N.P.D.) will be quite sufficient to produce correlated judgments; on the other hand, some slight similarity of eye-sight, of ear, of temperament, may be sufficient to associate the judgments of two observers. Whatever variety of small causes influence the judgment, it is clear that in actual practice they do not suffice to dominate some particular source of mental or physical likeness which leads to this correlation in judgments.

Our first series of experiments show that the actual instantaneous environment is not necessarily the source of likeness in judgment. The same lines were not dealt with by the three observers at one and the same instant. Thus it is on some quite definite, but probably quite undiscoverable, likeness of temperament that we must largely rely to account for this correlation of judgment.

To the naturalist, who has to observe, whether he be physicist, astronomer, or biologist, this genuine correlation of judgments is of equal significance with the "spurious" correlation, and, like the latter, almost invariably disregarded. A and B are two independent observers, making an experiment of the same character, or observing the same phenomena. As a rule their judgments, however, will not be independent. The importance of this conclusion in modifying the weight which must be given to a series of observations of the same phenomena made by two "independent" observers will be manifest. Once we admit that the judgments of independent

We have, if n be the number of judgments in the series:—

$$\left.\begin{aligned}\sigma_{01}^2 &= \frac{1}{n}\,\mathrm{S}\{p_{01} - (x_1 - \xi)\}^2,\\ \sigma_{02}^2 &= \frac{1}{n}\,\mathrm{S}\{p_{02} - (x_2 - \xi)\}^2,\\ \sigma_{21}^2 &= \frac{1}{n}\,\mathrm{S}\{p_{12} - (x_2 - x_1)\}^2\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(i.)}$$

where S denotes a summation for every judgment of the series.

Obviously the goodness of an observer is measured by two characters:

(i.) The smallness of his personal equation, p_{01}.

(ii.) The smallness of the variability of his judgment, σ_{01}.

The first determines the average error of his judgment, the second the constancy or stability of his judgment.

The latter is often quite as important a feature of the mental worth of an observer as the former.

This steadiness or reliability of judgment, which I shall term stability of judgment, will be defined as follows:—The relative stability of two observers for a given class of observations is measured by the inverse ratio of their standard deviations. Or, if we are speaking of the same class of observation the absolute stability of judgment is $\frac{1}{\sigma_{01}}$. In the case of relative judgments, $\frac{1}{\sigma_{21}}$ will measure the steadiness in relative appreciation of two observers; it serves as a measure of their degree of approximation to like estimates, and may be called their relative stability. It by no means follows, however, that two observers with a large degree of relative stability have necessarily large individual absolute stabilities in judgment, nor that their absolute personal equations are small. This remark is of considerable importance, for we are apt to think that if two out of three observers have a small relative personal equation and a large relative stability, then their conclusions are worth more than those of a third observer with whom they have large relative personal equations and smaller relative stabilities.

No conclusion of this kind can be admitted, if we find that the absolute judgments of independent observers are correlated; for, as will be shown later, the higher this correlation, *i.e.*, the less independence in judgment, the greater becomes the relative stability of the two observers. The more marked this association in judgment, the less are we able to set the judgment of two observers against a third.

The correlation in absolute judgments between two observers* is given by

$$r_{12} = \frac{\mathrm{S}\{(p_{01} - (x_1 - \xi))(p_{02} - (x_2 - \xi))\}}{n\sigma_{01}\sigma_{02}} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(ii.)}$$

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* 'Roy. Soc. Proc.,' vol. 60, p. 480 *et seq.*

The correlation in the relative judgments of two observers, 1 and 2, both referred to a standard observer 3, is given by

$$\rho_{3,12} = \frac{S\{(p_{31} - (x_3 - x_1))(p_{32} - (x_3 - x_2))\}}{n\sigma_{31}\sigma_{32}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iii.}).$$

So far as I can make out it is usually assumed that r_{12} is zero, and the existence, if r_{12}, &c., be zero, of very sensible values in the case of $\rho_{3,12}$ has always been disregarded.

The probable errors* of personal equations, variability in judgments, and correlations in judgments, as determined by the formulæ (i.) (ii.) (iii.) above, are :—

$$\left.\begin{array}{lll}
\text{Per cent. of } p_{01} & = & \cdot 67449\ \sigma_{01}/\sqrt{n} \\
\quad ,, \quad ,, \ p_{02} & = & \cdot 67449\ \sigma_{02}/\sqrt{n} \\
\quad ,, \quad ,, \ p_{21} & = & \cdot 67449\ \sigma_{12}/\sqrt{n} \\
\quad ,, \quad ,, \ \sigma_{01} & = & \cdot 67449\ \sigma_{01}/\sqrt{2n} \\
\quad ,, \quad ,, \ \sigma_{02} & = & \cdot 67449\ \sigma_{02}/\sqrt{2n} \\
\quad ,, \quad ,, \ \sigma_{21} & = & \cdot 67449\ \sigma_{21}/\sqrt{2n} \\
\quad ,, \quad ,, \ r_{12} & = & \cdot 67449\ (1 - r_{12}^2)/\sqrt{n} \\
\quad ,, \quad ,, \ \rho_{3,12} & = & \cdot 67449\ (1 - \rho_{3,12}^2)/\sqrt{n}
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (\text{iv.}).$$

If any investigation of personal equation is to have validity these probable errors must be small relatively to the quantity measured. Accordingly, no determination of personal equation is of the slightest value which does not give σ as well as p, for without this we do not know the weight to be attributed to the determination of p. My own experience would seem to show that ten to thirty observations, on which number some estimates of personal equation have been formed, are very insufficient. Further, astronomers rarely publish the data on which the personal equation has been determined so as to enable one to judge of its degree of stability, or of the degree of independence in the judgments of different observers.

We shall have to investigate whether there are methods of finding σ_{01} and σ_{02} when only the relative personal equations and relative variabilities are given, and we shall have to see how the correlation of absolute and relative judgments may be determined.

Personally it appears to me that without a knowledge of all these quantities we cannot profitably combine the observations of different observers or determine their individual independence and stability of judgment.

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* 'Phil. Trans.,' A, vol. 191, pp. 239–245.

(3.) *Current Theory of Errors of Observation.*

The assumption usually made is that the error of an observation is due to the result of the combined action of a great number of independent sources of error; each source follows a permanent law and attributes equal probability of occurrence to numerically equal errors. From this statement, or some modified form of it,* is deduced the well-known normal curve of error frequency:—

$$y = y_0 e^{-x^2/2\sigma^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v.}).$$

An important point to be considered is, therefore, whether actual errors of observation in any case are such that they may be supposed to be a random sampling of errors obeying this law. I have in a recent paper† obtained a criterion for the probability of any system being the result of a random sampling from a series following any law of frequency, and I have shown that it is most highly improbable that the series cited by **Airy** and **Merriman** as evidence of the suitability of the normal curve can really have been random samples from material actually obeying such a distribution.

Assuming the applicability of the normal curve, or, indeed, the independence of judgments of independent observers,‡ we have at once

$$\left.\begin{array}{ll} & \sigma_{21}^2 = \sigma_{01}^2 + \sigma_{02}^2, \\ \text{Similarly:} & \sigma_{32}^2 = \sigma_{02}^2 + \sigma_{03}^2, \\ \text{and,} & \sigma_{13}^2 = \sigma_{03}^2 + \sigma_{01}^2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vi.}).$$

Hence we deduce:—

$$\left.\begin{array}{l} \sigma_{01}^2 = \dfrac{\sigma_{21}^2 + \sigma_{13}^2 - \sigma_{32}^2}{2}, \\[2ex] \sigma_{02}^2 = \dfrac{\sigma_{32}^2 + \sigma_{21}^2 - \sigma_{13}^2}{2}, \\[2ex] \sigma_{03}^2 = \dfrac{\sigma_{13}^2 + \sigma_{32}^2 - \sigma_{21}^2}{2} \end{array}\right| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vii.}).$$

It was this simple result which led to the whole of the present investigation. I had not seen it noticed before, and it seemed of wide-reaching importance. I mean in the following manner: The astronomer, and often the physicist, can, as a rule, only determine relative and not absolute judgments. He cannot deduce the absolute

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[*] These are really additional assumptions. See pp. 271-273 later.

[†] 'Phil. Mag.,' July, 1900, p. 157 et seq.

[‡] If x_1 and x_2 be judgments of two observers and x_{12} their relative judgment, δx_1, δx_2, δx_{12} errors measured from the means of the respective systems, then $\delta x_{12} = \delta x_1 - \delta x_2$, whence the result follows at once if the correlation $= \dfrac{S(\delta x_1 \times \delta x_2)}{n\sigma_{01}\sigma_{02}}$ be zero.

correlated. Independence of absolute judgments connotes correlation of relative judgments. This is, of course, an instance of what I have termed "spurious" correlation, but it is none the less important that it should not be overlooked. When we cannot form absolute judgments, but refer our observations to a special observer as standard, then the observations so reduced of two independent observers will certainly be correlated. I am not aware that attention has hitherto been paid to this point when the observations of different observers relative to a standard man have been combined.

One result of the actual correlation of independent judgments is that the values experimentally determined for the ρ's are not those given by (viii.). A genuine correlation is superposed on the spurious correlation, and the total correlation observed may be greater or less than the values indicated in (viii.).

(4.) *New Theory of Errors of Observation.*

Let us suppose that the correlations r_{32}, r_{13}, r_{21} are not zero, then, provided we calculate the standard deviations of the absolute and relative judgments, we can find at once these correlations. We have

$$\left.\begin{aligned} r_{23} &= \frac{\sigma_{02}{}^2 + \sigma_{03}{}^2 - \sigma_{23}{}^2}{2\sigma_{02}\sigma_{03}}, \\ r_{31} &= \frac{\sigma_{03}{}^2 + \sigma_{01}{}^2 - \sigma_{31}{}^2}{2\sigma_{03}\sigma_{01}}, \\ r_{12} &= \frac{\sigma_{01}{}^2 + \sigma_{02}{}^2 - \sigma_{12}{}^2}{2\sigma_{10}\sigma_{02}} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\text{ix.}).$$

We are no longer able to find the absolute variabilities from the relative variabilities, and we require direct experiments in which the errors of absolute judgment are known in order to determine the correlations.

Turning now to the correlations between relative judgments, we easily deduce from (iii.)

$$\begin{aligned} \rho_{3,\,12} &= \frac{\sigma_{03}{}^2 + r_{12}\sigma_{01}\sigma_{02} - r_{31}\sigma_{03}\sigma_{01} - r_{32}\sigma_{03}\sigma_{02}}{\sqrt{(\sigma_{03}{}^2 + \sigma_{01}{}^2 - 2\sigma_{03}\sigma_{01}r_{31})}\sqrt{(\sigma_{03}{}^2 + \sigma_{02}{}^2 - 2\sigma_{03}\sigma_{02}r_{32})}} \\ &= \frac{\sigma_{31}{}^2 + \sigma_{32}{}^2 - \sigma_{12}{}^2}{2\sigma_{31}\sigma_{32}}, \end{aligned}$$

since $$\sigma_{31}{}^2 = \sigma_{03}{}^2 + \sigma_{01}{}^2 - 2\sigma_{03}\sigma_{01}r_{31},$$

and similar relations hold.

We have thus the series:

$$\left.\begin{aligned} \rho_{1,23} &= \frac{\sigma_{12}^2 + \sigma_{13}^2 - \sigma_{23}^2}{2\sigma_{12}\sigma_{13}} \\ \rho_{2,31} &= \frac{\sigma_{23}^2 + \sigma_{21}^2 - \sigma_{31}^2}{2\sigma_{23}\sigma_{21}} \\ \rho_{3,12} &= \frac{\sigma_{31}^2 + \sigma_{32}^2 - \sigma_{12}^2}{2\sigma_{31}\sigma_{32}} \end{aligned}\right\} \quad \text{(x.)}.$$

These suffice to find the ρ's as soon as a series of experiments giving relative judgments has been carried out. They will not suffice to differentiate the real and the spurious parts of the correlation between the relative judgments.

These results are, of course, quite independent of any theory of normal distribution. The correlation coefficients will give the probable value of an error of judgment which A will make when we know the error that B has made in the same observation. Thus, if e_{02} be the average error made by a second observer when a first makes the error e_{01}, we shall not have e_{02} equal to the personal equation of the second observer, but given by

$$e_{02} = p_{02} - p_{01} r_{12} \frac{\sigma_{02}}{\sigma_{01}} + e_{01} r_{12} \frac{\sigma_{02}}{\sigma_{01}} \quad \text{(xi.)}.$$

Again, if e_{12} be the average error made by a second observer relative to a first, when a third observer makes an error relative to the first of e_{13}, then e_{12} will not be equal to the relative personal equation of the second observer, but must be determined from

$$e_{12} = p_{12} - p_{13}\, \rho_{1,23} \frac{\sigma_{12}}{\sigma_{13}} + e_{13}\, \rho_{1,23} \frac{\sigma_{12}}{\sigma_{13}} \quad \text{(xi. } bis\text{)}.$$

It will thus be clear that the reduction of isolated observations to a common standard depends essentially on a discovery of the intensity of correlation for absolute or relative errors. $e_{02} \doteq p_{02}$ will only be true when judgments have been shown to be perfectly independent. $e_{12} = p_{12}$ will practically be never true, for the ρ's can only vanish in the exceptional case in which the spurious and real correlations just balance each other's influence.

We shall find as we advance need to develop this theory in certain directions, but its main features have now been sufficiently indicated, and we can turn to the experimental results.

(5.) *General Description of the Experiments.*

The first series of experiments were made in the summer of 1896 by Dr. Alice Lee, Mr. G. U. Yule, and myself. They were very simple in character. Sheets of white paper ruled with faint blue lines were taken, such as are sold for "scribbling," and on each blue line two segments of a line were obtained by pricking with a needle point. This was done in triplicate by running the needle point through three adjusted sheets. These segments formed a random distribution of lengths placed on a series of horizontal lines. Each observer now took 500 such lines—the series being the same for each—struck a pencil stroke with a fine pencil through the needle points terminating each segment, and then bisected that segment with a third pencil stroke at sight. We thus obtained three series of estimates of the midpoints of the same group of lines by three apparently independent observers. The judgments were made in the same room, under practically the same conditions of light for each individual, but each experimenter was not necessarily bisecting the same line at the same instant of time. The common factors were the length of the line and its position relative to the edge of the paper, which latter varied from line to line. It does not appear to me that these factors are more or less influential than the sameness of influences which must ever arise when two or more individuals judge the same phenomenon.

The actual length of the lines and the distance from the left-hand terminal of the point guessed as midpoint were now very carefully measured ; whatever errors occur in these measurements, and of course such must exist, they are of a totally different order of magnitude to the errors of midpoint judgment.* The letter u will be used to denote the length of any line, x for the distance from the left-hand terminal to the experimental bisection, $x' = x - \frac{1}{2}u$ will stand for the error in placing the midpoint, considered positive when towards the right. The subscript 1 refers to Dr. LEE's judgment, the subscript 2 to my judgment, and the subscript 3 to Mr. YULE's judgment. I should have liked to have taken 1000 instead of 500 judgments, but the labour of experimenting, and especially also of arithmetical reduction is so great that we had to limit ourselves to the smaller number. Even that, I believe, is far greater than has yet been used in the determination of personal equation.

À priori, it seemed reasonable to me that the longer the line the greater would be the error of its bisection. Accordingly x'/u, or the ratio of the error to the length of the line, was taken in the first place as the quantity to be tabulated. I call this quantity X′. Dr. LEE spent several months of the summer of 1896 in the reduction of the observations on this basis, and the series of diagrams giving the frequency curves were drawn for X′. The reduction, however, showed at once that the values of X′ for different observers were correlated. Such correlation of what I then thought must be independent judgments led me to more closely investigate the matter. I attributed this correlation of independent judgments to spurious correlation due to the use of indices, and I determined to reconsider the subject on an entirely different experimental plan, after developing the theory of spurious correlation.†

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* That judgment was made rapidly as soon as the needle points terminating the line had been marked so as to be visible.

† See 'Roy. Soc. Proc.,' vol. 60, p. 489.

observers, and was practically absorbed, being invisible until a white strip was placed on the black screen; then the bright line was visible in a half-darkened room so long as it fell on the white strip. This white strip was 32·6 centims. long and 6 centims. broad; it could be placed anywhere on a scale painted in red on the black screen, and quite invisible to the observers.

The method of experimenting was as follows: The pendulum was brought to rest in a vertical position and the hammer was then moved up so as to touch the bell without resting against it, *i.e.*, it did not change its position when the pendulum was withdrawn. The line of light from the lantern now reflected from the stationary mirror fell on the scale on the black screen, which was adjusted by a fourth person so as to give a definite equilibrium position. The pendulum was now drawn back and clamped at a definite angle, which gave a very considerable range to the line of light. The three observers looking at the screen now saw no light at all, only 6 feet by 2 of black cloth. The fourth person now attached the strip of white card to the black cloth by aid of a drawing pin, so that its top coincided with any division on the scale known to himself only. He was thus able to make a record of the position on the strip occupied by the bright line when the hammer struck the bell. No doubt slight errors of adjustment occurred, but they were of much higher order than the errors of judgment. The equilibrium position of the beam was tested at the end of every twenty experiments, as well as the proper contact of the hammer.

A series of positions for the bright line on the strip were selected so as to cover fairly well the possible range, but the order in which these were taken was quite unknown to the observers. Of course, if the bell rung when the bright line just appeared on the strip, the latter was not moving as fast as if it rung when the bright line was just leaving the strip; but the range of the bright line was very considerable compared with the length of the strip, and I doubt whether this difference of speed was sufficient to sensibly influence the judgment.* The shifting of the strip on the screen was only adopted after it had been found that to adjust the equilibrium position of the bright line between each experiment to a fresh position on the screen-scale would mean an expenditure of time which it was impossible to provide for. It was easy enough to shift the equilibrium position, but it required two persons, one at the pendulum and one at the screen, to adjust the equilibrium position to a definite point of the scale, and the one at the screen instructing the other at the pendulum how to raise or lower the line of light in adjustment was likely, besides the evil of tediousness, to have far more influence upon the judgment of the observers than the fairly small shift of the strip while it was hidden from sight by the body of the adjuster.

Each observer was provided with a white sheet of paper on which were twenty

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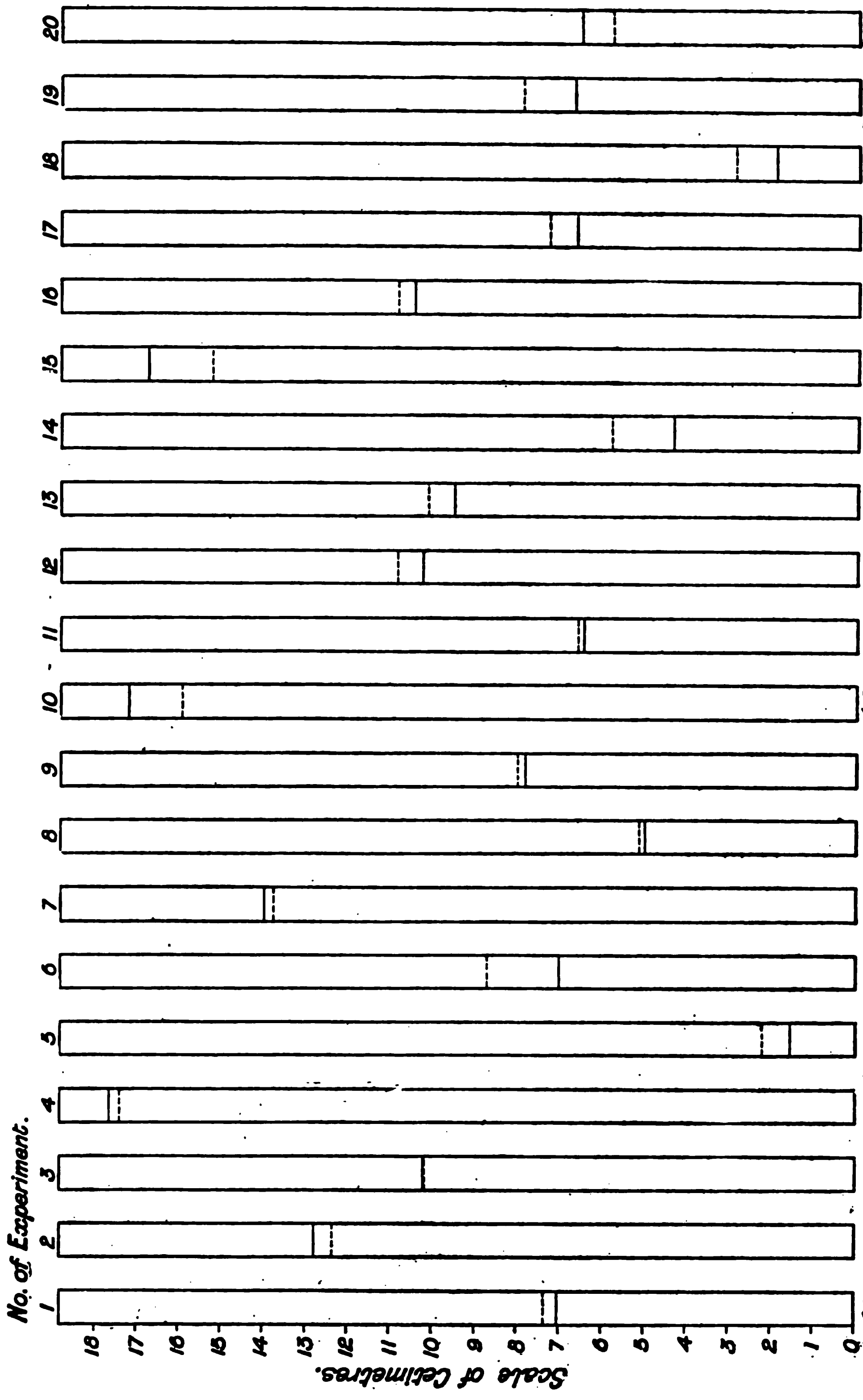
* If the correlations of judgments had been solely due to an "external cause" such as this, then it would not have been possible for the correlation to have been sensibly zero between two observers, but finite between the third observer and each of them.



Observer. K.P.

Sample of recording strips. { Dotted Lines give true position of bright line. Continuous Lines give estimated position of ditto.

The scale on the left was added after the observations had been recorded.

rectangles similar to the white strip on the black screen, and he drew across these "recording" strips a line in the position he considered the line of light to have on the observation strip when the bell sounded. Every strip already used was covered up before a new observation was made, so that it might not influence the next judgment; the lines were drawn from left to right and all measurements taken on the left-hand side of the strip. A facsimile of one of the sheets of observations accompanies this paper, and will give graphically an idea of the nature of the errors of judgment made. These errors were then scaled off to the nearest tenth of a millimetre, and formed the basis of the second series of errors of judgment. The line of light travelled *down* the strip, and if the estimated line is below the real line on the recording strip the error was considered positive. If the personal equation were solely due to reaction time, this positive error would represent a lag of the judgment, *i.e.*, the bright line would be recorded as occupying a position posterior to what it really occupied when the bell sounded. A glance at the observations, however, shows that reaction time must have had very small influence on the total magnitude of the personal equation; two observers made rather large negative mean errors, and the third only a very small positive mean error.

The experiments were carried out in about a week, not more than 2 hours being given to them at a time, to prevent over-fatigue. The observers were Dr. ALICE LEE, Dr. W. R. MACDONELL, and myself. Mr. K. TRESSLER kindly acted as adjuster of the scale. The observers were screened from each other, but the experiments being conducted in a long narrow room, the only one available, Dr. LEE was placed somewhat further from the observation strip than Dr. MACDONELL or myself. The only other differentiation between the observers, that I am aware of, was that I released the pendulum from its clamp with my left hand, drawing the recording line with my right; the bright line moved so slowly, however, that I was not at all conscious of being hurried, and, as a rule, I had my left hand on the table before the line of light had entered the strip.

As the arrangement of the pendulum seems likely to be of service for similar observations, especially in the psychological laboratory, it is figured on the opposite page.

In this series of experiments, which will be termed the "bright-line series" to distinguish it from the "bisection series," x represents the error of judgment considered positive as defined above, and the subscripts 1, 2, 3 refer respectively to me, Dr. MACDONELL, and Dr. LEE. Before entering into the details of these series, I shall consider some points bearing on the method of reducing material of this kind.

(6.) *On the Means and Standard Deviations of Grouped and Ungrouped Observations.*

It is well known that if the distribution of errors follows the normal law, the "best" method of finding the mean is to add up all the errors and divide by their number, the "best" method of finding the square of the standard deviation is to form the sum of the squares of the deviations from the mean and divide by their number, and the "best" method of finding a coefficient of correlation is to take the product of corresponding deviations from the respective means and divide by the product of the two standard deviations and the number of observations. These "best" methods become far too laborious in practice when the deviations run into hundreds or even thousands. The deviations are then grouped together, each group

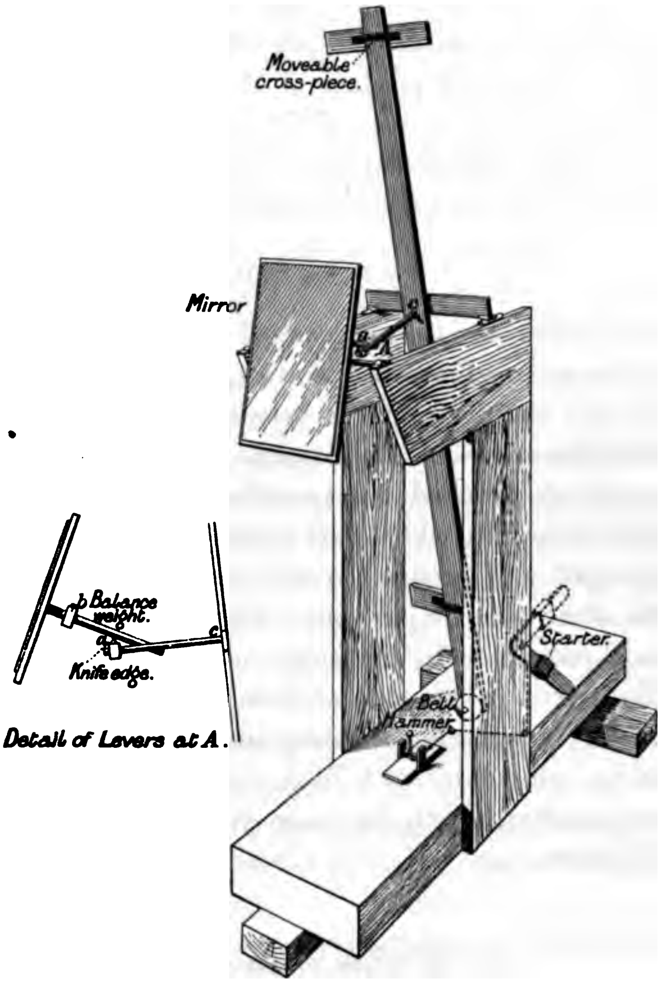


Fig. 1.—Apparatus for Personal Equation.

containing all deviations falling within a certain small range of quantity, and the means, standard deviations, and correlations are deduced from these grouped observations. If the means, standard deviations, and correlations be calculated from the grouped frequencies, as if these frequencies were actually the frequency of deviations coinciding with the midpoints of the small ranges which serve for the basis of the grouping, we do not obtain the same values as in the case of the ungrouped observations. It becomes of some importance to determine what corrective terms ought to be applied to make the grouped and ungrouped results accord. This point has been considered by Mr. W. F. SHEPPARD,* who has shown that from the square of the standard deviation we ought to subtract $\frac{1}{12}$th of the square of the base element of grouping, but that the mean and product of the grouped deviations should be left uncorrected. Thus corrected the values of the constants of the distribution as found from the ungrouped and grouped deviations will nearly, but not of course absolutely, coincide. In particular while the personal equation relation

$$p_{21} = p_{02} - p_{01}$$

will be absolutely satisfied for the ungrouped material, it will generally not be satisfied exactly for the grouped results. A test, however, of the practical justification for grouping is that the divergencies between the two methods ought to be of the order of the probable errors of the results. If this be so, then we may safely group. The fact that my grouped observations did not satisfy the relation cited above, led me to think it worth while that a comparison should at any rate be once made between ungrouped and grouped results on a large series of actual errors of observation. At the same time it gave me a means of verifying the accuracy of our very long arithmetical reductions by an independent investigation. The ungrouped observations were dealt with in the case of nine series involving 500 or 519 observations each. The labour of squaring so many individual deviations each read to four figures was lessened by using BARLOW's Tables, and the series were added up by aid of an American Comptometer, which for some years past we have found of great aid in statistical investigations.

(a.) *Bisection of Line Series.*

In Table I. will be found a comparison of the ungrouped and grouped results so far as the means and S.D.'s are concerned for our first series. X′ has been defined as the ratio of the error made in bisection to the length of the line bisected.

Here mean X_1' denotes that Dr. LEE made an average error of about 12/1000 of the length of the line in bisecting it, and that this error was to the *right* of the true midpoint. Mr. YULE and I made average errors of 4 to 5/1000 of a line in bisecting

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* 'London Math. Soc. Proc.,' vol. 29, pp. 368, 375.

TABLE I.

500 trials.	Absolute personal equation.			Relative personal equation.		
	X_1'.	X_2'.	X_3'.	$X_2' - X_3'$.	$X_3' - X_1'$.	$X_1' - X_2'$.
Mean, ungrouped . . .	+ ·01235 ± ·00074	− ·00444 ± ·00093	− ·00469 ± ·00079	+ ·00026	− ·01704	+ ·01679
„ grouped . . .	+ ·01230 ± ·00075	− ·00495 ± ·00093	− ·00377 ± ·00080	− ·00123 ± ·00098	− ·01589 ± ·00103	+ ·01712 ± ·00106
S.D., ungrouped . . .	·02464 ± ·00053	·03068 ± ·00065	·02618 ± ·00056			
„ grouped	·02455 ± ·00053	·03065 ± ·00065	·02625 ± ·00056	·03236 ± ·00069	·03376 ± ·00073	·03519 ± ·00075

it, and our errors were both to the *left* of the midpoint.* All these absolute equations are seen to be considerable multiples of their probable errors, or are undoubtedly significant. While Dr. LEE's personal equation is, roughly, three times as large as Mr. YULE's or mine, she is steadier in her judgment, our relative steadiness being as $\frac{1}{25} : \frac{1}{31} : \frac{1}{26}$ nearly, or about as 40 : 32 : 38.

The absolute personal equations show that the probable errors of the means and of the standard deviations are for all practical purposes identical, whether they are calculated from the standard deviations of the ungrouped or grouped observations. From these probable errors we see that the differences between the ungrouped and grouped results are in all cases but two less than the probable error of the quantity; in one of these cases, however, the difference is only very slightly greater, and accordingly it is not of any practical importance. In the other case, Mr. YULE's personal equation is insignificantly larger than mine for ungrouped results, and slightly smaller than mine for grouped results. The effect of this is that our relative personal equation swings round from negative to positive as we pass from ungrouped to grouped deviations. The total change is only ·00149, and as the probable error of the result is ·00098, we are perhaps hardly justified in holding that the grouped results are in disagreement with the ungrouped. I think all we could say is that our absolute personal equations are very nearly equal, and that we have sensibly no relative personal equation. The differences of the other relative personal equations as found by the two methods are less than their probable errors.†

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* The light fell from the left hand on the paper for all three experimenters during the bisections.

† The reader will notice at once that the relation $p_{23} = p_{02} - p_{03}$ no longer holds. If we deduce the relative from the absolute personal equations we find:

$$p_{23} = -\cdot 00118,\quad p_{31} = -\cdot 01607 \text{ and } p_{12} = +\cdot 01725 \text{ instead of}$$
$$-\cdot 00123 \qquad -\cdot 01589 \text{ and } \qquad +\cdot 01712 \text{ respectively.}$$

The differences are, however, quite insignificant, when we consider the probable errors.

So far, then, as this first series of experiments goes, we have ample justification for grouping our deviations.

(*b.*) *Bright Line Series.*

In this case I compared the results for ungrouped and grouped observations not only as far as concerns absolute personal equations, but also for the relative personal equations, and even for the coefficients of correlation. We have therefore a still wider basis for drawing inferences. This is done in Table II., x being now the error, positive if the bright line is recorded on the strip as being below its true position.

To find a length on the observation strip from that on the recording strip we have to multiply by the factor 1·734.

TABLE II.

519 observations.	Absolute personal equation.			Relative personal equation.		
	x_1.	x_2.	x_3.	$x_2 - x_3$.	$x_3 - x_1$.	$x_1 - x_2$.
Mean, ungrouped	+ ·06724 ± ·03538	− 1·14906 ± ·03480	− ·48563 ± ·05377	− ·66343 ± ·05170	− ·55287 ± ·05954	+ 1·21630 ± ·04928
„ grouped .	+ ·07774 ± ·03521	− 1·14483 ± ·03473	− ·44635 ± ·05393	− ·68275 ± ·05148	− ·61145 ± ·05943	+ 1·21518 ± ·04932
S.D. ungrouped .	1·19495 ± ·02502	1·17546 ± ·02461	1·81599 ± ·03802	1·74616 ± ·03656	2·01091 ± ·04210	1·66454 ± ·03485
„ grouped .	1·18913 ± ·02489	1·17289 ± ·02455	1·82146 ± ·03813	1·73883 ± ·03640	2·00717 ± ·04202	1·66597 ± ·03488
Correlations.	r_{23}.	r_{31}.	r_{12}.	$\rho_{1, 23}$.	$\rho_{2, 31}$.	$\rho_{3, 12}$.
Ungrouped . .	·3819 ± ·0253	·1571 ± ·0289	·0139 ± ·0296	·5625 ± ·0202	·3055 ± ·0268	·6154 ± ·0184
Grouped . . .	·3908 ± ·0251	·1624 ± ·0288	·0051 ± ·0296	·5653 ± ·0201	·3056 ± ·0268	·6127 ± ·0185

We see at once from this table that the probable errors of means, standard deviations, and correlations are for all practical purposes the same whether we group the observations or not. In the next place we find that, judged by these probable errors, the differences are less than would arise from the results of random sampling. Thus in all cases the differences are less than the probable errors, and in most cases very considerably less. The greatest divergence occurs in the relative personal equation of Dr. LEE and myself, but even in this case the difference is just less than the probable error. We may accordingly conclude that with such a number of length of the line bisected. On the other hand, it must be noted that the fluctuations in the personal equation when we come to deal with series of 25 are much larger than the probable error of a random sampling. The probable error of the personal equation of Dr. LEE, based on 25 experiments, is ± ·00331, but actually the personal equation as determined from two different sets of 25 experiments may amount to seven or eight times this amount. In other words, there is a significant difference in personal equation depending upon the individual 25 lines bisected

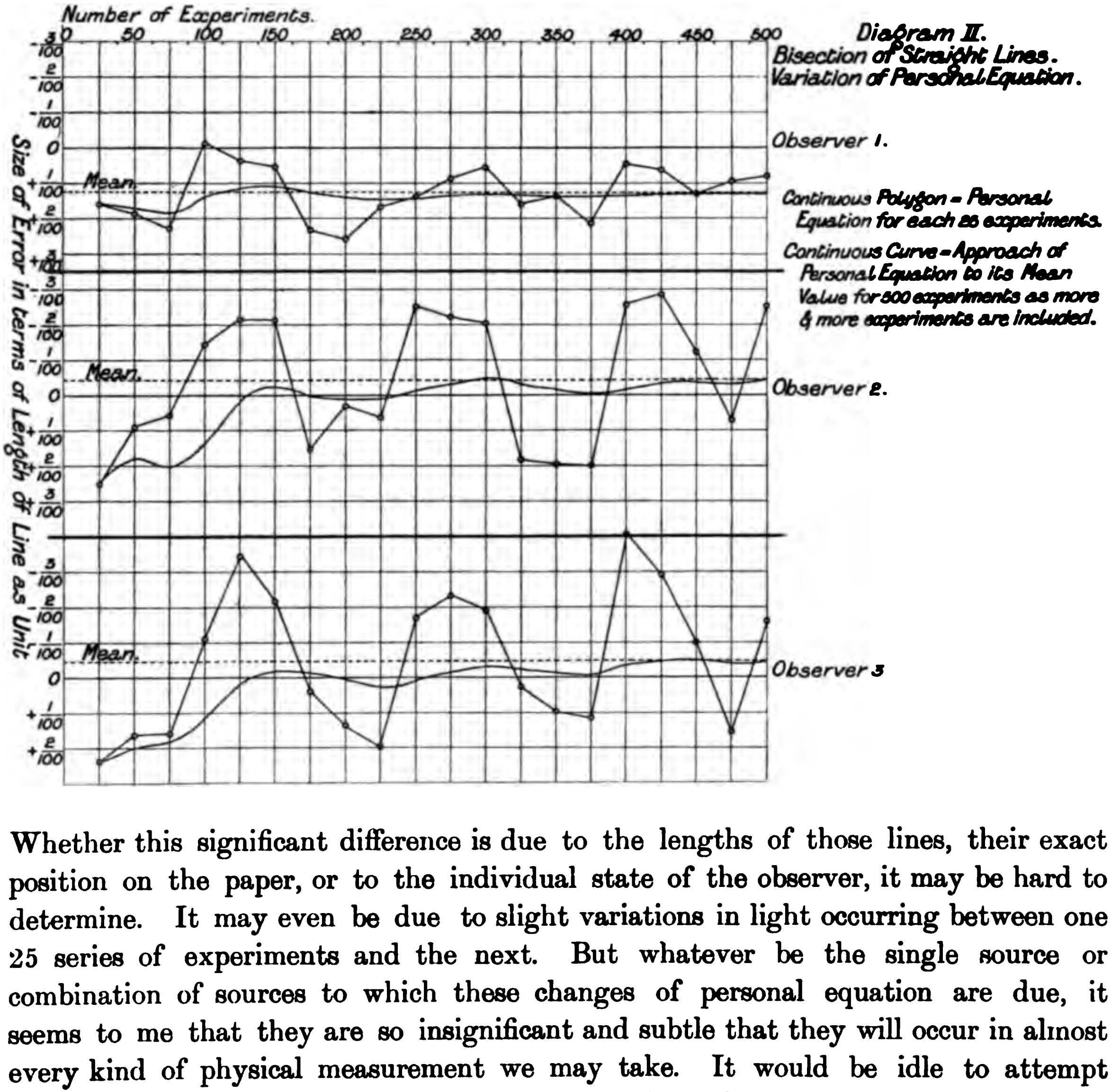
Diagram II.
Bisection of Straight Lines.
Variation of Personal Equation.

Whether this significant difference is due to the lengths of those lines, their exact position on the paper, or to the individual state of the observer, it may be hard to determine. It may even be due to slight variations in light occurring between one 25 series of experiments and the next. But whatever be the single source or combination of sources to which these changes of personal equation are due, it seems to me that they are so insignificant and subtle that they will occur in almost every kind of physical measurement we may take. It would be idle to attempt indeed to discover and eliminate such sources, for while it might be possible after elaborate investigation to eliminate them in an especially devised series of experiments, this could not be done in practice, where we must take our observer's experiments as they are given to us, and where we cannot possibly ensure uniformity in light, in mood, or health of observer, and as well as in all the features of the observed phenomena. The true conclusion appears to be that the range of data upon which the personal equation is based must be very wide, so as to swamp as far as possible these sources of variation due to the "atmosphere" of a short consecutive series. But if such a personal equation be found, what will be its value? It can hardly be applied satisfactorily to an isolated observation or to a short consecutive series of observations, for these will of course be influenced by their special atmosphere. It would only have value for a long series such as it was itself determined for, and such a series would rarely occur in practice. The fact that in both our series of experiments the differences between the values of the personal equation as found from short series are many times the probable error of sampling is very remarkable. I shall refer to it as "the influence of immediate atmosphere," where I understand the "atmosphere" to be compounded of all the little sources which affect either the observed thing or the observer more or less persistently during a short series. I am prepared to be told that the influence of immediate atmosphere was something peculiar to our own test experiments. But I shall require a good deal of the hard logic of experimental facts to be convinced that it has no existence in astronomical observations. There are many determinations of astronomical personal equation, but in published data I have been unable to discover enough material to determine how far the admitted variations in personal equation for short series are or are not of the order of deviations due to random sampling.

The following data will bring out the points of this discussion:—

TABLE V.—Personal Equation.

Observer.	First series.				Second series.			
	Experiments.	Bisection of lines.			Experiments.	Position of bright line.		
		1.	2.	3.		1.	2.	3.
Mean	500	+ ·01235	− ·00444	− ·00469	519	·06724	−1·14906	− ·48563
Ditto	1–250	+ 01424	− 00173	+ ·00065	1–266	·09571	−1·17282	− ·28940
Ditto	251–500	+ ·01046	− ·00714	− ·01004	267*–520	·03731	−1·12407	− ·69194
Standard deviation . .	500	·02464	·03068	·02618	519	1·19495	1·17546	1·81599
Probable error of mean	500	·00074	·00093	·00079	519	·03538	·03480	·05377
Ditto	250	·00105	·00132	·00112	260	·04998	·04917	·07596
Ditto	25	·00331	·00416	·00353	30	·14715	·14475	·22369

* Experiment 291 omitted.

This table shows us:—

(*a*.) That the probable errors for the personal equations deduced from twenty-five bisections are such that the fluctuations of personal equation given in Table III. or Diagram 2 are in very many cases significant.

TABLE III.—Personal Equation in Bisection of Lines.

Experiments.	Observer 1.		Observer 2.		Observer 3.	
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
1–25	+ ·01548	+ ·01548	+ ·02627	+ ·02627	+ ·02379	+ ·02379
26–50	+ ·01828	+ ·01688	+ ·00890	+ ·01759	+ ·01600	+ ·01989
51–75	+ ·02252	+ ·01876	+ ·00596	+ ·02056	+ ·01585	+ ·01855
76–100	- ·00149	+ ·01370	- ·01429	+ ·01342	- ·01149	+ ·01104
101–125	+ ·00340	+ ·01164	- ·02163	+ ·00104	- ·03513	+ ·00180
126–150	+ ·00498	+ ·01053	- ·02112	- ·00265	- ·02196	- ·00216
151–175	+ ·02326	+ ·01235	+ ·01524	- ·00010	+ ·00381	- ·00130
176–200	+ ·02576	+ ·01402	+ ·00290	+ ·00028	+ ·01326	+ ·00052
201–225	+ ·01629	+ ·01428	+ ·00649	+ ·00097	+ ·01979	+ ·00266
226–250	+ ·01396	+ ·01424	- ·02604	- ·00173	- ·01739	+ ·00065
251–275	+ ·00822	+ ·01370	- ·02217	- ·00359	- ·02387	- ·00158
276–300	+ ·00514	+ ·01298	- ·02105	- ·00504	- ·01977	- ·00309
301–325	+ ·01595	+ ·01321	+ ·01814	- ·00316	+ ·00274	- ·00264
326–350	+ ·01323	+ ·01321	+ ·01943	- ·00164	+ ·00944	- ·00178
351–375	+ ·02140	+ ·01376	+ ·02003	- ·00020	+ ·01132	- ·00091
376–400	+ ·00484	+ ·01320	- ·02636	- ·00183	- ·04012	- ·00336
401–425	+ ·00614	+ ·01279	- ·02902	- ·00343	- ·02904	- ·00487
426–450	+ ·01320	+ ·01281	- ·01227	- ·00392	- ·00978	- ·00514
451–475	+ ·00946	+ ·01263	+ ·00073	- ·00333	+ ·01512	- ·00408
476–500	+ ·00698	+ ·01235	- ·02548	- ·00444	- ·01643	- ·00469

TABLE IV.—Personal Equation for Position of Bright Line.

Experiments.	Observer 1.		Observer 2.		Observer 3.	
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
1–37	+ ·29973	+ ·29973	- ·27703	- ·27703	+ ·42243	+ ·42243
38–74	+ ·04838	+ ·17406	- ·50540	- ·44122	+ ·65919	+ ·54081
75–111	+ ·49432	+ ·28081	- 1·58568	- ·82270	+ ·22459	+ ·43541
112–148	+ ·13595	+ ·24459	- 1·41649	- ·97115	+ ·42162	+ ·43196
149–185	+ ·02459	+ ·20059	- 1·18027	- 1·01297	- 1·02459	+ ·14065
186–212	- ·03074	+ ·17113	- 1·61037	- 1·08906	- 1·15704	- ·02462
213–239	- ·09185	+ ·14142	- 1·60111	- 1·14690	- 1·73926	- ·21707
240–266	- ·30889	+ ·09571	- 1·40222	- 1·17282	- ·92963	- ·28940
*267–293	- ·40308	+ ·05130	- 1·21692	- 1·17675	- ·52346	- ·31024
294–320	- ·24519	+ ·02621	- 1·24593	- 1·18260	- ·67747	- ·34132
321–347	- ·54185	- ·01812	- ·97185	- 1·16616	- ·16556	- ·32760
348–374	+ ·08889	- ·01038	- 1·25704	- 1·17273	- ·66889	- ·35231
375–401	- ·04185	- ·01250	- ·92222	- 1·15582	- ·47518	- ·36060
402–435	- ·13735	- ·02228	- 1·07912	- 1·14982	- 1·22735	- ·42850
436–469	+ ·48117	+ ·01429	- ·86324	- 1·12900	- 1·34088	- ·49479
470–503	+ ·52029	+ ·04857	- 1·45265	- 1·15092	- ·39588	- ·48809
504–520	+ ·61882	+ ·06724	- 1·09412	- 1·14906	- ·41294	- ·48563

Columns (*a*) contain the personal equations determined from the experiments given in the first column. Columns (*b*) contain the personal equations determined from all the experiments up to and including the last given in the corresponding line of the first column.

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* Not including No. 291, which was *à priori* rejected. Hence the total number of experiments dealt with is one less after this than the number recorded to the right of the first column.

Although the personal equations in Table IV. are based upon series varying in number from 26 to 37, the probable errors for thirty observations of the position of a bright line suffice to show that the fluctuations in the values of the personal equations as given in Table IV. or in Diagram III., p. 270, are in many cases significant.

(*b.*) The probable errors for the personal equation in bisecting 250 lines show that there were significant changes in the personal equations of the three observers between the first and second moiety of the experiments. While Dr. LEE (1) bettered her judgment by ·004, Mr. YULE (3) swung over from ·001 to right of true midpoint to ·010 to left of midpoint, and I had a worse judgment by ·005 in the second moiety when compared with the first moiety of the results.

In the case of the second series with the bright line, Dr. MACDONELL (2) and I (1) have changes slightly for the better in our judgments between the 266 first experiments and the 253 second experiments; but having regard to the probable errors given for 260 experiments, it may be doubted whether these changes are significant. Dr. LEE (3) has, however, a quite significant change for the worse.

The fact that in some cases the personal equation grows less, in others greater, in the second half of the series seems to indicate that the changes in personal equation were by no means due to a secular improvement in judgment.* Nor do they admit of explanation on the assumption of increasing fatigue due to the exhaustion of the power of attention. It must be remembered that the experiments were spread out over a number of days, and this cause would only influence the latter experiments on each day. My worst experiments on the bright line are the Series 321–347 and 504–520 (Observer (1) Column (a) Table IV.), but they are much above the average in goodness for Dr. LEE (Observer (3) Column (a)), and above the average for Dr. MACDONELL. Dr. MACDONELL'S worst results are 186 to 239 (Observer (2) Column (a)), and these, especially 213 to 239, are bad for Dr. LEE, but they are very good results so far as I am concerned. If any fluctuation was accordingly due to fatigue, it did not affect us alike.

While these fluctuations in short series are significant, they by no means screen the general features of each observer's individuality. Dr. LEE is clearly in the habit of bisecting straight lines at a point some $\frac{1}{100}$ or more to the right of the true point of bisection, while I place it with a sensibly less error to the left. She places a line of light moving downwards over a vertical strip ·8 centim. above its true position, and I about ·1 centim. below its true position at any instant. Dr. MACDONELL, on the other hand, with the steadiest judgment of all three, displaces it 2 centims. above its true position.† The differences of personal equation in both series for all three observers are quite significant when compared with the probable errors of the differences, *i.e.*, there is a real individuality in observation which manifests itself in the personal equation.

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* It should be noted that in the cases of Dr. LEE, Mr. YULE, and myself we have for years been accustomed to reading scales and judging proportional parts by the eye.

† Table V., second series, gives lengths on recording strip. The actual values for observing strip are given on p. 253.

But the fluctuations in the personal equation are significant too, and they cannot offhand be attributed to anything like betterment with practice, or decadence with fatigue.

So long as the variations in the constants of an experimental series can be shown to be within the errors of random sampling we feel on safe ground; we know the number of experiments required to obtain a result with any required degree of accuracy. On the other hand, when we find significant fluctuations in the personal equation depending on the influence of immediate atmosphere, it becomes all the more important to show in each individual investigation that the personal equation itself is insignificant. Let me illustrate this point. A physicist makes twenty or thirty measurements of a quantity, say by aid of a bright line moving across a scale. He gives the mean value m of the result and also what he terms its probable error e. Now the use of this probable error I take it to be this. If the same experiments were to be repeated by the same man the same number of times with the mean result m', then we should expect to find $m' - m$ not a large multiple of the probable error of the difference $\sqrt{(e^2 + e^2)} = \sqrt{(2)}\,e$. e gives us a test of the closeness with which the result will repeat itself on repetition of the experiments. But the whole foundation of this statement is the hypothesis that the twenty or thirty experiments dealt with are a *random sampling* of all possible experiments that might be made. Now the variability in the results of the individual experiments includes the variability of personal error, and the hypothesis supposes that the personal errors are a random sampling of the observer's personal errors. Our investigations seem to indicate that the personal errors are far from being a random sampling but depend in some subtle manner on the influence of immediate atmosphere. Hence, unless it can be shown that the latter influence is small as compared with other sources of error in the measurement under consideration, the mere calculation of the probable error is by no means a security for the same observer reaching the same result on repeating the original series of experiments.

We, of course, for both series selected experiments in which the personal error would be large,* and accordingly could be easily dealt with. But the division of scale lengths by the eye and the estimated position of a bright line are fundamental in many types of physical observation. Further, large errors are for theoretical purposes quite as good as, for practical purposes much better than, small, when we wish to obtain an answer to the question: Are the fluctuations in personal equation merely the result of random sampling, or are they due to the influence of immediate atmosphere?

So important is it to realise that these fluctuations are not due to random sampling, that I have worked out all the constants for the second series, for the whole set of experiments, and for its first and second moiety.

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* As a matter of fact only at a maximum $\frac{1}{100}$ in dividing a line and $\frac{1}{10}$ in determining the position of a bright line between two scale marks.

They are given in the accompanying table.

TABLE VI.—Influence on Constants of Fluctuations in Personal Equation.

		All Observations. 1–520 (without 291).			*First Series.* 1–266 inclusive.			*Second Series.* 267–520 (without 291).		
	Observer.	1	2	3	1	2	3	1	2	3
Absolute Judgments.	Mean . .	·0672 ± ·0354	−1·1491 ± ·0348	− ·4856 ± ·0537	·0957 ± ·0514	−1·1728 ± ·0520	− ·2894 ± ·0778	·0373 ± ·0484	−1·1241 ± ·0459	− ·6919 ± ·0728
	S.D. . .	1·1949 ± ·0250	1·1755 ± ·0246	1·8160 ± ·0380	1·2428 ± ·0363	1·2563 ± ·0367	1·8815 ± ·0550	1·1417 ± ·0342	1·0833 ± ·0325	1·7205 ± ·0516
	Correlation	·3819 ± ·0253	·1571 ± ·0289	·0139 ± ·0296	3677 ± ·0355	·2594 ± ·0386	·0530 ± ·0412	·4123 ± ·0352	·0256 ± ·0424	− ·0368 ± ·0423
	Observers.	3–2	1–3	2–1	3–2	1–3	2–1	3–2	1–3	2–1
Relative Judgments.	Mean . .	·6634 ± ·0517	·5529 ± ·0595	−1·2163 ± ·0493	·8834 ± ·0760	·3851 ± ·0814	−1·2685 ± ·0711	·4321 ± ·0683	·7292 ± ·0865	−1·1614 ± ·0680
	S.D. . .	1·7462 ± ·0366	2·0109 ± ·0421	1·6645 ± ·0348	1·8385 ± ·0538	1·9676 ± ·0575	1·7198 ± ·0503	1·6114 ± ·0483	2·0406 ± ·0612	1·6026 ± ·0481
	Correlation	·5625 ± ·0202	·3055 ± ·0268	·6154 ± ·0184	·5085 ± ·0307	·3900 ± ·0351	·5935 ± ·0268	·6323 ± ·0255	·1938 ± ·0408	·6375 ± ·0252

In the row in absolute judgments, entitled "Correlation," the correlation, r_{32}, of the judgment of the second and third observers is entered in column (1), r_{13} in column (2), r_{21} in column (3). In the row in relative judgments, entitled "Correlation," the correlation of the judgments of the second and third observers referred to the first observor as a standard, or $\rho_{1,\,23}$, is entered in column (1), $\rho_{2,\,31}$ in column (2), and $\rho_{3,\,12}$ in column (3).

The figures in antique type give the probable errors of each constant, and the probable errors in the differences of the constants can be found in the usual way as the square root of the sum of the squares of these.

Dealing first with the absolute observations, we note that the personal equations of Dr. MACDONELL and myself, (2) and (1), are within the limits of the probable errors the same for either half series and for the whole series. Both of us appear to have improved by about ·03, but whether this is a real improvement between the first and second series it is impossible to say, for the probable error of the half series is as much as ·05. In my own case, in the second series my personal equation is less than its probable error, and accordingly on the basis of 253 experiments—a number be it noted far larger than could ever be made in actual practice—it would be impossible to say whether I had a personal equation or not. I mention this point, because it seems to me a *sine quâ non* of all investigations of personal equation that the probable error of the results should be given, and in most cases one seeks for it in vain.

Dr. LEE's personal equation has increased substantially between the first and second series. All three observers have grown apparently steadier in their judgment. The probable errors, however, of the S.D.'s do not allow of the assertion that the steadiness has substantially increased. The variations in correlations of judgments are noteworthy. Judged from the first series, or the second series, or the whole series, the correlation between the judgments of Dr. LEE and Dr. MACDONELL remains sensibly the same, *i.e.*, ·4 within the limits of the probable error; there is sensibly no correlation between the judgments of Dr. MACDONELL and myself as given by any of the three series. Between Dr. LEE and myself there is on the whole series a substantial correlation of ·16 ± ·03, but the two half series show us that it was on the wane during the course of the experiments, having fallen from the comparatively high value of ·26 to practically zero between the two half series. Whatever causes therefore produced the marked divergence of personal equation between Dr. MACDONELL and myself, they seemed to have been combined in Dr. LEE, and—to speak metaphorically—the dominant set for Dr. MACDONELL became after a struggle dominant for Dr. LEE; her methods of judging in the course of the experiments became more and more like Dr. MACDONELL'S and less like mine.

We turn now to the relative judgments. These it must be remembered are the only data which would be generally known in practice. Here it is only in the difference of Dr. MACDONELL'S and my judgments (column 2–1) that there is any real approach to constancy in the relative personal equation. The differences of our judgments have sensibly the same value for the first, the second, and the whole series. The same remark applies also to relative steadiness of judgment.* On the other hand, the relative personal equations of Dr. LEE and Dr. MACDONELL, or of Dr. LEE and me, differ substantially between the first half and the second half series. The relative steadinesses of judgment are less altered, being sensibly constant for Dr. LEE and myself, but possibly varying slightly for Dr. LEE and Dr. MACDONELL.

When we turn to the correlation of relative judgments, that of Dr. MACDONELL'S and my judgments, referred to Dr. LEE'S as standard, shows sensible constancy throughout the three series; that of Dr. MACDONELL'S and Dr. LEE'S, referred to mine as standard, shows not very large but sensible change; and finally that of Dr. LEE'S and mine referred to Dr. MACDONELL'S, shows very substantial modification.

Now judged by size of personal equation I stand first and Dr. MACDONELL last, judged by steadiness Dr. MACDONELL and I are almost equal (within the limits of the probable error), and Dr. LEE last. The most constant results for absolute personal equation are found—as we might *à priori* expect they would be—where the steadiness is greatest. But if we wish to obtain relative judgments whose relationship to each other will remain at closely the same value during a long series, then apparently we ought to refer not to the most steady, but to the least steady of the observers as a standard.

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* I may remind the reader of what this exactly means: The differences of 1·72 and 1·60, the standard deviations for (2–1) in the first and second series, from 1·66, the standard deviation in the whole series, are about ·06, and this is just about the magnitude of the probable error of these differences, *i.e.*, $\sqrt{\{(\cdot035)^2 + (\cdot050)^2\}} = \cdot061$.

It may be asked how, when as in practice we only know the relative judgments, are we to find out the degree of steadiness of the individual observers? This is a very important problem, and the answer would be perfectly clear if the old theory on p. 240 of this memoir were correct. Unfortunately the correlation of judgments comes in, and deprives us of any means of judging from a knowledge of relative steadinesses what the absolute steadinesses are. Let me illustrate this: The relative variabilities are greatest in the cases of 3–2 and 1–3, we might therefore suppose 3 to be least steady; the relative variabilities are least for 3–2 and 2–1, we might therefore suppose 2 to be most steady, and we should thus reach the actual scale of steadiness in absolute judgments—Dr. MACDONELL, myself, Dr. LEE. But now turn from the bright-line experiments in Table VI. to the bisection experiments in Table I. The relative judgment standard deviations are greatest for 3–1 and 1–2 and least for 2–3 and 3–1, we should therefore suppose that 1 was least steady and 3 most steady, or the order of steadiness 3, 2, 1, *i.e.*, Mr. YULE, myself, Dr. LEE. But an examination of the absolute standard deviations shows us that the real order is quite different, being Dr. LEE, Mr. YULE, and myself. In other words, no argument can be drawn, owing to the correlation in judgments, from relative to absolute steadiness.

It seems therefore impossible without experiments *ad hoc* to determine which observer is steadiest in judgment from a knowledge of relative personal equations.

We can only conclude that, at any rate in our own cases, the fluctuations in personal equation are such that, even in what are—for practical purposes—very large series, we cannot invariably assume them to be due to random sampling. We cannot attribute sensible changes in our own case to practice or to fatigue, but the high correlation of judgments suggests an "influence of the immediate atmosphere," which may work upon two observers for a time in the same manner.

(8.) *On the Interdependence of Judgments of the same Phenomenon.*

(i.) *The Bright-line Experiments.*

In the preceding paragraphs of this paper we have already had occasion to frequently refer to the correlation of the judgments of independent observers. Relations (vi.) of p. 240 are not fulfilled, nor even approximately fulfilled. For example, in Table II. we find $\sigma_{23} = 1{\cdot}74$, about, which is actually *less* than $\sigma_{03} = 1{\cdot}82$, about, when, if the theory of p. 240 were correct, $\sigma_{23} = \sqrt{(\sigma_{03}{}^2 + \sigma_{02}{}^2)}$! An examination of Table II. show us substantial correlations in two out of the three cases between absolute judgments. Now it is well to put somewhat more definitely what is meant by this correlation. Astronomers have already found that the brightness of a star influences the personal equation.* This in the language of the present writer produces a correlation of judgments, for "every one of the observers records the time of transit of faint stars later than that of bright stars." Hence if a number of observations were made on stars of varying magnitude, the judgment being a function of the magnitude, we should have a series of correlated errors. Again it is quite possible that the rate of transit of a bright line in our experiments might tend to correlate judgments, although the Cape observers did not find the personal equation to vary with stars of very different declination. It is not, however, contended that the correlation of jugdments is not due to one cause or another. The point of the present writer is this, that when every effort is made to eliminate large causes, such as varying brightness or rate of motion of the line in our own experiments, there still remains a multitude of small causes which produce correlation. It might be possible in an *ideal* series still further to eliminate some of these, but in practical observation we have to take a given phenomenon as it is, and we cannot possibly subtract from it the whole of its characteristic atmosphere. The next point to be noticed is, that whatever be these lesser causes of the characteristic atmosphere, *e.g.*, possibility of judging better the position of a bright line when it is nearer to one or another part of its range of visibility, or of bisecting a line of one length better than of another length—*they affect different observers in quite different manners.* Unlike the brightness of stars, the fluctuations of personal equation due to these causes are in themselves personal. Dr. MACDONELL and I have within the limits of error no correlation in our judgments of the position of a bright line. Dr. LEE and Dr. MACDONELL have a correlation as high as that of a measure made on a pair of brothers. In other words, correlation of judgments is a personal matter, just as personal equation itself. We could no doubt increase it by introducing variety in the observed phenomena—degree of brightness, degrees of speed—but beyond such causes capable of differentiation, there appear to be others, which I have classed as the influence of the immediate atmosphere, and which appeal to different personalities in different ways, and where there is a resemblance between certain features of two personalities produce correlation in their judgments. For example, A and B are alike in their sight, being slightly short-sighted we will say, B and C are alike in their nervous temperament, being able to judge more correctly if the bell rings after the bright line has been visible a rather longer time. There is thus an element of personality the same in A and B and another the same in B and C. The result would be that A's and B's judgments would be correlated, and also B's and C's judgments would be correlated, but not necessarily A's and C's. Something like this probably is what actually occurs in the case of Dr. MACDONELL, Dr. LEE, and myself. But it would be practically hopeless to try and discover the common elements in our personalities, and what in the immediate atmosphere of the experiments affected such elements. Even if, in a long and laborious series of experiments and reductions, we could discover the subtle causes of our correlations or non-correlations, the results would be of small value, for they would be personal to ourselves, and in actual observation they could not be eliminated from our own future experiments, nor could the like causes be determined for other observers. We are forced to admit, I think, that correlation is a personal character of every pair of observers, and to look upon it as a personal constant to be determined by experiment.

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* 'Monthly Notices of the Roy. Astron. Soc.,' vol. 60, November, 1899.

Here again, however, arises the very same point as we have considered in discussing absolute steadiness of judgment—we do not in practice know the absolute judgments, and so cannot find the correlation of absolute judgments, r_{23}, r_{31}, r_{12}. All we can do is to refer the judgments of two observers to a third as standard, and then measure the correlation of relative judgment. In this case we have a result which is not purely personal; we have superposed on the correlation due to a common element in personality, an element of "spurious correlation."

Taking the bright-line experimental result from Table VI., we have for 519 observations:—

$$r_{32} = \cdot 3819 \pm \cdot 0253, \qquad \rho_{1,32} = \cdot 5625 \pm \cdot 0202,$$
$$r_{13} = \cdot 1571 \pm \cdot 0289, \qquad \rho_{2,13} = \cdot 3055 \pm \cdot 0268,$$
$$r_{21} = \cdot 0139 \pm \cdot 0296, \qquad \rho_{3,21} = \cdot 6154 \pm \cdot 0184.$$

The latter series, all that we should usually know, enables us to form no opinion at all about the former. The absolute judgments of Dr. MACDONELL and myself have sensibly no correlation; our relative judgments have the greatest correlation of all—such are the masking effects of spurious correlation when judgments are referred to a third observer as standard!

If, from the values of the standard deviations of the absolute judgments, we calculate what would be the spurious correlations on the assumption that the absolute judgments are not correlated, we have by the method of p. 241:—

$$\bar{\rho}_{1,32} = \frac{\sigma_{01}^2}{\sqrt{(\sigma_{01}^2 + \sigma_{02}^2)}\sqrt{(\sigma_{01}^2 + \sigma_{03}^2)}} = \cdot 3918,$$

$$\bar{\rho}_{2,13} = \frac{\sigma_{02}^2}{\sqrt{(\sigma_{02}^2 + \sigma_{03}^2)}\sqrt{(\sigma_{02}^2 + \sigma_{01}^2)}} = \cdot 3811,$$

$$\bar{\rho}_{3,21} = \frac{\sigma_{03}^2}{\sqrt{(\sigma_{03}^2 + \sigma_{01}^2)}\sqrt{(\sigma_{03}^2 + \sigma_{02}^2)}} = \cdot 7013.$$

Hence $\rho_{1,32} - \bar{\rho}_{1,32} = \cdot 1707$, $\rho_{2,13} - \bar{\rho}_{2,13} = - \cdot 0756$, and $\rho_{3,21} - \bar{\rho}_{3,21} = - \cdot 0859$, or the effect of the correlation of absolute judgments is to increase in one case and decrease in the other two the spurious correlation. Without direct experiments *ad hoc* I see no way of determining from the usual data of the personal equation how much of the observed correlation of judgments may be due to a common element in the personality, and how much is really spurious. The two causes sometimes work in the same, sometimes in opposite directions.

The bright-line experiments show in a perfectly direct and simple manner that the correlation of absolute judgments is not wholly due to some external source influencing all observers in the same way, but is the result of a common element in the personalities of two observers. They further demonstrate the extreme difficulty in actual observations of separating without experiments *ad hoc* this psychological from the spurious correlation. These are precisely the points they were designed to elucidate.

(ii.) *The Bisection Experiments.*

I have indicated that it was the correlation in judgments of independent observers in the case of bisection that led to the second series or bright-line experiments. After these had demonstrated that the correlation of judgments was not wholly spurious correlation, it seemed desirable to reconsider the bisection experiments with a view to analysing more fully the character of the correlation exhibited by them.

The reader will remember that the error in judgment in their case was taken to be the displacement to the right of the true midpoint measured as a fraction of the total length of the line bisected. The following are the values of the correlations between the absolute judgments and between the relative judgments thus measured:—

TABLE VII.

$r_{23} = \cdot 3627 \pm \cdot 0262$	$\rho_{1,\,23} = \cdot 5615 \pm \cdot 0207$
$r_{31} = \cdot 1139 \pm \cdot 0298$	$\rho_{2,\,31} = \cdot 4980 \pm \cdot 0227$
$r_{12} = \cdot 2053 \pm \cdot 0289$	$\rho_{3,\,12} = \cdot 4379 \pm \cdot 0244$

Thus in every case the correlation has a quite sensible value.

I have pointed out that the absolute displacement of the midpoint by the experimenter was divided originally by the length of the line, because *à priori* we supposed that errors of bisection would be proportional to the length of the line bisected. But that when I had more fully realised the meaning of spurious correlation I saw that the whole of the above correlations might be really spurious in character, for they were the correlations of ratios having the same denominator. The experiments were accordingly put on one side until the bright-line experiments were concluded. It then seemed desirable to determine the correlations between the absolute displacements of the midpoints, and to find the magnitude of the correlation between the lengths of the lines experimented on and the errors made in their bisection. The labour of reducing again all the data would be excessive, and a very little consideration showed me that it was really unnecessary, if we knew the variation and distribution of the lengths of the lines bisected. Let u stand for the length of any one of the bisected lines, which as we have seen were a random sample. Then we have the following distribution :—

TABLE VIII.—Distribution of the 500 Experimental Lines.

Magnitude in ½ inches.	Frequency.	Magnitude in ½ inches.	Frequency.
3·00	5	5·75	44
3·25	7	6·00	37
3·50	4	6·25	22
3·75	12	6·50	18
4·00	20	6·75	13
4·25	28	7·00	9
4·50	30	7·25	13
4·75	48	7·50	0
5·00	58	7·75	3
5·25	64	8·00	4
5·50	58	8·25	3

Here the frequency corresponding to any magnitude m in half-inches denotes all the lines whose lengths fall between $m - {\cdot}125$ and $m + {\cdot}125$ half-inches. The lengths of the experimental lines had before this grouping been read off to the nearest $\frac{1}{100}$ of an inch.

From this frequency we found :—

$$m_u = \text{mean value of } u \qquad = 5{\cdot}3165 \text{ half-inches.}$$
$$\sigma_u = \text{standard deviation of } u = {\cdot}9513 \text{ half-inch.}$$
$$v_u = \sigma_u/m_u \qquad = {\cdot}1789.$$

Now let x'_q = distance of experimental point of bisection from real midpoint of line, positive if it fall to the right, and x_q = distance from left-hand terminal of line to experimental point of bisection in the case of the qth observer. Let us write $X'_q = x'_q/u$, then X'_q is the ratio error which we had previously dealt with, and $X_q = x_q/u$.

Clearly
$$x'_q = x_q - \tfrac{1}{2}u,$$

and if m_z denote the mean value of a variant z, we at once find :

$$\left.\begin{aligned} m_{X_q} &= m_{X'_q} + {\cdot}5 \\ \sigma_{X_q} &= \sigma_{X'_q} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \text{(xii.)}$$

Now
$$X'_q = X_q - {\cdot}5 = x_q/u - {\cdot}5.$$

Treating in the usual way variations as differentials, whose squares and products may be neglected, we have :—

$$\delta X'_q = \frac{\delta x_q}{m_u} - \frac{m_{x_q}}{m^2_u}\,\delta u \quad \ldots\ldots\ldots\ldots \quad \text{(xiii.)},$$

whence squaring, summing for all possible values, and remembering the definitions of standard deviation and correlation coefficient we find :

$$\sigma^2_{x'_q}/(m_{x'_q} + \cdot 5)^2 = v^2_{x_q} + v^2_u - 2v_{x_q}v_u r_{ux_q} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)},$$

where v_{x_q} stands for σ_{x_q}/m_{x_q}.

But the left-hand side of this equation is known from the previous reductions, $\sigma_{x'_q}$, $m_{x'_q}$ having the values in Table I.; v_u has just been determined. Hence a knowledge of v_{x_q} would enable us to find r_{ux_q} without the labour of further correlation tables. The values of x_1, x_2, x_3 had of course been measured in order to find x'_1, x'_2, and x'_3, so that all we required were their frequency distributions. They were as follows :—

TABLE IX.—Table of Frequencies of x_q.

Magnitude in ½ inches.	Observer			Magnitude in ½ inches.	Observer		
	1.	2.	3.		1.	2.	3.
1·35	—	1	1	3·00	52	39	31
1·50	5	8	6	3·15	31	28	30
1·65	8	5	6	3·30	20	23	20
1·80	6	15	12	3·45	21	17	16
1·95	16	29	21	3·60	12	10	10
2·10	29	36	39	3·75	10	6	10
2·25	42	47	49	3·90	3	4	3
2·40	41	54	60	4·05	2	2	2
2·55	56	62	63	4·20	3	1	2
2·70	75	55	56	4·35	—	1	1
2·85	68	55	62	4·50	—	2	—

Here the unit of grouping is ·15 half-inch, and a magnitude m covers all the frequency between $m - \cdot 075$ and $m + \cdot 075$ half-inches. From these data we deduced m_x and σ_x being in half-inch units.

TABLE X.

Quantity.	1.	2.	3.
Mean, m_x	2·7216	2·6379	2·6445
S.D., σ_x	·4909	·5253	·5072
$m_x/\sigma_x = v_x$	·1804	·1991	·1918

Since $$x'_q = x_q - \tfrac{1}{2}u,$$

we have $$\delta x'_q = \delta x_q - \tfrac{1}{2}\delta u,$$

and $$\sigma^2_{x'_q} = \sigma^2_{x_q} + \tfrac{1}{4}\sigma^2_u - \sigma_u\sigma_x r_{ux_q} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xv.)}.$$

Now, σ_x, $\sigma_{x'}$, σ_u and r_{ux} are all known. Hence the correlation of the absolute displacements will be known as soon as we find $r_{x_px_q}$.

But $$X_q/X_p = x_q/x_p,$$

or, $$\delta X_q/m_{X_q} - \delta X_p/m_{X_p} = \delta x_q/m_{x_q} - \delta x_p/m_{x_p}.$$

Hence squaring and summing we find:

$$v^2{}_{X_q} + v^2{}_{X_p} - 2v_{X_p}v_{X_q}r_{X_pX_q} = v^2{}_{x_q} + v^2{}_{x_p} - 2v_{x_p}v_{x_q}r_{x_px_q}.$$

But since $X_p = X'_p + \cdot 5$, $\delta X_p = \delta X'_p$, whence we have at once :—

$$\sigma_{X_p} = \sigma_{X'_p},$$

and $$r_{X_pX_q} = r_{X'_pX'_q}.$$

Thus we find :—

$$r_{x_px_q} = \tfrac{1}{2}\left\{\frac{v^2{}_{x_q} + v^2{}_{x_p}}{v_{x_q}v_{x_p}} - \frac{\dfrac{\sigma^2{}_{X'_q}}{(m_{X'_q} + \cdot 5)^2} + \dfrac{\sigma^2{}_{X'_p}}{(m_{X'_p} + \cdot 5)^2} - \dfrac{2\sigma_{X'_q}\sigma_{X'_p}r_{X'_qX'_p}}{(m_{X'_q} + \cdot 5)(m_{X'_p} + \cdot 5)}}{v_{x_q}\,v_{x_p}}\right\}. \quad . \quad \text{(xviii.)}.$$

Here v_x is given by Table X., $m_{X'}$, $\sigma_{X'}$ and $r_{X'_qX'_p}$ are all entered in Table I., so that $r_{x_px_q}$ can be found. Hence from (xvii.) we find $r_{x_p'x'_q}$ the correlation of the absolute displacements.

Substituting the numerical values we easily find the following results :—

TABLE XII.

$r_{x_2x_3} = \cdot 9445$	$r \quad = \cdot 3596 \pm \cdot 0263$
$r_{x_3x_1} = \cdot 9359$	$r_{x_3x_1'} = \cdot 1242 \pm \cdot 0297$
$r_{x_1x_2} = \cdot 9358$	$r_{x_1'x_2'} = \cdot 2223 \pm \cdot 0287$

There are thus seen to be substantial correlations between the errors in the absolute displacements, not reduced to the length of the bisected line as unit.

To find the correlations between the relative displacements not reduced to the length of the bisected line as unit, we have to find first

$$\sigma^2{}_{x'_qx'_p} = \sigma^2{}_{x'_q} + \sigma^2{}_{x'_p} - 2\sigma_{x'_q}\sigma_{x'_p}r_{x'_qx'_p},$$

using Tables XI. and XII., and thence find

$$\rho_{1,23} = \frac{\sigma^2{}_{x'_1x'_2} + \sigma^2{}_{x'_1x'_3} - \sigma^2{}_{x'_2x'_3}}{2\sigma_{x'_1x'_2}\,\sigma_{x'_1x'_3}}.$$

There results, the standard deviations being in half-inch units :

Table XIII.

$\sigma_{x'_2 x'_3}$ = ·1729	$\rho_{1, 23}$ = ·5503 ± ·0210
$\sigma_{x'_3 x'_1}$ = ·1793	$_{2, 31}$ = ·5002 ± ·0226
$\sigma_{x'_1 x'_2}$ = ·1852	$\rho_{3, 12}$ = ·4478 ± ·0241

We have now the complete data requisite for analysing the experiments on the bisection of straight lines. We place all the correlation coefficients together in Table XIV. for comparison of the two methods of deducing results.

Table XIV.—Correlation in the Judgments as to Midpoint of Lines.

	Errors measured in terms of length of line.	Errors measured absolutely.
r_{23}	·3627 ± ·0262	·3596 ± ·0263
r_{31}	·1139 ± ·0298	·1242 ± ·0297
r_{12}	·2053 ± ·0289	·2223 ± ·0287
$\rho_{1, 23}$	·5615 ± ·0207	·5503 ± ·0210
$\rho_{2, 31}$	·4980 ± ·0227	·5002 ± ·0226
$\rho_{3, 12}$	·4379 ± ·0244	·4478 ± ·0241

We conclude at once that:—

(i.) Within the limits of the probable errors of the observations the correlations of the errors in judgment, whether measured absolutely or in terms of the length of the line bisected, are sensibly the same; and this is true not only for the correlation in absolute (r) but also for the correlation (ρ) in relative judgments.

(ii.) Thus while we have shown that the error in bisecting a line is not proportional to the length of the line, and indeed not at all or only slightly correlated with it, yet the observed correlation of judgments cannot arise solely from the use of a ratio or index. For this correlation still exists, if we deal with the absolute errors. It is thus not a purely spurious correlation.

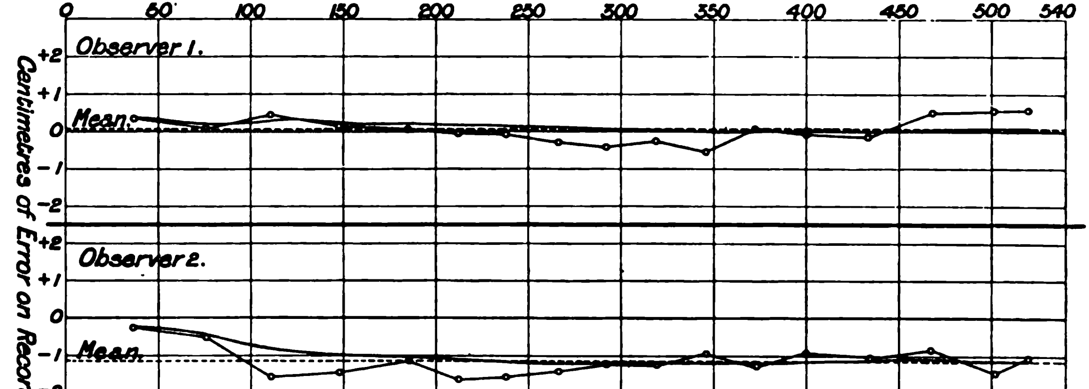
(iii.) The correlation varies considerably from one pair of observers to a second. We thus are forced to conclude that it is not a result of a common varying external cause, but must in part or wholly be due to a common element in the personalities of two experimenters, which is affected in the same way, and differently from some other common element in the personalities of another pair of observers.

Thus the bisection experiments entirely confirm the conclusions we have formed as a result of the bright-line experiments. In both cases there is a real personal correlation of judgment, only in the two series it is differently masked by or combined with spurious correlation according to the special manipulation used in the reduction of the errors.

Taking into account what we have learnt as to the nature of fluctuations in personal equation, I think we may conclude broadly as follows :—

The errors of judgment of apparently independent observers are not as a rule independent. The immediate atmosphere of each single observation or of each short series of observations affects in a differential manner the factors of the personality, causing variations in the personal equation which are not of the order of those due to random sampling. Certain factors affected by the immediate atmosphere seem to be common elements of two or more personalities, and there results from this a tendency in each pair of observers to judge in the same manner. If we enlarge the concept of "immediate atmosphere" to embrace not only the objective side of the phenomena, but the physical and mental state of the percipient, we may simply state that certain elements of this immediate atmosphere are common to each pair of observers and produce a correlation between their judgments. Their personal equations fluctuate in sympathy. This sympathetic fluctuation of personal equations leading to correlation of judgments is really visible on inspection, as the reader will at once see on examining Diagrams II. and III.

Diagram III. Personal Equation
Motion of Bright Line.



Continuous Polygon = Values of Personal Equation as based upon 27 to 37 Observations.
Continuous Curve = Approach " " " to its Mean Value for 519 " .

This quite sub-conscious sympathetic fluctuation of personal equation in the case of apparently independent observers is not only of fundamental importance when we have to combine observations of the same phenomenon by different observers, and assign the weight of the combination, but it appears to have an even wider bearing when we have to consider to what degree the testimony of a number of apparently independent witnesses of the same event is strengthened by the concurrence of their judgments as to what actually took place. Without some estimate of the correlation of judgments we cannot assert what weight is to be given to combined testimony.

(9.) *On the Nature of the Frequency Distribution in the case of Errors of Judgment.*

Having completed our investigation of the nature of fluctuations in personal equation and of the correlation between judgments—an investigation which demands no hypothesis as to the form of their law of distribution—we now turn to a consideration of the manner in which errors of judgment are distributed.

In Tables XV. and XVI. will be found the frequencies for the two series of experiments, the results being *grouped* (see pp. 272–273).*

The question to be answered is this: Is the general nature of these distributions capable of being described by the "normal" curve of errors, on the assumption that they are random samplings of the whole "populations" of errors that the observers respectively would produce if they continued to experiment indefinitely under the same conditions? So far as I am aware no thorough investigation has yet been made as to how far actually observed errors are capable of being described by the normal curve of errors. In most text-books on the theory of errors certain axioms are laid down as ruling the distribution of errors of judgment, and on the basis of these axioms the normal curve of errors is deduced. One or two limited series of errors of observation are then cited, and the axioms declared to be satisfactory by comparing a graph of the theoretical with the observed distribution, or by a table comparing the observed and theoretical frequencies of errors occurring within each small range. As a rule a vague inspection of the amount of agreement is the only thing appealed to to test the accordance of theory and experiment. So far as I am aware writers on the theory of errors have quite overlooked the point that that theory itself provides a perfectly general test of whether the accordance between theory and experiment is a reasonable or an unreasonable one. It is not a question of whether there is a "practical accordance" between the two, whatever that may mean, but of the degree of probability that a given system of errors or deviations is a random sampling from an indefinitely large distribution of errors obeying the axioms from which the normal curve of errors has been deduced. To talk of "practical accordance" between theory and observation is simply to shuffle out of an examination of the truth, when the odds are 3000 to 1, or even 70 to 1, against the observed results being a random sample of errors obeying certain fundamental axioms.† Now in the

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* In Table XV. a group such as 4·755 embraces all the frequency between 4·505 and 5·005; and in Table XVI. a group such as ·04 embraces all the frequency between ·035 and ·045.

† A recent writer on statistics seems to find that an agreement measured by the odds of 3000 to 1 is *very satisfactory*, and one against which the odds are 70 to 1 represents with all *practicable accuracy* the observed frequency. Comment is needless.

TABLE XV.—Frequency of Absolute and Relative Errors in Bright-line Series of Experiments. Number, 519

Size of error.	(1)	(2.)	(3.)	(3-2.)	(1-3.)	(2-1.)
7·755	—	—	—	1	—	—
7·255	—	—	—	1	—	—
6·755	—	—	—	—	1	—
6·255	1	—	1	—	1	—
5·755	—	—	—	4	2	—
5·255	—	1	—	1	5	—
4·755	1	—	—	4	5	—
4·255	—	—	—	12	17	—
3·755	1	—	1	15	18	—
3·255	6	—	12	19	17	1
2·755	4	—	19	20	29	1
2·255	12	—	18	26	35	8
1·755	22	3	26	34	41	13
1·255	57	8	42	49	37	19
·755	71	31	48	62	41	23
·255	97	35	44	77	60	54
- ·245	85	73	48	90	41	72
- ·745	69	76	46	41	56	55
- 1·245	56	96	60	21	35	50
- 1·745	23	79	42	19	34	59
- 2·245	7	60	36	9	21	58
- 2·745	4	30	36	5	10	38
- 3·245	1	17	20	5	6	30
- 3·745	1	5	12	1	5	15
- 4·245	—	3	5	3	2	7
- 4·745	1	1	2	—	—	7
- 5·245	—	—	1	—	—	3
- 5·745	—	—	—	—	—	2
- 6·245	—	—	—	—	—	2
- 6·745	—	1	—	—	—	—
- 7·245	—	—	—	—	—	—
- 7·745	—	—	—	—	—	1
- 8·245	—	—	—	—	—	—
- 8·745	—	—	—	—	—	—
- 9·245	—	—	—	—	—	—
- 9·745	—	—	—	—	—	—
- 10·245	—	—	—	—	—	1

present investigation we have no less than twelve frequency distributions, six absolute distributions and six relative distributions; the latter being of course of the type which will usually occur in astronomical or physical observations where the absolute errors cannot be measured. We have then material *enough* to discuss the problem: Is it suitable for the purpose? It seems to me that there is nothing peculiar to our data which marks them off from other series of observational errors, except their rather extensive character, which was necessary if safe conclusions were to be drawn. There were four independent observers, three of whom at least had been long used to making observations and measurements; the fourth, less accustomed, turned out in the sequel to have the steadiest judgment. Further, the investigations were begun with

TABLE XVI.—Frequency of Absolute and Relative Errors referred to Length of Line as Unit in Bisection Experiments.* Number = 500.

Size of error.	(1.)	(2.)	(3.)	(2–3.)	(3–1.)	(1–2.)
− ·12	—	—	—	—	—	2
− ·11	—	—	—	—	—	1
− ·10	—	—	—	1	—	3
− ·09	1	—	—	2	—	7
− ·08	4	1	—	1	1	9
− ·07	8·5	3	1	6	2	23·5
− ·06	12	11	7·5	13	6·5	21·5
− ·05	13·5	14·5	9·5	16·5	14·5	35
− ·04	45	21·5	22	26	10	58
− ·03	61	30	40·5	45·5	26	45
− ·02	76	47	43·5	36	43	59
− ·01	90·5	51·5	51	66	33	64·5
·00	74·5	72	68·5	55·5	42	43
+ ·01	50	65·5	75	61	53·5	37·5
+ ·02	30·5	53	70·5	52·5	60·5	26·5
+ ·03	21·5	50·5	61	41·5	61	27·5
+ ·04	7	28·5	25·5	34·5	52	16
+ ·05	3	27	13·5	20	34·5	7
+ ·06	2	13·5	10	13	27	11
+ ·07	—	7·5	1	4	17·5	1
+ ·08	—	—	—	2	5	1
+ ·09	—	1	—	2	7·5	1
+ ·10	—	—	—	1	3·5	—
+ ·11	—	2	—	—	—	—

no intention of considering the problem of normal frequency; they were designed to demonstrate what appeared a remarkable and valuable result flowing from the theory of errors as usually expounded (see p. 240). Each of us made our individual judgments with care and without any theoretical bias. We were of course, during the work of the observations, liable to physical and psychological influences, to the subtle changes of daily health and of sense-keenness. But I contend that all such things affect every observer, and that it is idle to propound a theory which would hold for an ideal observer of perfectly equable temperament and physical fitness observing under a perfectly equable environment for a number of days or even weeks an exactly identical phenomenon. Such a theory could not be verified, and if verified would have no practical application. Our observations seem to me a perfectly fair sample of actual errors of judgment, and I believe no objections can be taken to them which would not apply with even increased force against most of the series of errors of judgment with which physicists or astronomers have to deal.

There are two classes of considerations which arise in our view of frequency distributions:—

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[*] In this table and in the diagrams X. to XV. the error has been given the opposite sign to its value in Table I.

(*a.*) General physical characters of the nature of the distribution without regard to the special frequency of errors of particular sizes.

(*b.*) Agreement between theory and observation in the general distribution of errors of each particular size.

I propose to investigate these classes of considerations separately.

(10.) (*a.*) *General Physical Characters of a Normal Distribution.*

While the analytical processes by means of which the normal curve is deduced are extremely varied—sometimes very simple (HAGEN), sometimes very complex (POISSON), there is confessedly or tacitly involved an axiom of the following kind :—

(α.) *Positive and negative errors of the same size are equally frequent.* Sometimes this result is disguised by assuming that the actual error is the sum of an indefinitely great number of small elementary errors which are equally likely to be positive or negative. Whatever process of proof be followed the result is the same—the normal distribution gives a symmetrical distribution of errors, and this is its first general physical character. Now in an immense number of cases of deviations from the mean, such as occur in organic nature, this symmetry is quite unknown ; such distributions I have spoken of as skew frequency distributions,* and their characteristic feature is that the mode or position of the maximum frequency diverges from the mean. The ratio of the distance of the mode from the mean to the standard deviation I have treated as a measure of the " skewness " of the distribution. It will vanish when the curve is symmetrical or when the sums of all odd powers of the errors are zero. Thus if n be the number of observations, $n\mu_p$ the sum of the pth powers of the errors, $\mu_p = 0$ for a normal distribution if p be odd. For most practical purposes the labour of investigation compels us to confine our attention to the question of whether μ_3 is sensibly zero.

But the normal distribution not only involves a condition as to the odd moments, but also one which must sensibly hold in the case of each pair of even moments.† The simplest of such relations is expressed by

$$\mu_4 = 3\mu_2^2.$$

Now this relation and its extensions to higher moments have nothing whatever to do with the symmetry of the normal distribution—with the equal frequency of errors of the same size, whether positive or negative. They depend really upon two additional axioms, which are again confessedly or tacitly assumed in the course of the proof, namely :—

(β.) That there are an indefinitely great number of cause-groups associated in producing each individual error.

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* 'Phil. Trans.,' A, vol. 186, p. 343 *et seq.*

† 'Phil. Trans.,' A, vol. 185, p. 108.

(γ.) That the contributions towards any individual error of these cause-groups are not correlated among themselves.

It is not my purpose at present to consider the philosophical arguments for or against these axioms. I have considered the matter at length in a paper not yet published, but I want to indicate the source of such relations as those just referred to. If actual distributions of error do not sensibly satisfy $\mu_3 = 0$, then axiom (α) is not true; if they do not sensibly satisfy $\mu_4 = 3\mu_2^2$, then either (β) or (γ) or both are invalid.

The nature of the relation $\mu_4 = 3\mu_3^2$ deserves a little fuller consideration from the physical side.

Let m_1 and m_2 be the number of errors of magnitudes x_1 and x_2 respectively, and let $n'\mu_2'$ and $n'\mu_4'$ represent the second and fourth moments of the remainder of the errors. Let $m_1 + m_2 = m'$. Then

$$n\mu_2 = n'\mu_2' + m_1x_1^2 + m_2x_2^2,$$

$$n\mu_4 = n'\mu_4' + m_1x_1^4 + m_2x_2^4.$$

Hence we find :—

$$n\mu_4 = n'\mu_4' + m'x_2^4 + (n\mu_2 - n'\mu_2' - m'x_2^2)(x_1^2 + x_2^2).$$

Now without altering the total frequency, *i.e.*, keeping m' constant, take part of the frequency m_2 and transfer it from x_2 to x_1; do this equally on both sides of the mean, so that the position of the mean be not altered. Now in order that μ_2 should also not be altered, x_2 being supposed constant, we must have :—

$$\delta m_1/m_1 = (2x_1\delta x_1)/(x_2^2 - x_1^2),$$

or if x_2 be $> x_1$, δx_1 must be positive. Thus if we bring a part of the outlying frequency inward to a point nearer the mean, we can still retain the same mean and the same standard deviation, *i.e.*, get the same normal curve, if we shift the inlying frequency group a little outward. The whole effect of such a change will be to flatten the frequency curve at its summit by a reduction of its tails, which increases the middle part of the curve. Now, looking at the above value of μ_4 we see that since $x_1^2 < x_2^2$, $n\mu_2 - n'\mu_2' - m'x_2^2$ is negative, and therefore that when x_1 increases μ_4 decreases.

We conclude accordingly that symmetrical or nearly symmetrical curves which have the same mean and standard deviation as a normal curve will be flatter topped if μ_4 be $< 3\mu_2^2$, and steeper at the top if μ_4 be $> 3\mu_2^2$.

Take, for example, the details of shots at a target given by MERRIMAN, 'Method of Least Squares,' p. 14 : here*

$$\mu_2 = 2{\cdot}402{,}343\,, \qquad \mu_4 = 14{\cdot}578{,}491\,,$$

and

$$3\mu_2^2 = 17{\cdot}313{,}752.$$

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* Using SHEPPARD's corrective terms, 'London Math. Soc. Proc.,' vol. 29, p. 369.

Accordingly μ_4 is $< 3\mu_2^2$ by a considerable amount, and the observations are immensely flatter than the normal curve with which they can be fitted. Actually the normal curve has a maximum ordinate which rises some 15 to 20 per cent. above the corresponding ordinate of the observations. Hence quite apart from the question of equal negative and positive errors, we should assert that because $\mu_4 = 3\mu_2^2$ is not sensibly satisfied, it follows that one or other or both the axioms (β), (γ) cannot be true for this distribution of hits.

I propose to look a little more closely into the probable errors of the quantities connected with a normal distribution. I take d to be the distance from the mean to the mode, and define the skewness, Sk., as in earlier memoirs, to be the ratio of d to σ. I write $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$. Mr. FILON and I have dealt with the probable errors of skew frequency curves in a special memoir,* and deduced those for the normal curve as the limit to those of a skew curve of what I have termed Type III. The disadvantage of this procedure is that it supposes the deviations from symmetry to take place along a class of curve for which

$$2\mu_2(3\mu_2^2 - \mu_4) + 3\mu_3^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xix.)}.$$

μ_4 is thus known in terms of μ_2 and μ_3. The result is that when μ_3 is put zero to reach the normal case, the error of μ_4 is found to be absolutely correlated with that of μ_2, and the probable value of this error to be deducible from that of μ_2 by means of the relation

$$\mu_4 = 3\mu_2^2.$$

To obtain perfectly general results we must use not Type III., but Type I. or Type IV. of that memoir, curves in which μ_2, μ_3, and μ_4 are absolutely independent of each other. Our results can easily be deduced by aid of certain elegant formulæ due to Mr. W. F. SHEPPARD.† In our notation these are :—

Probable error of $\mu_p = {}^{\cdot}67449\, \Sigma_{\mu_p}$ is given by

$$\Sigma^2_{\mu_p} = \frac{\mu_{2p} - \mu^2_p + p^2 \mu^2_{p-1} \mu_2 - 2p\mu_{p+1} \mu_{p-1}}{n} \quad . \quad . \quad . \quad . \quad . \quad \text{(xx.)}.$$

$R_{\mu_p \mu_q}$ = Correlation of errors in μ_p and μ_q is given by

$$\Sigma_{\mu_p} \Sigma_{\mu_q} R_{\mu_p \mu_q} = \frac{\mu_{p+q} - p\mu_{p-1} \mu_{q+1} - q\mu_{p+1} \mu_{q-1} + pq\mu_{p-1} \mu_{q-1} \mu_2 - \mu_p\mu_q}{n} \quad . \quad \text{(xxi.)}.$$

These results are perfectly general whatever be the law of the frequency. As special cases we have, when $\mu_1 = 0$:

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* 'Phil. Trans.,' A, vol. 191, pp. 229–311, especially p. 276.

† 'Phil. Trans.,' A, vol. 192, p. 126.

$$\left.\begin{aligned}
\Sigma^2_{\mu_2} &= (\mu_4 - \mu_2^{\,2})/n \\
\Sigma^2_{\mu_3} &= (\mu_6 - \mu_3^{\,2} + 9\mu_2^{\,2} - 6\mu_4\mu_2)/n \\
\Sigma^2_{\mu_4} &= (\mu_8 - \mu_4^{\,2} + 16\mu_3^{\,2}\mu_2 - 8\mu_5\mu_3)/n \\
\Sigma_{\mu_2}\,\Sigma_{\mu_3}\,R_{\mu_2\mu_3} &= (\mu_5 - 4\,\mu_3\mu_2)/n \\
\Sigma_{\mu_3}\,\Sigma_{\mu_4}\,R_{\mu_3\mu_4} &= (\mu_7 - 3\mu_2\mu_5 - 5\mu_4\mu_3 + 12\mu_2^{\,2}\mu_3)/n \\
\Sigma_{\mu_2}\,\Sigma_{\mu_4}\,R_{\mu_2\mu_4} &= (\mu_6 - 4\mu_3^{\,2} - \mu_2\mu_4)/n
\end{aligned}\right\} \quad . \quad . \quad . \quad \text{(xxii.)}.$$

For the special case of the normal curve, since

$$\mu_4 = 3\mu_2^{\,2} = 3\sigma^4, \quad \mu_6 = 15\sigma^6, \quad \mu_8 = 105\sigma^8$$
$$\mu_7 = \mu_5 = \mu_3 = 0,$$

we have:—

$$\text{Probable error of } \mu_2 = \cdot 67449 \times \sqrt{2}\sigma^2/\sqrt{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxiii.)}.$$

$$\text{,, \quad ,,} \quad \mu_3 = \cdot 67449 \times \sqrt{6}\sigma^3/\sqrt{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxiv.)}.$$

$$\text{,, \quad ,,} \quad \mu_4 = \cdot 67449 \times \sqrt{96}\sigma^4/\sqrt{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxv.)}.$$

$$R_{\mu_2\mu_3} = 0 \quad . \quad . \quad \text{(xxvi.)}. \qquad R_{\mu_3\mu_4} = 0 \quad . \quad . \quad \text{(xxvii.)}. \qquad R_{\mu_2\mu_4} = \tfrac{1}{2}\sqrt{3} \quad . \quad . \quad \text{(xxviii.)}.$$

In a further memoir on skew variation,* not yet published, I show that 'if the differential equation to the frequency curve be

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1x}{b_0 + b_1x + b_2x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxix.)},$$

then whatever be the form of the curve, the distance d between the mean and the mode, and the skewness are always given by

$$d = \frac{\mu_3(\mu_4 + 3\mu_2^{\,2})}{2(5\mu_2\mu_4 - 6\mu_3^{\,2} - 9\mu_2^{\,3})} = \frac{1}{2}\,\frac{\sqrt{\mu_2}\sqrt{\beta_1}(\beta_2 + 3)}{5\beta_2 - 6\beta_1 - 9} \quad . \quad . \quad . \quad . \quad \text{(xxx.)}.$$

$$\text{Sk.} = \frac{\mu_3(\mu_4 + 3\mu_2^{\,2})}{2\sqrt{\mu_2}(5\mu_2\mu_4 - 6\mu_3^{\,2} - 9\mu_2^{\,3})} = \frac{1}{2}\,\frac{\sqrt{\beta_1}(\beta_2 + 3)}{5\beta_2 - 6\beta_1 - 9} \quad . \quad . \quad . \quad . \quad \text{(xxxi.)}.$$

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* My original memoir ('Phil. Trans.,' A, vol. 186, p. 343), being much misunderstood, has been alternately over- and under-rated. I had found that the ordinary theory of errors was far from describing frequencies within the limits of error imposed by a random sampling. My object was then to discover a series of curves which would enable me in a very great number of cases to do this. I did not select these curves at random, but endeavoured to see where the usual hypotheses failed and must be generalised. My hypergeometrical series was not empirically chosen, but on the grounds of the axioms (α), (β), (γ) above. I chose a system where the positive and negative errors were not equally probable, where there was not an infinite number of cause-groups, and lastly, one where these cause-groups did not contribute independent but correlated elements to the total error. All these points as well as criticisms, mostly due to complete misunderstanding of what random sampling means, I have considered in a further memoir.

Hence, since β_1 and β_2 can always be found, we have general expressions for d and Sk. in the case of the first four moments being arbitrary.

Further, if we increase our β series and write :—

$$\beta_1 = \mu_3^2/\mu_2^3, \quad \beta_2 = \mu_4/\mu_2^2, \quad \beta_3 = \mu_3\mu_5/\mu_2^4,$$
$$\beta_4 = \mu_6/\mu_2^3, \quad \beta_5 = \mu_7\mu_3/\mu_2^5, \quad \beta_6 = \mu_8/\mu_2^4 \quad . \quad . \quad . \quad . \quad \text{(xxxii.)},$$

we have the following perfectly general results, the law of frequency being any whatever :—

$$n\Sigma_{\beta_1}^2 = \beta_1(4\beta_4 - 24\beta_2 + 36 + 9\beta_1\beta_2 - 12\beta_3 + 35\beta_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxiii.)}.$$

$$n\Sigma_{\beta_2}^2 = \beta_6 - 4\beta_2\beta_4 + 4\beta_2^3 - \beta_2^2 + 16\beta_2\beta_1 - 8\beta_3 + 16\beta_1 \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxiv.)}.$$

$$\Sigma_{\beta_1}\Sigma_{\beta_2}R_{\beta_1\beta_2} = 2\beta_5 - 3\beta_4\beta_1 - 4\beta_3\beta_2 + 6\beta_2^2\beta_1 + 3\beta_1\beta_2 - 6\beta_3 + 12\beta_1^2 + 24\beta_1 \quad . \quad \text{(xxxv.)}.$$

Lastly if $$K = 6 + 3\beta_1 - 2\beta_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxvi.)}.$$

$$\begin{aligned} n\Sigma_K^2 = {} & 4\beta_6 - 16\beta_2\beta_4 + 16\beta_2^3 + 72\beta_1\beta_4 - 24\beta_5 - 72\beta_1\beta_2^2 + 48\beta_2\beta_3 + 81\beta_1^2\beta_2 \\ & - 108\beta_1\beta_3 - 4\beta_2^2 - 188\beta_1\beta_2 + 72\beta_3 + 171\beta_1^2 + 100\beta_1 \quad . \quad . \quad . \quad . \quad . \text{(xxxvii.)}. \end{aligned}$$

Thus the probable errors ·67449 Σ_{β_1}, ·67449 Σ_{β_2}, ·67449 Σ_K of the quantities β_1, β_2, and what I have termed the criterion K, can be found *whatever be the law of frequency.*

Let $\epsilon_1 = \sqrt{\beta_1} = \mu_3/(\mu_2)^{\frac{3}{2}}$, then

$$\Sigma_{\epsilon_1} = \tfrac{1}{2}\Sigma_{\beta_1}/\sqrt{\beta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxviii.)},$$

and its value can be found from (xxxiii.). Knowing Σ_{β_1}, Σ_{β_2}, and the correlation of errors in β_1 and β_2, *i.e.*, $R_{\beta_1\beta_2}$, we can find at once the probable errors in d and Sk. from (xxx.) and (xxxi.). Thus with very great generality as to the law of frequency, we can test how far the distribution is a random sample from a population following any law whatever.

Applying the above general results to the special case of the normal curve, we find since

$$\beta_1 = 0, \quad \beta_2 = 3, \quad \beta_3 = 0, \quad \beta_4 = 15, \quad \beta_5 = 0, \quad \beta_6 = 105,$$

Probable error of $\beta_1 = 0$

„ „ $\beta_2 = ·67449 \times \sqrt{\frac{24}{n}}$ . . . . (xxxix.),

„ „ $\sqrt{\beta_1} = ·67449 \times \sqrt{\frac{6}{n}}$ . . . . (xl.),

$R_{\beta_1\beta_2} = 0$

„ „ $K = ·67449 \sqrt{\frac{96}{n}}$ . . . . (xli.),

„ „ $d = ·67449 \sqrt{\frac{3}{2n}}\,\sigma$ . . . . (xlii.),

„ „ Sk. $= ·67449 \sqrt{\frac{3}{2n}}$ . . . . (xliii.).

The result that the probable error of β_1 is zero, but that of $\sqrt{\beta_1}$ is finite, may appear paradoxical, but it is due to the fact that the errors are treated as small quantities and β_1 involves μ_3^2, the square of a small quantity, or one zero, if the distribution were truly normal.

Of these results, Mr. FILON and I have already published with a less general proof (xxiii.), (xxiv.), (xxvi.), (xxvii.), (xlii.), and (xliii.). Instead of (xxv.) and (xxviii.), we found

$$\text{Probable error of } \mu_4 = \cdot 67449\ \sqrt{72}\sigma^4/\sqrt{n}, \text{ and } R_{\mu_2\mu_4} = 1,$$

i.e., replacing $\sqrt{96} = 9\cdot 8$ by $\sqrt{72} = 8\cdot 5$, and $\cdot 866$ by 1. This was due to the fact that we considered the variations from normality to be given by a distribution of Type III. (see p. 276). The differences, however, are not such as to invalidate arguments based on the general order of the probable error of μ_4.*

We have now general relations enough to answer the following questions :—

(i.) Does the value of d found from (xxx.) differ from zero by an amount large as compared with the probable error of d given in (xlii.)?

(ii.) Does the skewness found from (xxxi.) differ from zero by an amount large as compared with the probable error of the skewness as given in (xliii.)?

(iii.) Does the value of μ_3 differ from zero by a value large as compared with the probable error of μ_3 given in (xxiv.)?

In all these questions we have a test of whether the distribution is really a random selection from a symmetrical distribution, *i.e.*, from material obeying axiom (α). The same thing is again dealt with by testing the error of $\sqrt{\beta_1}$ as given by (xl.).

(iv.) Is the condition $\mu_4 = 3\mu_2^2$, or $\beta_2 = 3$ satisfied for the distribution, *i.e.*, does β_2 differ from 3 by a quantity which is not large as compared with its probable error as given by (xxxix.)?

If (i.) to (iii.) are satisfied, but not (iv.), the distribution is still not a random selection from material obeying the normal law, *i.e.*, axioms (β) and (γ) cannot both be true for it.

Lastly, if the material does not obey the normal law, does the criterion K differ sensibly from zero, and therefore form a characteristic to be regarded?

I turn first to the motion of the bright line and give in Table XVII. the constants for this series of distributions.

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* See 'Phil. Trans.,' A, vol. 191, p. 277.

Table XVII.--Motion of Bright Line.

	1.	2.	3.	3–2.	1–3.	2–1.
μ_2	5·6561	5·5027	13·2709	12·0942	16·1136	11·1019
μ_3	+4·7424 ±·9755	+·5452 ±·9361	+5·5761 ±3·5060	+21·0453 ±3·0502	+16·2877 ±4·6909	-20·4072 ±2·6827
μ_4	160·3883	152·1762	456·8883	583·8991	693·2790	569·4060
β_1	·1243 ·0000	·0018 ·0000	·0133 ·0000	·2504 ·0000	·0634 ·0000	·3044 ·0000
β_2	5·0135 ±·1450	5·0256 ±·1450	2·5942 ±·1450	3·9919 ±·1450	2·6701 ±·1450	4·6918 ±·1450
d	·2193 ±·0862	·0247 ±·0851	·3020 ±·1321	·6432 ±·1261	·7219 ±·1456	·5706 ±·1088
Sk.	·0922 ±·0363	·0105 ±·0363	·0829 ±·0363	·1850 ±·0363	·1798 ±·0363	·1713 ±·0363
$\sqrt{\beta_1}$	·3524 ±·0725	·0422 ±·0725	·1153 ±·0725	·5004 ±·0725	·2518 ±·0725	·5517 ±·0725
K	-3·6541 ±·2901	-4·0460 ±·2901	·8514 ±·2901	-1·2327 ±·2901	·8501 ±·2901	-2·3266 ±·2901

N.B.—The units of this table are ½ centim. on the recording strip—not on the observation strip; these are the units of our grouping in Table XV. In most of my previous tables the unit has been taken as 1 centim. of the recording strip. The ½ centim. is retained here, as it will be required in the plotted diagrams as unit of grouping.

Now let us examine these results, remembering that on the basis of a random sampling the odds against a quantity exceeding its supposed value by twice, thrice, four, five times its probable error, are 10 to 1, 49 to 1, 332 to 1, 2700 to 1 respectively, in round numbers.

In the first place, μ_3 differs from zero by 4 to 6 times the probable error in (1), (3–2), (1–3), and (2–1). Further, d differs from zero by 2·5 to 5 times the probable error in the same cases, and the skewness also by 2·5 to 5 times its probable error. $\sqrt{\beta_1}$ differs from zero by 3 to nearly 8 times its probable error in the same four distributions. I consider that it is really impossible to look upon these distributions as random samplings from symmetrical material. On the other hand, (2) and (3) or the absolute personal equation of Dr. MACDONELL and Dr. LEE might well be *symmetrical* distributions. Do they, however, fulfil the conditions for normality $\beta_2 = 3$, $K = 0$? The deviations of β_2 from 3 are in the two cases 2·0256 and 2·4058, or nearly 14 and 2·8 times the probable errors respectively. Further, their values of K differ from zero by nearly 14 and 2·9 times their probable errors. Thus the odds are enormous against Dr. MACDONELL'S judgment being a random sampling from a normal distribution of errors, and are about 300 to 1 against Dr. LEE'S being such! Of the other distributions the odds are enormously against normal distribution in cases (1), (3–2), and (2–1). They are less marked in (1–3), β_2 only differing from 3 and K from 0 by about 2·1 and 2·9, their probable errors respectively—but this case has already been excluded from normality on account of its sensible skewness.

Thus while two of the cases might reasonably be considered symmetrical distributions, *i.e.*, to fulfil axiom (α), and one of the cases might with some improbability (10 or 11 to 1) be supposed to have $\beta_2 = 3$, *i.e.*, to fulfil axioms (β) and (γ), no single case can be supposed, with any reasonable degree of probability, to fulfil all three, or to be capable of representation by a normal curve. The third moment, the distance from mean to mode, the skewness and the magnitude and sign ot the criterion K are quantities which, one or more or all, are in each individual case sensible and quite inconsistent with the result of random sampling from "normal material."

I now give a similar table for the bisection experiments.

TABLE XVIII.—Bisection of Lines.

	1.	2.	3.	2–3.	3–1.	1–2.
μ_2	6·0258	9·3966	6·8915	10·4745	11·3987	12·3817
μ_3	−1·7326	·9779	−3·8272	2·2447	−4·6402	2·6583
	±1·0929	±2·1283	±1·3367	±2·5048	±2·8435	±3·2191
μ_4	118·0048	267·1088	124·7219	301·1530	352·0688	450·2880
β_1	·0137	·0012	·0448	·0044	·0145	·0037
	·0000	·0000	·0000	·0000	·0000	·0000
β_2	3·2499	3·0251	2·6261	2·7448	2·7096	2·9372
	± ·1478	± ·1478	± ·1478	± ·1478	± ·1478	± ·1478
d	+ ·1254	− ·0512	+ ·4045	·1310	+ ·2605	− ·1125
	± ·0907	± ·1132	± ·0970	± ·1196	± ·1247	± ·1300
Sk.	·0511	·0006	·1541	·0405	·0772	·0511
	± ·0369	± ·0369	± ·0369	± ·0369	± ·0369	± ·0369
$\sqrt{\beta_1}$	·1171	·0340	·2115	·0662	·1206	·0610
	± ·0739	± ·0739	± ·0739	± ·0739	± ·0739	± ·0739
K	− ·4587	− ·0468	+ ·8821	+ ·5235	+ ·6243	+ ·1368
	± ·2955	± ·2955	± ·2955	± ·2955	± ·2955	± ·2955

Now it will be seen by a most cursory glance at this table that the distribution of errors in the case of the bisection of right lines can be far more nearly represented by a normal curve than in the case of judgment as to the position of a bright line. In the case of every one of the constants for the distribution of my own errors of judgment (*i.e.*, (2)), they differ by less than their probable error from their value on the normal theory. I can therefore treat my judgments as following the normal law and represent them by this curve. In Mr. YULE'S case (*i.e.*, (3)), the distance from his mode to his mean is more than four times its probable error; or, only once in 332 trials, say, should we expect such a divergence from normality in a random selecting. It is thus very improbable that his judgments follow the normal law so far as symmetry is concerned. Further, the value of β_2 differs from 3 by about 2·53 times its probable error, or the odds against such a value are about 22 to 1, or, since β_2 can, unlike d and Sk., differ from its normal value either in excess or defect, say 10 or 11 to 1. On both counts, then, Mr. YULE'S judgments form improbably a normal distribution. Lastly, turning to Dr. LEE'S (*i.e.*, (1)), we see that the greatest divergence from normality is about 1·7 times the probable error in the case of β_2. The divergence from symmetry is measured by about 1·5 times the probable error. Thus the odds against such a random selection are on the two counts 4 to 1 and 3 to 1, about, or since the two results, given normality, are independent, the combination gives about 12 to 1. These are certainly only moderately long odds, and we must conclude that though a skew curve describes Dr. LEE'S judgment considerably better than a normal curve, yet a normal curve might, without any great improbability, be adopted.

The results for the absolute judgments indicate what we may expect to find for the relative judgments. The relative judgment of Dr. LEE and myself can well be described by a normal curve (see 2–1); the constants differing by less or by very little more from their normal values than the probable errors. On the other hand, Mr. YULE'S and Dr. LEE'S (see 1–3) relative judgments differ from normality on the score both of asymmetry and of flat-toppedness, by odds of more than 10 to 1 in each case, or of 100 (or at least 50) to 1 in the combination. Lastly, the odds against Mr. YULE'S and my relative judgments (see 3–2) on the basis of a random distribution are only about 4 to 1 on the more unfavourable way of considering them, so that 3–2 might pass as a normal distribution.

We thus conclude that while two out of the six distributions in the bisection series are very improbably random selections from normal material, two others are capitally represented by normal curves, while the remaining two are not very favourable cases.

Taking these results in connection with those for the bright-line distributions we must conclude: That the distribution of errors of judgment can diverge in a very sensible way, both on account of asymmetry and of flat-toppedness, from the Gaussian curve of errors; but that cases can be found which approximate with all probability to random sampling from normal material. Consequently it is necessary to select a type or types of frequency curve which, while allowing for these points of sensible divergence, will yet pass into the normal distribution in certain special cases where within the limits prescribed by the probable error the skewness and $\beta_2 - 3$ are sensibly zero.

Since it is incontestible that, if axioms (α), (β), and (γ) are adopted, our distribution of errors must be normal, we must conclude that one or other or all of these axioms are not universally true. When therefore we get material for which the skewness is sensibly not zero, or β_2 is sensibly not three, we are quite at liberty to assert that the sources producing these errors do not fulfil axiom (α) or axioms (β) or (γ) respectively.*

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* It is very necessary to insist upon this. A recent critic has asserted that I have argued in the second memoir of my evolution series ('Phil. Trans.,' A, vol. 186, p. 343) in an illegitimate manner on the nature of the sources which lead to particular types of distribution. He denies that it is possible to state anything

(11.) (*b*.) *Agreement between Theory and Observation in the General Distribution of Errors of each particular size.*

Now in the 'Philosophical Magazine' for July, 1900, I have worked out a very simple criterion for the goodness of fit of any frequency distribution to a theoretical curve. I have measured the probability that the divergence from a given curve is one which may be attributed to random sampling. The test is of the following kind: Calculate the squares of the differences of the observed and theoretical frequencies, and divide each such square by the corresponding theoretical frequency; the sum of all such results, written χ^2, is the constant from which we can easily determine whether the probability, P, is large or small that the observed system of divergences or a still more divergent system would arise by random sampling. In Table XIX. below are recorded for the case of the bright-line experiments the values of χ^2, n', or the number of frequency groups, and P, the above-mentioned probability.

TABLE XIX.—Motion of Bright Line.

		1.	2.	3.	3–2.	1–3.	2–1.
Skew curve	n'	18	16	20	24	23	23
	χ^2	12·07	19·72	15·88	60·24	20·37	40·17
	P	·7959	·1829	·6653	·000,035	·5599	·0103
Normal curve	n'	18	16	20	24	24	23
	χ^2	42·85	83·50	21·82	154·41	34·46	99·79
	P	·0006	·000,000	·2933	·000,000	·0441	·000,000

Some words are necessary as to the meaning of this table. n' gives the number of groups of frequency upon which the determination of χ^2 was based. This had to be somewhat arbitrary when there were outlying observations, as in cases (2), (3), (2–1). The calculation of the frequencies within the range of each group was found partly by mechanical integration of carefully drawn diagrams of the

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as to these sources. When one advances into a new country one is apt not to see all things at once in their due proportions, and I may well have laid more stress than was justifiable on the importance of range, for example. This was not because a determination of range, if it exists, is not of most primary importance, but because I had not till the fourth memoir of the series ascertained a method of determining the probable error of the determination of range, and seen that in certain cases it is considerable. The critic—to whom I hope to reply elsewhere—seems to have failed to perceive the aim of my investigations *i.e.*, to find a simple description of frequency, which will describe the great bulk of cases *within the errors of random sampling.*

theoretical curves, partly by reading ordinates where the contour of the curves was nearly straight, and partly by quadrature near the tails. Till tables have been calculated for the skew curves, such processes are all that are available; but they are quite sufficient, for we do not want great exactness in the determination of χ^2. We merely desire to know whether the observations are with reasonable probability the result of a random sampling from the proposed theoretical distribution. Now let us examine the results. We see that for (2), (3–2), and (2–1) the normal curve is for practical purposes "impossible." As a matter of fact, we might have gone to the tenth figure without the probability being sensible in these cases. Further, (1) is highly improbable, the odds being about 1666 to 1 against its occurrence as a random sample. In the case of (1–3) the odds are 22 to 1 against a deviation as bad or worse than this, so that this is an improbable result. Lastly in case 3, and this only, we find the odds short, only 2·4 to 1, about, against it; a case such as this would occur on the average about twice in five trials. It is really the only case in which, under our present test, we could admit the normal curve.

Turning to the skew curve, we see that in three out of the six cases the odds are in its favour, namely, in (1), (3), and (1–3). It is not improbable in (2), the odds being only about 5 to 1 against it. It is improbable in (2–1) and very improbable in (3–2); in both of these cases, however, it is at least a million times as probable as the normal curve. Thus the skew curve is always markedly and often immensely superior as a method of describing the frequency to the normal curve.

Nor is it hard to discover grounds for its failure in cases (3–2) and (2–1), or for its lesser success in (2). The skew curve depending, as its constants do, on the fourth moment, takes much more account of outlying observations than the normal curve does. Let us consider how the χ^2 of these distributions is made up.

Absolute Equation (2). If the reader will look at the diagram (p. 294) of this distribution, he will observe the outlying observation on the left. There is a less marked one on the right. The skew curve endeavours, and fairly successfully endeavours, to account for these outlying observations by thinning its peak and stretching its tails—it thus becomes a much worse fit for the body of the observations than the normal curve. Beyond 3·5 on the left the skew curve leads us to expect ·3, about, of an observation, the normal curve only ·013 of an observation. Thus the outlying observation increases the χ^2 of the skew curve by about 3, but the χ^2 for the normal curve by about 73! In other words the outlying observation is not very probable from the standard of the skew curve; it is improbable enough to be considered practically impossible from the standpoint of the normal curve. If we reject this outlying observation as due to a momentary eccentricity of the observer, then with the *same* values of the constants of the curves the χ^2's are as 17 to 10, about, or the normal curve fits the body of the observations better than the skew curve. But this position is, of course, again entirely reversed if we fit the two curves afresh, recalculating the constants without including the outlying observation in our data.

Relative Equation (3–2). A glance at the diagram shows the great irregularity of the distribution in this case. The outlying group on the left is here quite easily accounted for by the skew curve. It is immensely improbable on the basis of a normal distribution. The outlying group of three observations on the right contributes 17 to the χ^2 of the skew curve and only 2·7 to that of the normal curve. The peak costs the normal curve 29 and the skew curve 16. If we were to cut off the two extreme groups the χ^2 for the skew curve would be reduced to about 40, and for the normal curve, to about 75. Thus the skew curve, without re-calculation of constants, would still be immensely more probable than the normal curve. There is little doubt, however, that there is some source of change in the personal equation of Dr. LEE which has produced the anomalies in the relative judgment of Dr. MACDONELL and herself.

Relative Equation (2–1). The small probability of the skew curve and the "practical impossibility" of the normal curve depend entirely on the existence of the outlying observation to the right. The χ^2's in both cases would be reduced to about 24, and thus give probable results on the basis of random samplings if this outlying observation were removed. A re-calculation of constants would set the skew curve far above the normal, for its constants are more widely modified by outlying observations.

As I have already pointed out the value of χ^2 depends largely on where the range for the grouping of the frequencies is taken, and the tails largely determine what its value will be. But I have endeavoured to be equally fair to both theories, and rough as the numbers must necessarily be, we may still safely conclude that the skew curve gives infinitely more probable results than the normal. Indeed, with the rejection of an outlying observation or two, we could bring the whole skew-series into the range of probable random samplings, but we should fail to achieve this in the case of the normal curve without much "doctoring," which would have to be applied in certain cases to the very body of the observations and not only to its tails.

Personally while considering that the value of χ^2 is a very good criterion for the rejection or not of outlying observations, *as soon as a probable law for the distribution of errors has been determined*, I have thought it right not to reject one single observation* after the constants had once been determined, because I had in view the comparison of two different theories, and such rejection might apparently favour one or the other theory.

I now turn to the results for the bisection of lines; the probabilities for the random sampling in these series are given in Table XX.

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* In the bright-line experiments 520 were originally made, as I supposed when we came to examine the recording strips some obvious slips or blunders would be found, and I left myself a margin of 20 for such. Only one experiment, however, No. 291, seemed to be a failure, the recording mark of one observer being in this case quite removed from the part of the scale occupied by the bright line.

TABLE XX.—Bisection of Lines.

		1.	2.	3.	3–2.	1–3.	2–1.
Skew curve	n'	16	21	16	17	21	23
	χ^2	11·36	17·27	14·90	8·84	20·28	23·99
	P	·7271	·6353	·4590	·9199	·4405	·3478
Normal curve	n'	17	20	17	17	22	23
	χ^2	13·30	22·04	20·31	9·34	21·38	25·65
	P	·6506	·2817	·2066	·8976	·4364	·2670

Now looked at from this standpoint, we see that not one of the distributions are improbable on the basis of either the skew or of the normal curve. The longest odds against a random sampling on the basis of a normal distribution are only 4 to 1, and on the basis of a skew distribution only 2 to 1. There is a clear and marked advantage in favour of the skew distribution, but it is nothing like so enormous as in the case of the bright-line series. If we take the distributions (1), (2), and (3) as giving independent probabilities of random sampling,* then the odds against these distributions as a result of random sampling from normal material are 24 to 1. Thus it seems that even in this case the normal law is somewhat improbable as a general law of distribution.

On the other hand, the combined odds against the system of distributions represented by the skew curves, are only 3·7 to 1; or looking at the problem rather differently: In the case of the normal distribution random sampling would give curves better than the observed in 20 per cent. of the trials, but in the case of skew distribution in only 5 per cent. of the trials. In other words, there is very great improvement in the closeness of fit produced by using skew distributions.

I place here the equations to the skew and normal distributions (a) and (b) respectively; remarking that the unit of y in either case is an observation, but the unit of x in the bright-line experiments is half a centimetre of the recording strip, and in the bisection experiments $\frac{1}{100}$ of the length of the line bisected.

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* As we have found correlation in judgments, there is, of course, some assumption in this hypothesis but it will serve as a rough comparative test.

BRIGHT-LINE EXPERIMENTS.

Absolute Judgments.

(1.)

(*a.*) $x = 5{\cdot}442{,}309 \tan\theta.$
$y = 92{\cdot}5307 \cos^{8\cdot386064}\theta\, e^{+1\cdot078805\,\theta}.$
Origin of curve at $-\ {\cdot}38195.$
(*b.*) $y = 87{\cdot}060$ expt. $(-\,x^2/11{\cdot}312{,}202).$
Origin of curve at the mean, $+\ {\cdot}07774.$

(2.)

(*a.*) $x = 5{\cdot}227{,}204 \tan\theta.$
$y = 100{\cdot}806 \cos^{7\cdot96724}\theta\, e^{\cdot112,2496\,\theta}.$
Origin at $-\ 1{\cdot}19399.$
(*b.*) $y = 88{\cdot}265$ expt. $(-\,x^2/11{\cdot}005{,}434).$
Origin at mean, $-\ 1{\cdot}14483.$

(3.)

(*a.*) $y = 53{\cdot}359\left(1 + \dfrac{x}{11{\cdot}0856}\right)^{3\cdot96821}\left(1 - \dfrac{x}{14{\cdot}4504}\right)^{5\cdot17262}$
Origin at the mode, $-\ {\cdot}59736.$
(*b.*) $y = 56{\cdot}837$ expt. $(-\,x^2/26{\cdot}541{,}754).$
Origin at the mean, $-\ {\cdot}44635.$

Relative Judgments.

(3–2.)

(*a.*) $x = 11{\cdot}17755 \tan\theta.$
$y = 21{\cdot}2674 \cos^{15\cdot34390}\theta\, e^{+5\cdot89124\,\theta}.$
Origin at $-\ 1{\cdot}78466.$
(*b.*) $y = 59{\cdot}537$ expt. $(-\,x^2/24{\cdot}188{,}480).$
Origin at the mean, $+\ {\cdot}68275.$

(1–3.)

(*a.*) $y = 48{\cdot}890\left(1 + \dfrac{x}{10{\cdot}4055}\right)^{3\cdot35160}\left(1 - \dfrac{x}{18\ 5911}\right)^{5\cdot98816}.$
Origin at the mode, $+\ {\cdot}2505.$
(*b.*) $y = 51{\cdot}580$ expt. $(-\,x^2/32{\cdot}227{,}254).$
Origin at the mean, $+\ {\cdot}61145.$

(2–1.)

(*a*.) $x = 8{\cdot}646433 \tan\theta$.
$y = 45{\cdot}7498 \cos^{10{\cdot}55017}\theta\, e^{-2{\cdot}976575\,\theta}$.
Origin at $+\,{\cdot}28986$.
(*b*.) $y = 62{\cdot}141$ expt. $(-\,x^2/22{\cdot}203{,}842)$.
Origin at mean $-\,1{\cdot}21518$.

BISECTION EXPERIMENTS.

Absolute Equations.

(1.)

(*a*.) $x = 12{\cdot}89913 \tan\theta$.
$y = 60{\cdot}959 \cos^{31{\cdot}24937}\theta\, e^{-4{\cdot}441{,}508\,\theta}$.
Origin at $+\,{\cdot}72873$ hundredth of line.
(*b*.) $y = 81{\cdot}260$ expt. $(-\,x^2/12{\cdot}051{,}534)$.
Origin at mean $-\,1{\cdot}230$ hundredths of line.

(2.)

(*a*.) $x = 48{\cdot}8323 \tan\theta$.
$y = 6{\cdot}038454 \cos^{261{\cdot}55}\theta\, e^{35{\cdot}61738\,\theta}$.
Origin at $+\,6{\cdot}2061$.
(*b*.) $y = 65{\cdot}072$ expt. $(-\,x^2/18{\cdot}793{,}284)$.
Origin at mean $+\,{\cdot}495$ hundredth of line.

(3.)

(*a*.) $y = 71{\cdot}56246\left(1 + \frac{x}{11{\cdot}349400}\right)^{5{\cdot}41665}\left(1 + \frac{x}{6{\cdot}998{,}136}\right)^{3{\cdot}33995}$
Origin at mode $+\,{\cdot}781519$ hundredth of line.
(*b*.) $y = 75{\cdot}987$ expt. $(-\,x^2/13{\cdot}783{,}076)$.
Origin at mean $+\,{\cdot}377$ hundredth of line.

Relative Equations.

(2–3.)

(*a*.) $y = 59{\cdot}43126\left(1 + \frac{x}{16{\cdot}16672}\right)^{9{\cdot}763{,}885}\left(1 - \frac{x}{13{\cdot}55284}\right)^{8{\cdot}185{,}180}$
Origin at the mode $+\,{\cdot}2662$ hundredth of line.
(*b*.) $y = 61{\cdot}633$ expt. $(-\,x^2/20{\cdot}949{,}076)$.
Origin at mean $+\,{\cdot}1230$ hundredth of line.

(3–1.)

$$(a.)\quad y = 56{\cdot}5924\left(1 + \frac{x}{16\ 28385}\right)^{8\cdot216,093}\left(1 - \frac{x}{12{\cdot}03998}\right)^{6\cdot074,832}.$$

Origin at mode + 1·8495 hundredth of line.

$$(b.)\quad y = 59{\cdot}083 \text{ expt. } (-x^2/22{\cdot}797,492).$$

Origin at mean + 1·5890 hundredths of line.

(1–2.)

$$(a.)\quad y = 56{\cdot}2136\left(1 + \frac{x}{28{\cdot}15012}\right)^{35\cdot390655}\left(1 - \frac{x}{37{\cdot}69023}\right)^{47\cdot384605}.$$

Origin at mode — 1·8245 hundredths of line.

$$(b.)\quad y = 56{\cdot}688 \text{ expt. } (-x^2/24{\cdot}763,446).$$

Origin at mean — 1·7120 hundredths of line.

Summing up the results of the above investigation as to random sampling, we conclude :—

(i.) That outlying observations render the skew curves a bad fit in one, and a very bad fit in a second case, and that in ten cases the observed results are very probable as random samplings from skew distributions.

(ii.) That the normal distribution is bad in one case and preposterously bad in four others ; it is probable in seven other cases, but in all cases less probable, and in five very much less probable, than the skew distribution.

We are thus led to much the same result as in our previous investigation of typical physical constants of the distribution, namely : that the axioms on which normality depends are not universally true, but that we require to use curves which will allow of a distinction between mode and mean, that will not assume an arbitrary relation between the fourth and second moments, yet which will pass gradually into the normal curve as we deal with material more and more nearly satisfying the fundamental axioms (α) (β) and (γ) (see pp. 274–275).

Such curves are supplied by the skew curves. If it be argued that these curves themselves involve relations between the first four and the higher moments, the answer is simply that we need only take such a number of independent moments that the bulk of frequency distributions can be represented as random samplings from our theoretical curves. It is idle to assert with **Lipps** that if we have n frequency groups we must take n independent moments to describe the distribution, the *sine quâ non* of the problem is to describe with the fewest possible constants the distribution of a very great number of groups. Nor will arbitrary curves with six or seven constants do as well as a well-chosen curve with three or four.* The normal curve in certain cases is a probable description, in a fair number of other cases it is a rough approximation, in many it is impossible. We must then start from its simple axioms (α), (β), (γ), and generalise in the next simplest manner. We assume that our contributory cause-groups are not indefinitely great in number, however numerous are the causes which determine the contribution of the group, that this contribution is not equally likely to be positive or negative, and finally that the contributions of the cause-groups are not independent but correlated quantities.* The simplest extension of the theory of GAUSS, LAPLACE, and POISSON in these directions leads us to the system of skew curves which have been applied in this memoir. I have treated them here purely from the experimental side. I have endeavoured to show in a fairly wide series of observations that the system of skew curves will, and the normal curve will not, satisfy the demands which we may fairly make on a theoretical frequency distribution. In another paper I shall consider the philosophical points which have been raised by EDGWORTH, LIPPS, and other recent writers. My present object has been to show certain failures in the ordinary theory of errors, and especially in its application to personal equation, and to show how existing theory may be widened so as to describe observations within the limits of the probable errors of the constants determined on the basis of random sampling.

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* Tested in a variety of ways in a memoir on the general theory of curve fitting, which will shortly appear in 'Biometrika.'

12. *Summary and Conclusions.*

If we attempt to sum up the results reached, their importance seems to rest on the amount of weight that is given to the experimental material. Can we look upon this as typical of the measurements usually made by physicists and astronomers? I am unable myself to differentiate it, or to see causes for the high correlation of judgments which are peculiar to our experiments, and not to observations such as are daily made in the physical laboratory or the observatory. If this be so, then we must conclude as follows :--

(*a.*) The personal equation, while tending to a constant value, appears subject to fluctuations far exceeding those of random sampling.

(*b.*) These fluctuations in the case of two or more observers, whether dealing at the same time with the same phenomenon or measuring at different times the same physical quantity, appear to be "sympathetic." Thus there may arise a very considerable correlation of judgments between two observers assumed *à priori* to observe independently.

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* Suppose, for example, that the cause-groups were those series of incalculable causes which determine (*a*) whether a coin shall fall head or tail uppermost; (*b*) whether an n-sided teetotum shall fall on one of p sides of one colour or not; (*c*) whether a card drawn out of a pack of np cards of p suits is of any particular suit. Then, if an indefinitely large number of coins be thrown together, the frequency distribution for heads satisfies all the fundamental axioms (α), (β), (γ) of the normal curve; if a finite number of teetotums be spun and the sides of the p-colour counted, we have satisfied (γ) only. If s cards be drawn simultaneously from our pack and the cards of one suit counted, then we have satisfied no one of the three fundamental axioms; there is correlation between the contributions of the cause-groups. This is only a rough illustration of the manner in which one or more of the fundamental axioms can be suspended artificially, but it is not without suggestion for the processes of nature.

(*c.*) In addition to this psychological or organic correlation occurring in the case of absolute judgments, there is a spurious correlation which arises when two observers are referred to either a third observer as standard or to a common time or space element in each measurement as unit.

(*d.*) Errors of judgment whether relative or absolute far from universally exhibit the normal distribution of frequency. It is necessary to generalise this law of distribution, and this can only be done by supposing some or all of the axioms on which the normal law is based to be modified until we have a sufficiently general theoretical distribution, which will enable us to look upon the great bulk of observational errors as random samplings from the theoretical frequencies.

Even then we may expect occasionally outlying observations due to mistakes of record, or the interference of special causes of isolated occurrence, to render our distribution as a random sample improbable. But this raises the question of the rejection of improbable observations, which is common to any theory of distribution.

Practically it would seem :

(i.) That the correlation of judgments is a necessary factor in our appreciation of personal equation. The weight to be given to a combined observation, or to the combination of observations of two observers, depends upon a knowledge of this factor.

(ii.) That we should attempt not only to find the personal equation of two observers, but also the variations and correlation of their judgments. For this purpose it may be needful to make experiments *ad hoc*, mimicking the actual observations to be made as closely as possible, for there appears no method of determining these quantities from the relative as distinguished from the absolute judgments.

(iii.) That the existence of this correlation in judgments appears to vitiate very largely the existing theory of the probability of testimony; that theory ought to be extended by the introduction of what we may term the psychological element; an element which many may more or less unconsciously have found wanting, when they considered the weight which had to be given on the mathematical theory to the testimony of "independent" witnesses of the same series of events.*

(iv.) That great care should be used in applying the current theory of errors to observations until it has been shown that within the fluctuations of random sampling these observations really follow the normal law. If they do not, then the physical distinction between mean and mode,* the probabilities of negative and positive errors of the same magnitude being quite different, the abnormal concentration of errors round the mode, are all characters of the distribution, which must be taken into consideration, and which it is important to describe.

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* If Dr. LEE and Dr. MACDONELL assert that a bright line was in certain positions when the bell rang, their united testimony is very far from having the weight it would have on the old mathematical theory that they are independent witnesses, and yet they record perfectly "independently."

In concluding this paper I desire to heartily thank those who have aided me in its preparation. In the first place my gratitude is due to Dr. LEE, Mr. G. U. YULE, and Dr. MACDONELL for the time and care they gave in experiment and observation. In the next place I owe Dr. LEE special thanks for the constant assistance I have received in the laborious computations she has aided me in, and which are hardly obvious on the face of this paper. To Mr. K. TRESSLER I am indebted for great assistance in the conduct of the bright-line experiments, especially in the preliminary adjustments we had to go through before we got our apparatus into efficient working order. He has also prepared from the calculations of Dr. LEE and myself the whole of the frequency diagrams. The work of experiment and reduction has extended over nearly six years, during which considerable progress has been made (*e.g.*, by Mr. SHEPPARD'S discovery of the best corrective terms for the moments) in statistical theory, and thus all our data have not been dealt with in an absolutely uniform manner;† but the divergences due to method are small as compared with the probable errors, and we have taken great care by duplication of calculations to avoid as far as possible arithmetical blunders.

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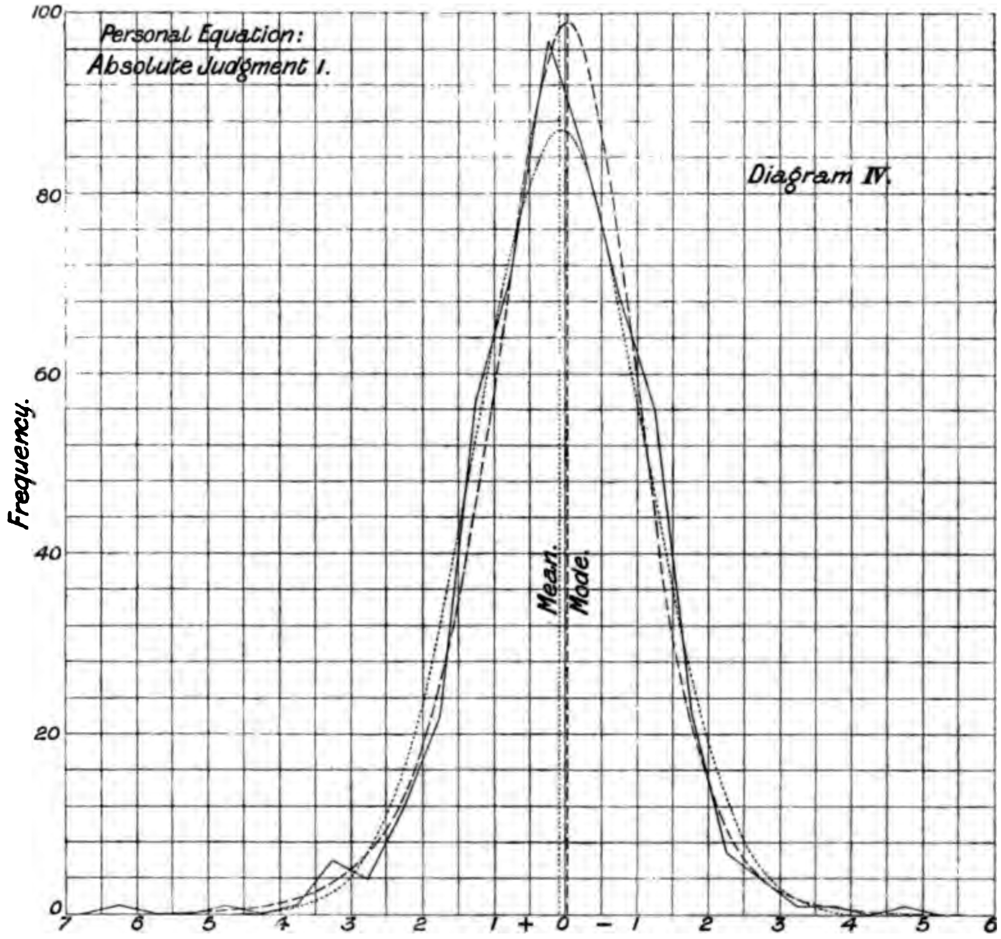
* Some American writers persist in taking the maximum group of observed frequency as the mode. But the fluctuations of random sampling make such a determination of the mode in many cases quite futile: see for examples my Diagrams VI., IX., and XII. The mode is where dy/dx vanishes for the theoretical frequency curve, and is not visible on mere inspection of the observations.

† The calculations for the bisection of lines were in part made on the grouped observations without SHEPPARD'S corrections, *i.e.*, with the value of the mean error as given in the usual theory.

Bright Line Experiments.



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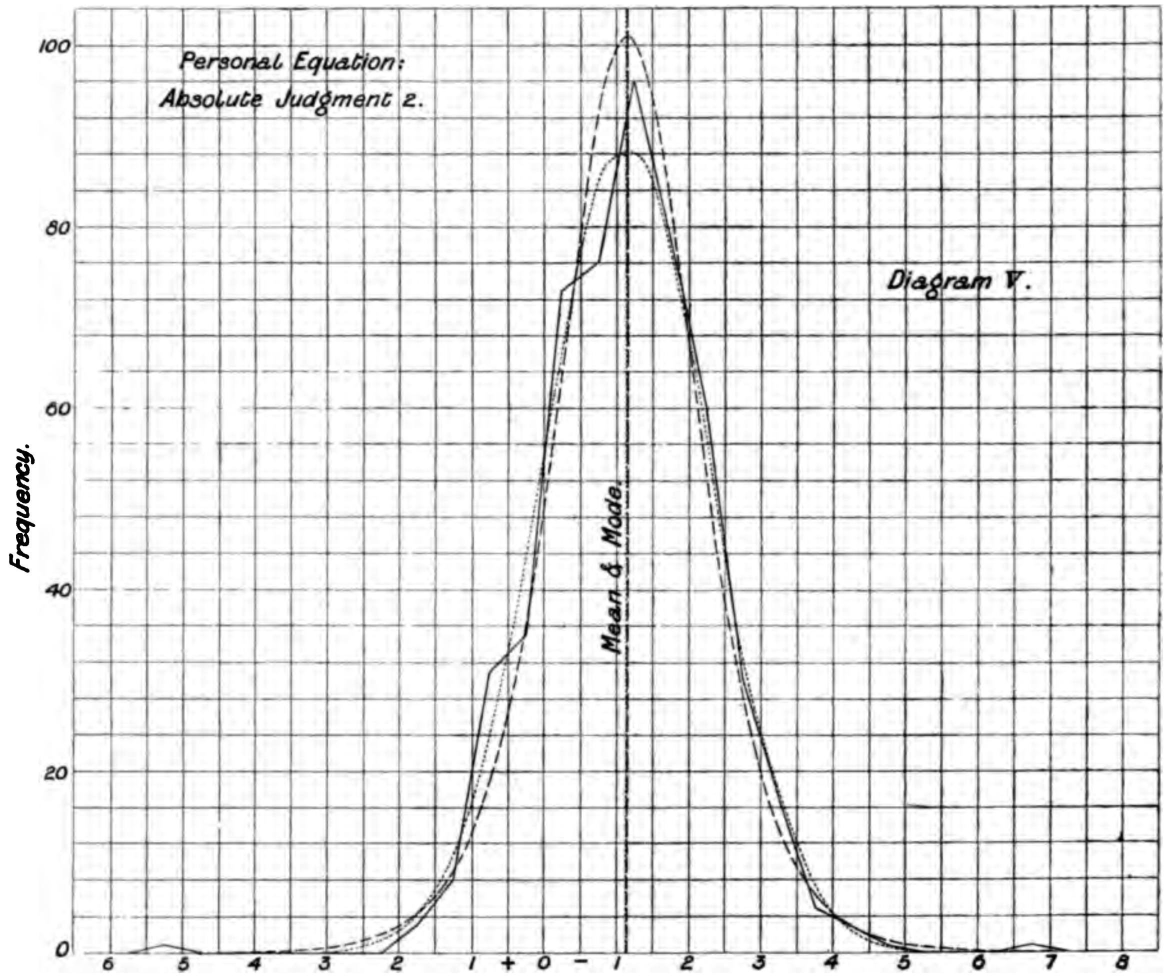

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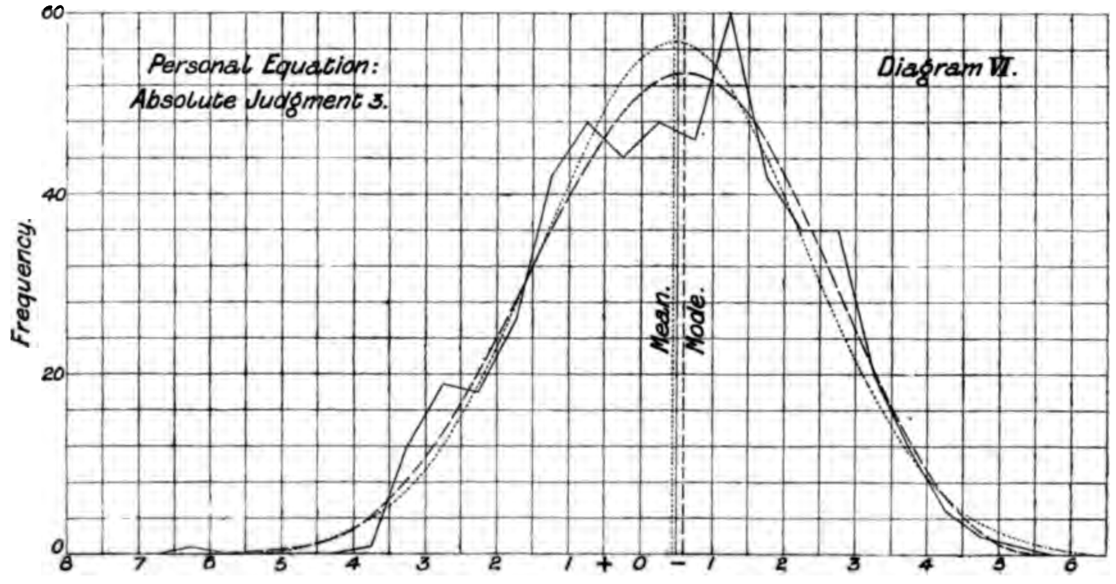

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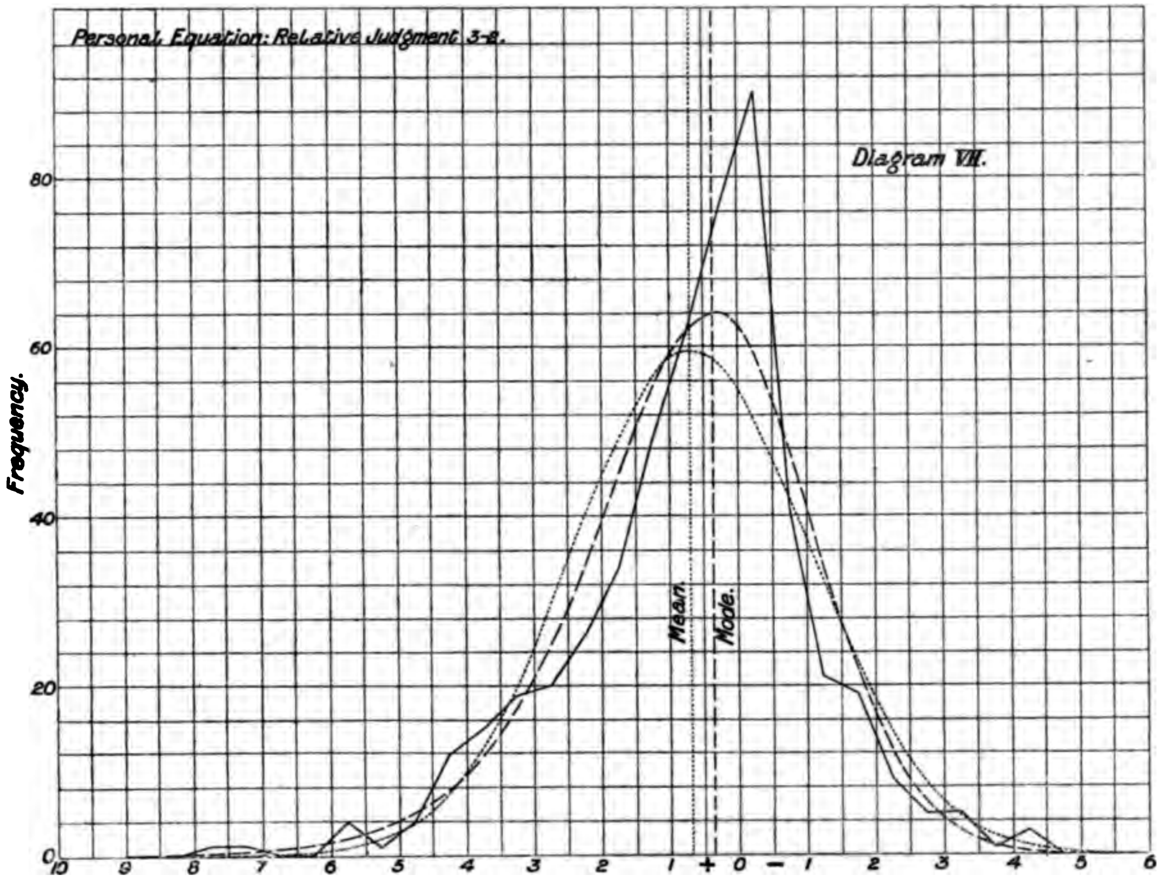

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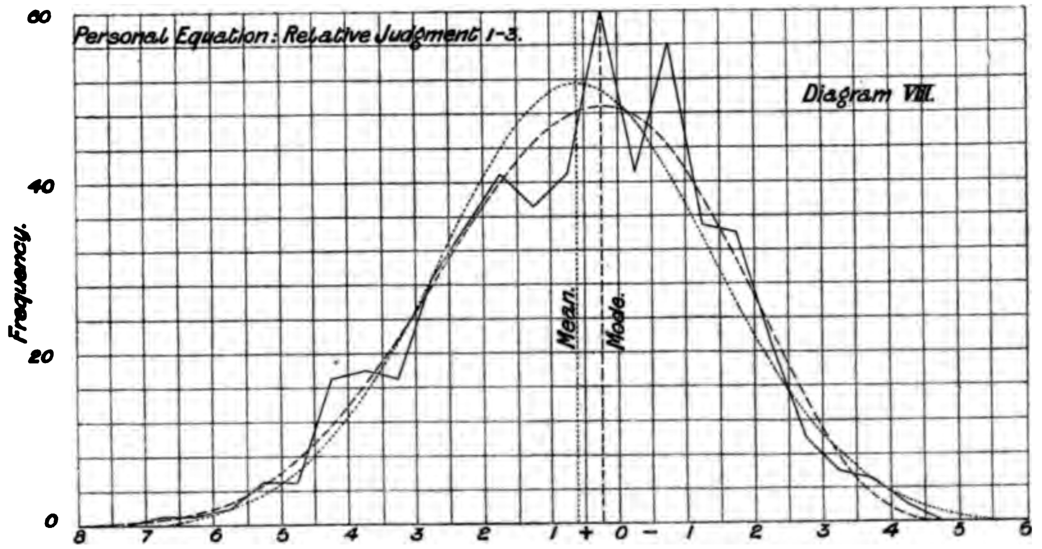

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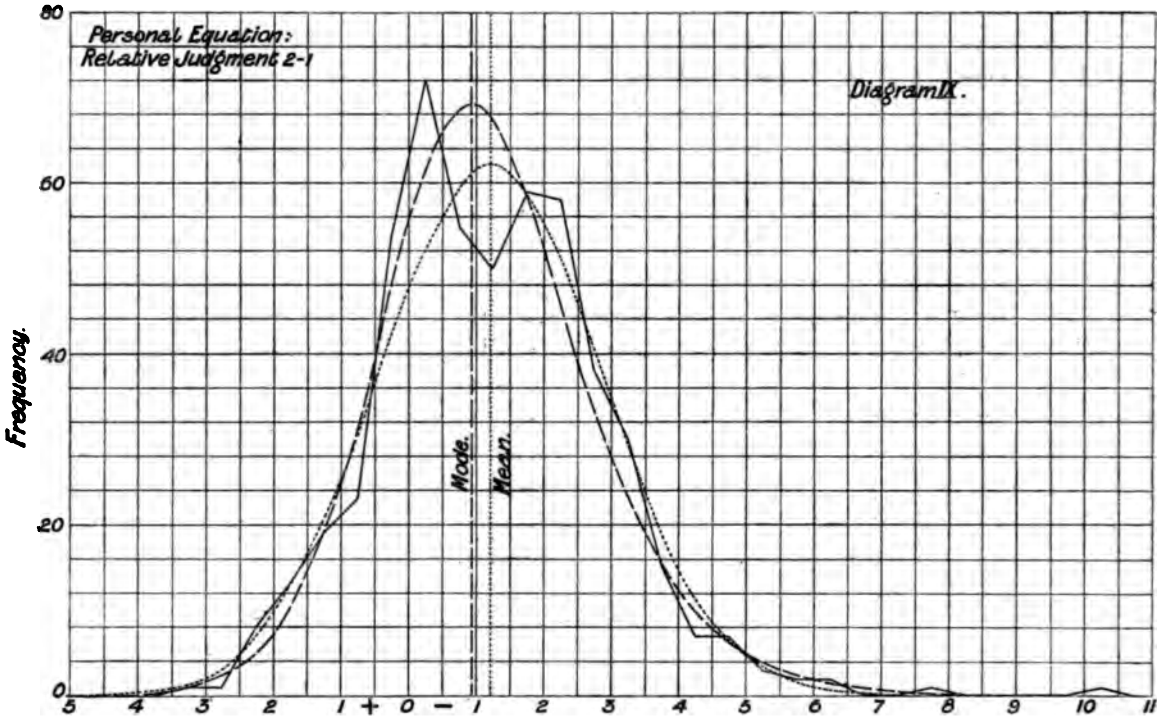
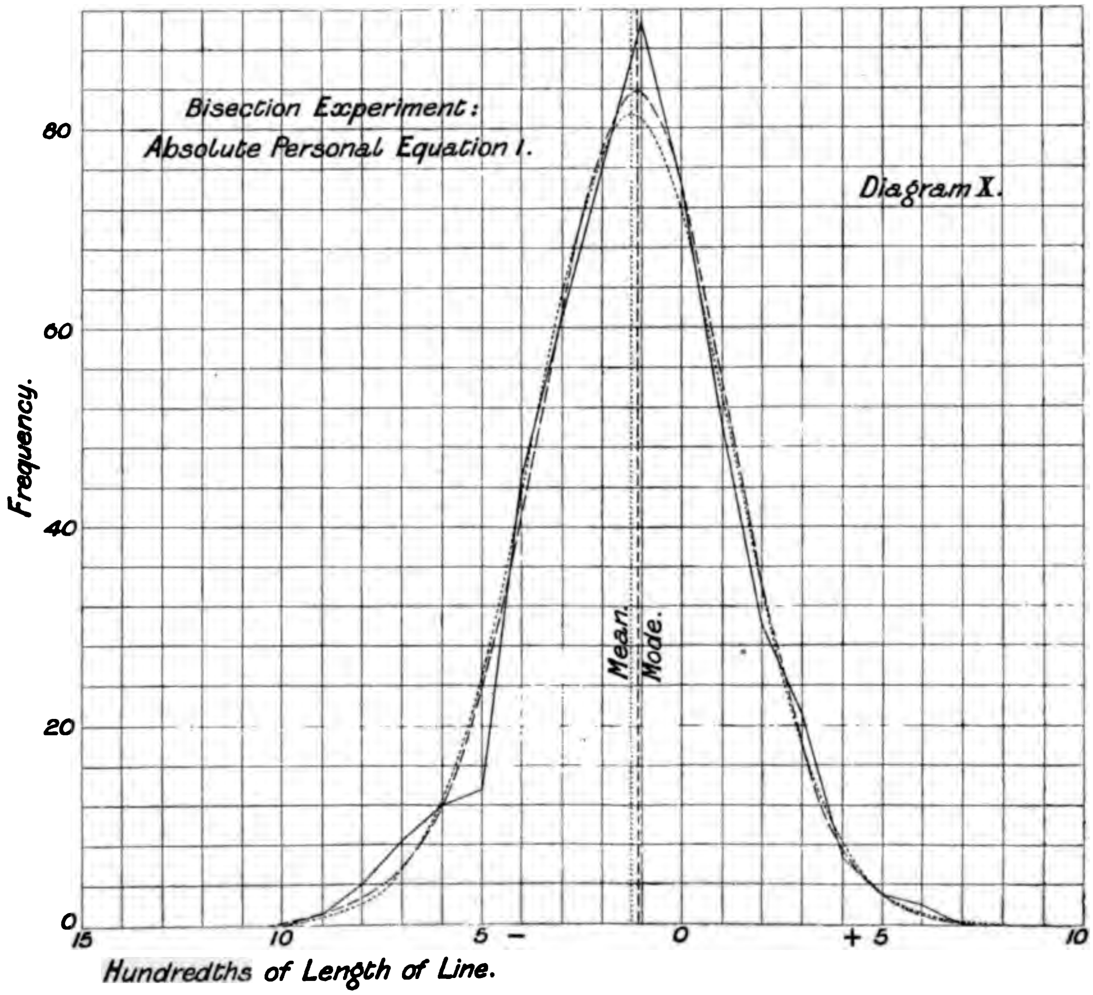
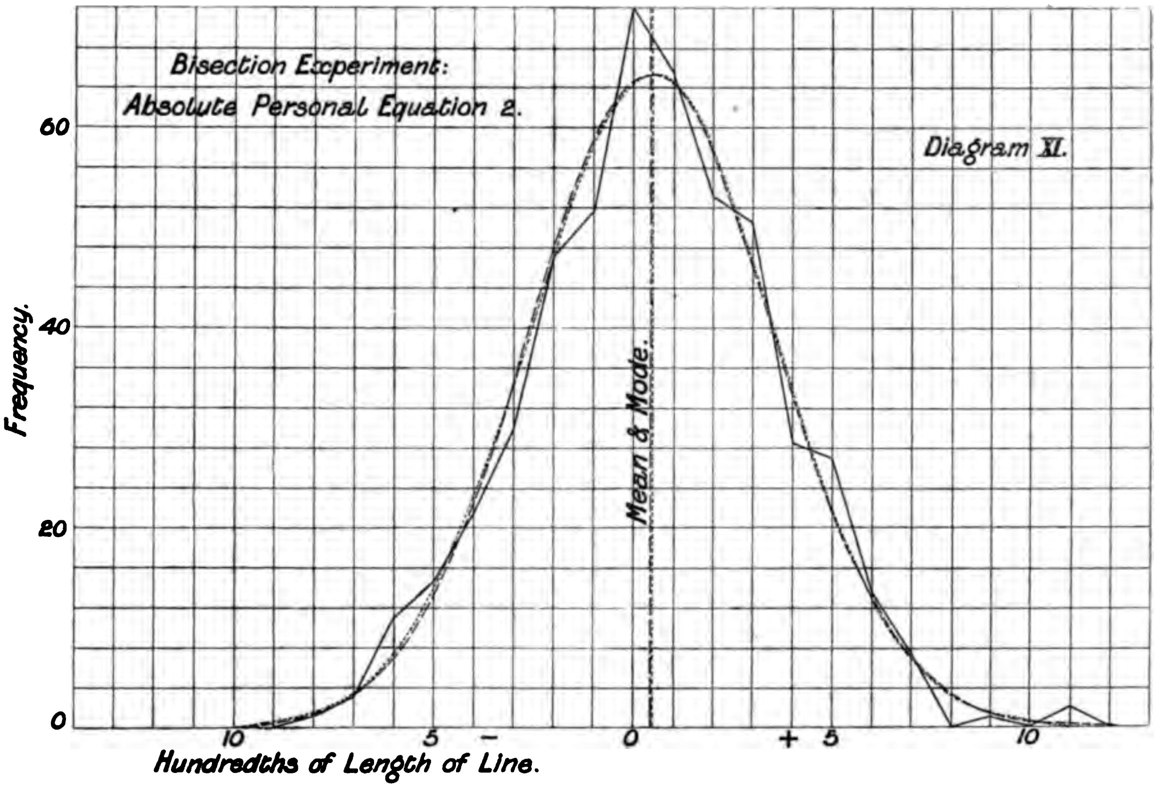
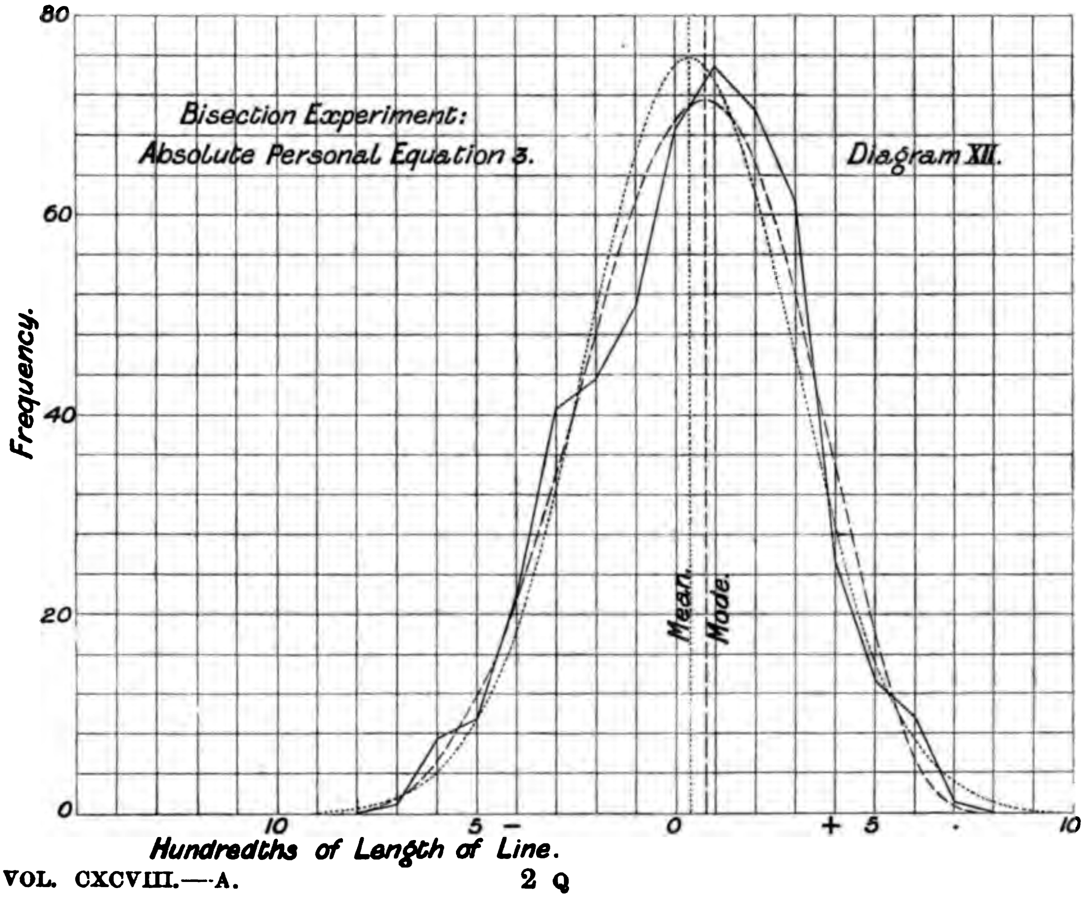





Bisection Experiment:
Absolute Personal Equation 2.
Diagram XI.
Frequency.
60
40
20
0
Mean & Mode.
10
5 −
0
+ 5
10
Hundredths of Length of Line.


80
Bisection Experiment:
Absolute Personal Equation 3.
Diagram XII.
Frequency.
60
40
20
0
Mean.
Mode.
10
5 −
0
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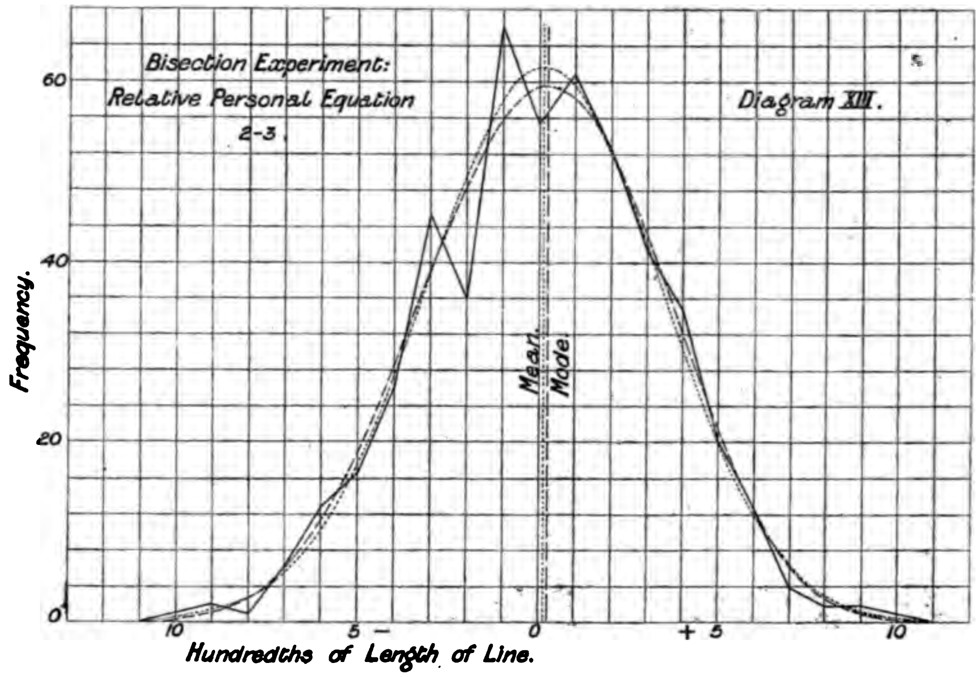
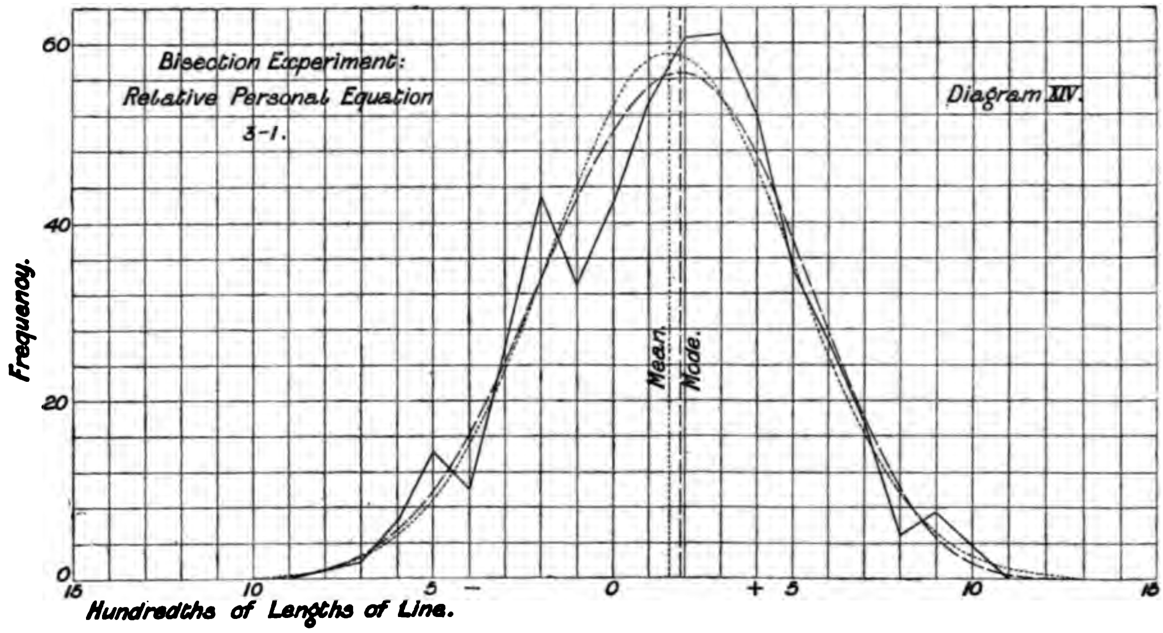
Hundredths of Length of Line.


Bisection Experiment:
Relative Personal Equation
2-3.
Diagram XIII.
Frequency.
60
40
20
0
Mean.
Mode.
10
5 −
0
+ 5
10
Hundredths of Length of Line.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Relative Personal Equation
3-1.
Diagram XIV.
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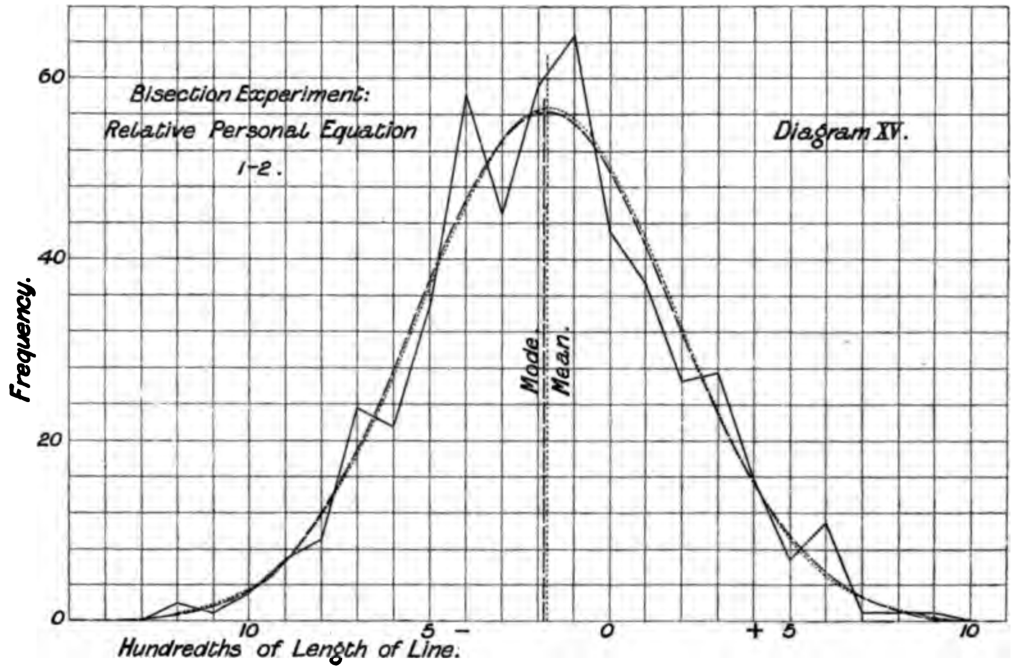
Mean.
Mode.
15
10
5 −
0
+ 5
10
15
Hundredths of Lengths of Line.


Bisection Experiment:
Relative Personal Equation
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20
0
10
5 −
0
+ 5
10
Hundredths of Length of Line.
Mode.
Mean.

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Phil. Trans., A, vol. 198, 1902, pp. 301–331.

Rotating Liquid Mass, Pear-shaped Figure of Equilibrium of.
Darwin, G. H. Phil. Trans., A, vol. 198, 1902, pp. 301–331.

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17.3.1902.

VI. *On the pear-shaped Figure of Equilibrium of a Rotating Mass of Liquid.*

By G. H. DARWIN, *F.R.S., Plumian Professor and Fellow of Trinity College, Cambridge.*

Received October 21,—Read November 21, 1901.

INTRODUCTION.

THIS is the sequel to a previous paper on "Ellipsoidal Harmonic Analysis" ('Phil. Trans..,' A, vol. 197, pp. 461–557). I here make use of the methods of that paper, and for brevity shall refer to it as "Harmonics."

The sections 1 to 4 are preparatory, and might have been included in "Harmonics," but seem more appropriate here. Section 5 is an independent investigation of so much of M. POINCARÉ's celebrated memoir on rotating liquid* as relates to the immediate object in view.

It is not necessary to say more here, since I give a short summary in the last section.

§ 1. *The Harmonics of the Third Degree.*

It was remarked in § 7 of "Harmonics" that this group of harmonics may be determined in rigorous algebraic form; I now proceed to do this.

The requisite formulæ are contained in § 6 of "Harmonics," and I take the several harmonics in succession. Throughout this section we have, of course, $i = 3$.

$s = 0$; type OEC, and $\mathfrak{P}_3(\nu) = q_0 P_3(\nu) + \beta q_2 P_3^{\,2}(\nu)$, with $q_0 = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{\frac{1}{2}\beta^2\{3,1\}\{3,2\}}{4+\beta\sigma} = \frac{15\beta^2}{1+\frac{1}{4}\beta\sigma}.$$

If we write

$$(B_1)^2 = 1 + 15\beta^2,$$

the proper solution of the quadratic equation is

$$\beta\sigma = 2(B_1 - 1).$$

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* 'Acta Mathematica,' vol. 7, 1885.

17.3.1902.

Also

$$\frac{q_2}{q_0} = \frac{1}{4 + \beta\sigma} = \frac{1}{2(B_1 + 1)}, \text{ with } q_0 = 1.$$

Then since

$$P_3(\nu) = \tfrac{5}{2}\nu^3 - \tfrac{3}{2}\nu, \qquad P_3{}^2 = 15\nu(\nu^2 - 1),$$

we find

$$\mathfrak{P}_3(\nu) = \frac{1}{2\beta}(B_1 - 1 + 5\beta)\,\nu\left(\nu^2 - \frac{4 - B_1}{5(1-\beta)}\right). \quad . \quad . \quad . \quad . \qquad (1).$$

$s = 1$; type OOC, and $\mathbf{P}_3{}^1(\nu) = \sqrt{\dfrac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}}\,(q_1{}'P_3{}^1(\nu) + \beta q_3{}'P_3{}^3(\nu))$, with $q_1{}' = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma + \tfrac{1}{2}\beta\,.\,3\,.\,4 = \frac{\frac{1}{4}\beta^2\{3, 2\}\,\{3, 3\}}{8 + \beta\sigma},$$

or

$$\beta\sigma + 6\beta = \frac{\frac{15}{4}\beta^2}{2 + \frac{1}{4}\beta\sigma}.$$

If we write

$$(B_2)^2 = 1 - \tfrac{3}{2}\beta(1 - \beta),$$

the proper solution of the quadratic equation is

$$\beta\sigma = 4(B_2 - 1 - \tfrac{3}{4}\beta).$$

Also

$$\frac{2q_3}{q_1} = \frac{1}{8 + \beta\sigma} = \frac{1}{4(B_2 + 1 - \frac{3}{4}\beta)}, \text{ and } \frac{q_3{}'}{q_1{}'} = 3\frac{q_3}{q_1}, \text{ with } q_1{}' = 1.$$

Then since $P_3{}^1(\nu) = \frac{3}{2}(5\nu^2 - 1)(\nu^2 - 1)^{\frac{1}{2}}$, $P_3{}^3(\nu) = 15(\nu^2 - 1)(\nu^2 - 1)^{\frac{1}{2}}$, we find

$$\mathbf{P}_3{}^1(\nu) = \frac{6}{\beta}(B_2 - 1 + 2\beta)\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(\nu^2 - \frac{3 - \beta - 2B_2}{5(1-\beta)}\right) \quad . \quad . \quad . \quad (2).$$

$s = 1$, type OOS and $\mathfrak{P}_3{}^1(\nu) = q_1 P_3{}^1(\nu) + \beta q_3 P_3{}^3(\nu)$, with $q_1 = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma - \tfrac{1}{2}\beta\,.\,3\,.\,4 = \frac{\frac{1}{4}\beta^2\{3, 2\}\,\{3, 3\}}{8 + \beta\sigma}.$$

A comparison with the last case shows that we have only to change the sign of β; accordingly if

$$(B_3)^2 = 1 + \tfrac{3}{2}\beta(1 + \beta),$$

we have

$$\beta\sigma = 4(B_3 - 1 + \tfrac{3}{4}\beta),$$

$$\frac{2q_3}{q_1} = \frac{1}{4(B_2 + 1 + \frac{3}{4}\beta)}, \text{ with } q_1 = 1.$$

On substitution we find

$$\mathfrak{P}_3{}^1(\nu) = \frac{2}{\beta}(B_3 - 1 + 3\beta)(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{3 + \beta - 2B_3}{5(1-\beta)}\right) : \quad . \quad . \quad . \quad (3).$$

$s = 2$; type OEC, $\mathfrak{P}_3{}^2(\nu) = \beta q_0 P_3(\nu) + q_2 P_3{}^2(\nu)$, with $q_2 = 1$.
The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{-\frac{1}{2}\beta^2\{3, 1\}\{3, 2\}}{4 - \beta\sigma} = \frac{-15\beta^2}{1 - \frac{1}{4}\beta\sigma}.$$

We have already defined $(B_1)^2$ as $1 + 15\beta^2$, and find the proper solution of the quadratic equation to be

$$\beta\sigma = -2(B_1 - 1).$$

Also

$$\frac{2q_0}{q_2} = \frac{-\{3, 1\}\{3, 2\}}{4 - \beta\sigma} = \frac{-4(B_1 - 1)}{\beta^2}, \text{ with } q_2 = 1.$$

With the known values of P_3 and of $P_3{}^2$, we find

$$\mathfrak{P}_3{}^2(\nu) = \frac{5}{\beta}(1 + 3\beta - B_1)\,\nu\left(\nu^2 - \frac{4 + B_1}{5(1-\beta)}\right) \quad . \quad . \quad . \quad . \quad . \quad (4).$$

A comparison with (1) for $s = 0$ shows that the last factors in each only differ in the sign of B_1.

$s = 2$; type OES, $\mathsf{P}_3{}^2(\nu) = \sqrt{\frac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}} \cdot P_3{}^2(\nu)$.
Since $P_3{}^2(\nu) = 15\nu(\nu^2 - 1)$, we have at once

$$\mathsf{P}_3{}^2(\nu) = 15\nu(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$s = 3$; type OOC, $\mathsf{P}_3{}^3(\nu) = \sqrt{\frac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}} \cdot [\beta q_1{}' P_3{}^1(\nu) + q_3{}' P_3{}^3(\nu)]$, with $q_3{}' = 1$.

The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{-\frac{1}{4}\beta^2\{3, 2\}\{3, 3\}}{8 - \beta\sigma - 6\beta} = \frac{-15\beta^2}{8 - \beta\sigma - 6\beta}.$$

We have already defined

$$(B_2)^2 = 1 - \tfrac{3}{2}\beta(1 - \beta),$$

and find for the proper solution

$$\beta\sigma = 4(1 - \tfrac{3}{4}\beta - B_2).$$

Also

$$\frac{2q_1}{q_3} = \frac{-\{3, 2\}\,\{3, 3\}}{8 - \beta\sigma - 6\beta} = -\frac{16}{\beta^2}\left(B_2 - 1 + \tfrac{3}{4}\beta\right),$$

and

$$\frac{q_1'}{q_3'} = \frac{q_1}{3q_3}, \text{ with } q_3' = 1.$$

Whence on substitution

$$\mathbf{P}_3^{\,3}(\nu) = \frac{20}{\beta}(1 - B_2)\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(\nu^2 - \frac{3 - \beta + 2B_2}{5(1-\beta)}\right) \quad . \quad . \quad . \qquad (6).$$

$s = 3$; OOS, $\mathfrak{P}_3^{\,3}(\nu) = \beta q_1 P_3^{\,1}(\nu) + q_3 P_3^{\,3}(\nu)$, with $q_3 = 1$.
The equation for $\beta\sigma$ is

$$\beta\sigma = \frac{-\frac{1}{4}\beta^2\{3, 2\}\,\{3, 3\}}{8 - \beta\sigma + 6\beta}.$$

We may derive the result from the last case by introducing

$$(B_3)^2 = 1 + \tfrac{3}{2}\beta(1 + \beta),$$

and changing the sign of β, so that

$$\beta\sigma = 4\left(1 + \tfrac{3}{4}\beta - B_3\right),$$

$$\frac{2q_1}{q_3} = -\frac{16}{\beta^2}\left(B_3 - 1 - \tfrac{3}{4}\beta\right), \text{ with } q_3 = 1.$$

Whence on substitution

$$\mathfrak{P}_3^{\,3}(\nu) = \frac{60}{\beta}(1 + \beta - B_3)(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{3 + \beta + 2B_3}{5(1-\beta)}\right) \quad . \quad . \quad . \quad . \qquad (7).$$

The forms of the corresponding functions of μ are the same, except that $(1 - \mu^2)^{\frac{1}{2}}$ and $\left(\frac{1+\beta}{1-\beta} - \mu^2\right)^{\frac{1}{2}}$ replace the corresponding factors.

I have not determined the cosine- and sine-functions, because they may be written down at once from the results already obtained. The three roots of the fundamental cubic are ν^2, μ^2, and $\frac{1-\beta\cos 2\phi}{1-\beta}$. Hence we have only to replace ν^2 by this last function in the seven formulæ (1)—(7) in order to obtain functions *proportional* to the seven cosine- and sine-functions. If the definition of the latter functions is to agree with that given in "Harmonics," the factors must be determined appropriately, but the question as to the value of the factor will not arise here.

§ 2. *Change of Notation.*

It will be convenient, with a view to future work, to change the notation, and I desire to adopt a notation which shall not only agree in the main with that used in "Harmonics," but shall also facilitate reference to a previous paper on Jacobi's ellipsoid ('Roy. Soc. Proc.,' vol. 41, pp. 319–336).

I write

$$\kappa^2 = \frac{1-\beta}{1+\beta}, \quad \kappa'^2 = 1 - \kappa^2.$$

It may be noted that what I here write κ was denoted by κ' in "Harmonics," and *vice versâ.*

I have in general written the current co-ordinates ν, μ, ϕ, and the ellipsoid of reference ν_0, so that the squares of the semi-axes are

$$k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right), \quad k^2(\nu_0^2 - 1), \quad k^2\nu_0^2.$$

I now propose to write as the squares of three semi-axes of the ellipsoid of reference

$$c^2\cos^2\gamma, \quad c^2(1 - \kappa^2\sin^2\gamma), \quad c^2.$$

Comparing these two we see that

$$k = c\kappa\sin\gamma, \text{ and } \nu_0 = \frac{1}{\kappa\sin\gamma}.$$

For the current co-ordinates I retain ϕ and write

$$\nu = \frac{1}{\kappa\sin\psi}, \quad \mu = \sin\theta.$$

The three roots of the fundamental cubic are therefore

$$\nu^2 = \frac{1}{\kappa^2\sin^2\psi}, \quad \mu^2 = \sin^2\theta, \quad \frac{1-\beta\cos2\phi}{1-\beta} = \frac{1}{\kappa^2}(1 - \kappa'^2\cos^2\phi).$$

The rectangular co-ordinates x, y, z are therefore now expressible as follows :—

$$\left.\begin{aligned} x &= \frac{c\sin\gamma}{\sin\psi}\,.\cos\psi\,(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}\cos\phi, \\ y &= \frac{c\sin\gamma}{\sin\psi}\,.(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}\cos\theta\sin\phi, \\ z &= \frac{c\sin\gamma}{\sin\psi}\,.\sin\theta\,(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

These give

$$\frac{x^2}{\cos^2\psi} + \frac{y^2}{1 - \kappa^2 \sin^2\psi} + z^2 = \frac{c^2 \sin^2\gamma}{\sin^2\psi}.$$

At the surface $\psi = \gamma$, and we have

$$\frac{x^2}{\cos^2\gamma} + \frac{y^2}{1 - \kappa^2 \sin^2\gamma} + z^2 = c^2.$$

In the formulæ for the third harmonics, in every case but one, and in two out of the five harmonics of the second degree, there occurs a factor of the form (ν^2 — constant); in each such case I write that constant in the form q^2/κ^2, and $q'^2 = 1 - q^2$. Thus q will have a different value for each harmonic.

It has been already remarked that for most purposes it is immaterial by what constants the several functions are multiplied. Although it would be easy to determine the constant in each case so as to make the function agree with its value as defined in "Harmonics," yet I shall not take that course, and shall omit factors as being in most cases redundant.

For the sake of completeness I will give the first and second harmonics in the new notation, as well as the third.

Since the harmonics of the first degree are expressed by

$$\mathfrak{P}_1(\nu) = \nu, \qquad \mathsf{P}_1^{\,1}(\nu) = \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \qquad \mathfrak{P}_1^{\,1}(\nu) = (\nu^2 - 1)^{\frac{1}{2}},$$

it is clear that in the new notation

$$\left.\begin{array}{lll} \mathfrak{P}_1(\nu) = \dfrac{1}{\sin\psi}, & \mathsf{P}_1^{\,1}(\nu) = \cot\psi, & \mathfrak{P}_1^{\,1}(\nu) = \dfrac{(1 - \kappa^2 \sin^2\psi)^{\frac{1}{2}}}{\sin\psi}, \\ \mathfrak{P}_1(\mu) = \sin\theta, & \mathsf{P}_1^{\,1}(\mu) = (1 - \kappa^2 \sin^2\theta)^{\frac{1}{2}}, & \mathfrak{P}_1^{\,1}(\mu) = \cos\theta, \\ \mathsf{C}_1(\phi) = (1 - \kappa'^2 \cos^2\phi)^{\frac{1}{2}}, & \mathfrak{C}_1^{\,1}(\phi) = \cos\phi, & \mathfrak{S}_1^{\,1}(\phi) = \sin\phi \end{array}\right\} \quad (9).$$

It appears from § 12 of "Harmonics" that

$$\mathfrak{P}_2(\nu) = \nu^2 + \frac{\gamma}{\alpha}, \qquad \mathfrak{P}_2^{\,2}(\nu) = \nu^2 + \frac{\gamma'}{\alpha'},$$

where $$\frac{\gamma}{\alpha} = \frac{B-2}{3(1-\beta)}, \quad \frac{\gamma'}{\alpha'} = -\frac{B+2}{3(1-\beta)}, \quad \text{and} \quad B^2 = 1 + 3\beta^2.$$

In accordance with the notation suggested above, let

$$\frac{q^2}{\kappa^2} = \frac{2 \mp B}{3(1-\beta)}.$$

Then substituting $\frac{1-\kappa^2}{1+\kappa^2}$ for β, we find

$$q^2 = \tfrac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}],$$

and for both cases

$$\kappa^2 = q^2\frac{2-3q^2}{1-2q^2}.$$

Hence

$$\left.\begin{aligned} \mathfrak{P}_2(\nu) \text{ and } \mathfrak{P}_2{}^2(\nu) &= \frac{1-q^2\sin^2\psi}{\sin^2\psi}, \\ \mathfrak{P}_2(\mu) \text{ and } \mathfrak{P}_2{}^2(\mu) &= -\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\ \mathfrak{C}_2(\phi) \text{ and } \mathfrak{C}_2{}^2(\phi) &= 1 - \frac{\kappa'^2}{q'^2}\cos^2\phi \end{aligned}\right\} \quad \ldots\ldots (10),$$

where $\kappa^2 = q^2\,\frac{2-3q^2}{1-2q^2}$, and $q^2 = \frac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$, with upper sign for the first and the lower sign for the second.

It appears from (19) and (20), § 7, of "Harmonics" that

$$\left.\begin{aligned} \mathsf{P}_2{}^1(\nu) &= \frac{\cos\psi}{\sin^2\psi}, \\ \mathsf{P}_2{}^1(\mu) &= \sin\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}, \\ \mathsf{C}_2{}^1(\phi) &= \cos\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad \ldots\ldots (11),$$

and from (21) and (22) that

$$\left.\begin{aligned} \mathfrak{P}_2{}^1(\nu) &= \frac{(1-\kappa^2\sin^2\psi)^{\frac{1}{2}}}{\sin^2\psi}, \\ \mathfrak{P}_2{}^1(\mu) &= \sin\theta\cos\theta, \\ \mathsf{S}_2{}^1(\phi) &= \sin\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad \ldots\ldots (12).$$

Lastly, from (25) and (26)

$$\left.\begin{aligned} \mathsf{P}_2{}^2(\nu) &= \frac{\cos\psi(1-\kappa^2\sin^2\psi)^{\frac{1}{2}}}{\sin^2\psi}, \\ \mathsf{P}_2{}^2(\mu) &= \cos\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}, \\ \mathfrak{S}_2{}^2(\phi) &= \sin\phi\cos\phi \end{aligned}\right\} \quad \ldots\ldots (13).$$

Turning to the harmonics of the third degree, we found that in the two cases where the type is OEC,

$$\mathfrak{P}_3(\nu) \text{ and } \mathfrak{P}_3{}^2(\nu) = \nu\left(\nu^2 - \frac{4 \mp B_1}{5(1-\beta)}\right).$$

If we put $\frac{q^2}{\kappa^2} = \frac{4 \mp B_1}{5(1-\beta)}$, we find

$$q^2 = \tfrac{2}{5}\left[1 + \kappa^2 \mp \left(1 - \tfrac{7}{4}\kappa^2 + \kappa^4\right)^{\frac{1}{2}}\right],$$

and $$\kappa^2 = q^2 \cdot \frac{4 - 5q^2}{3 - 4q^2}.$$

Therefore, with the above alternative form for q^2,

$$\left.\begin{aligned}
\mathfrak{P}_3(\nu) \text{ and } \mathfrak{P}_3^{\,2}(\nu) &= \frac{1 - q^2 \sin^2\psi}{\sin^3\psi}, \\
\mathfrak{P}_3(\mu) \text{ and } \mathfrak{P}_3^{\,2}(\mu) &= -\sin\theta\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\
\mathsf{C}_3(\phi) \text{ and } \mathsf{C}_3^{\,2}(\phi) &= \left(1 - \frac{\kappa'^2}{q'^2}\cos^2\phi\right)\left(1 - \kappa'^2\cos^2\phi\right)^{\frac{1}{2}}
\end{aligned}\right\} \quad . \quad . \quad . \quad (14).$$

Again in the two cases where the type is OOC we found

$$\mathsf{P}_3^{\,1}(\nu) \text{ and } \mathsf{P}_3^{\,3}(\nu) = \left(\nu^2 - \frac{3 - \beta \mp 2B_2}{5(1-\beta)}\right)\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$$

Putting $$\frac{q^2}{\kappa^2} = \frac{3 - \beta \mp 2B_2}{5(1-\beta)},$$

we find $$q^2 = \tfrac{1}{5}\left(1 + 2\kappa^2 \mp \left(1 - \kappa^2 + 4\kappa^4\right)^{\frac{1}{2}}\right),$$

and $$\kappa^2 = q^2 \frac{2 - 5q^2}{1 - 4q^2}.$$

Therefore, with the above alternative form for q^2,

$$\left.\begin{aligned}
\mathsf{P}_3^{\,1}(\nu) \text{ and } \mathsf{P}_3^{\,3}(\mu) &= \frac{\cos\psi\,(1 - q^2\sin^2\psi)}{\sin^3\psi}, \\
\mathsf{P}_3^{\,1}(\mu) \text{ and } \mathsf{P}_3^{\,3}(\mu) &= -\left(1 - \kappa^2\sin^2\theta\right)^{\frac{1}{2}}\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\
\mathfrak{C}_3^{\,1}(\phi) \text{ and } \mathfrak{C}_3^{\,3}(\phi) &= \cos\phi\left(1 - \frac{\kappa'^2}{q'^2}\cos^2\phi\right)
\end{aligned}\right\} \quad . \quad . \quad . \quad (15).$$

In the two cases where the type is OOS we found

$$\mathfrak{P}_3^{\,1}(\nu) \text{ and } \mathfrak{P}_3^{\,3}(\nu) = \left(\nu^2 - \frac{3 + \beta \mp 2B_3}{5(1-\beta)}\right)\left(\nu^2 - 1\right)^{\frac{1}{2}}.$$

Putting $$\frac{q^2}{\kappa^2} = \frac{3 + \beta \mp 2B_3}{5(1-\beta)},$$

we find $$q^2 = \tfrac{1}{5}\left(2 + \kappa^2 \mp (4 - \kappa^2\kappa'^2)^{\frac{1}{2}}\right),$$

and $$\kappa^2 = q^2 \frac{4 - 5q^2}{1 - 2q^2}.$$

Therefore, with the above alternative form for q^2,

$$\left.\begin{aligned} \mathfrak{P}_3^{\,1}(\nu) \text{ and } \mathfrak{P}_3^{\,3}(\nu) &= \frac{(1 - \kappa^2 \sin^2\psi)^{\frac{1}{2}}(1 - q^2\sin^2\psi)}{\sin^3\psi}, \\ \mathfrak{P}_3^{\,1}(\mu) \text{ and } \mathfrak{P}_3^{\,3}(\mu) &= -\cos\theta\left(1 - \frac{\kappa^2}{q^2}\sin^2\theta\right), \\ \mathfrak{S}_3^{\,1}(\phi) \text{ and } \mathfrak{S}_3^{\,3}(\phi) &= \sin\phi\left(1 - \frac{\kappa'^2}{q'^2}\cos^2\phi\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (16).$$

The seventh of these harmonics, which is of type OES, stands by itself. We had

$$\mathbf{P}_3^{\,2}(\nu) = \nu(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$$

This gives in the new notation

$$\left.\begin{aligned} \mathbf{P}_3^{\,2}(\nu) &= \frac{\cos\psi(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}{\sin^3\psi}, \\ \mathbf{P}_3^{\,2}(\mu) &= \sin\theta\cos\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}, \\ \mathbf{S}_3^{\,2}(\phi) &= \sin\phi\cos\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

The formulæ (9) to (17) give the fifteen sets of three functions constituting the fifteen harmonic functions of the first three degrees. It would be easy, although somewhat tedious, to find the coefficient by which each function is to be multiplied so that its definition may agree with that of the previous paper.

§ 3. *Expressions for the Solid Harmonics in Rectangular Co-ordinates.*

The three roots of the original cubic equation were ν^2, μ^2, $\frac{1 - \beta\cos 2\phi}{1-\beta}$, and in the new notation the three roots of

$$\frac{x^2}{\omega^2 - 1/\kappa^2} + \frac{y^2}{\omega^2 - 1} + \frac{z^2}{\omega^2} = c^2\kappa^2\sin^2\gamma \quad \text{are} \quad \frac{1}{\kappa^2\sin^2\psi}, \ \sin^2\theta, \ \frac{1 - \kappa'^2\cos^2\phi}{\kappa^2}.$$

Hence it follows that we have the identity

$$\frac{x^2}{\omega^2-1/\kappa^2}+\frac{y^2}{\omega^2-1}+\frac{z^2}{\omega^2}-c^2\kappa^2\sin^2\gamma=c^2\kappa^2\sin^2\gamma\,\frac{\left(\frac{1}{\kappa^2\sin^2\psi}-\omega^2\right)(\sin^2\theta-\omega^2)\left(\frac{1-\kappa'^2\cos^2\phi}{\kappa^2}-\omega^2\right)}{(1/\kappa^2-\omega^2)(1-\omega^2)\,\omega^2}.$$

Putting $\omega^2=\dfrac{q^2}{\kappa^2}$,

$$\frac{x^2}{q'^2}+\frac{y^2}{\kappa^2-q^2}-\frac{z^2}{q^2}+c^2\sin^2\gamma=\frac{c^2\sin^2\gamma}{(\kappa^2-q^2)\sin^2\psi}(1-q^2\sin^2\psi)\left(1-\frac{\kappa^2}{q^2}\sin^2\theta\right)\left(1-\frac{\kappa'^2}{q'^2}\cos^2\phi\right).$$

This expression, together with those for x, y, z in (8), enables us to write down the results at once. As before, I drop the several factors as being redundant for most purposes.

From (9)

$$\mathfrak{P}_1(\nu)\,\mathfrak{P}_1(\mu)\,\mathsf{C}_1(\phi)=z,\quad \mathsf{P}_1{}^1(\nu)\,\mathsf{P}_1{}^1(\mu)\,\mathfrak{C}_1{}^1(\phi)=x,\quad \mathfrak{P}_1{}^1(\nu)\,\mathfrak{P}_1{}^1(\mu)\,\mathfrak{S}_1{}^1(\phi)=y \quad . \quad . \quad . \quad (18).$$

From (10)

$$\mathfrak{P}_2(\nu)\,\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi)\ \text{and}\ \mathfrak{P}_2{}^2(\nu)\,\mathfrak{P}_2{}^2(\mu)\,\mathfrak{C}_2{}^2(\phi)=q^2x^2+\frac{q^2q'^2}{\kappa^2-q^2}y^2-q'^2z^2 + c^2q^2q'^2\sin^2\gamma \quad . \quad (19),$$

where $\qquad q^2=\frac{1}{3}[1+\kappa^2\mp(1-\kappa^2\kappa'^2)^{\frac{1}{2}}]$, and $\kappa^2=q^2\dfrac{2-3q^2}{1-2q^2}$,

so that $\qquad \dfrac{q^2q'^2}{\kappa^2-q^2}=1-2q^2$.

From (11), (12), and (13)

$$\mathsf{P}_2{}^1(\nu)\,\mathsf{P}_2{}^1(\mu)\,\mathsf{C}_2{}^1(\phi)=xz,\quad \mathfrak{P}_2{}^1(\nu)\,\mathfrak{P}_2{}^1(\mu)\,\mathsf{S}_2{}^1(\phi)=yz,\quad \mathsf{P}_2{}^2(\nu)\,\mathsf{P}_2{}^2(\mu)\,\mathfrak{S}_2{}^2(\phi)=xy \quad . \quad . \quad . \quad (20).$$

From (14)

$$\mathfrak{P}_3(\nu)\,\mathfrak{P}_3(\mu)\,\mathsf{C}_3(\phi)\ \text{and}\ \mathfrak{P}_3{}^2(\nu)\,\mathfrak{P}_3{}^2(\mu)\,\mathsf{C}_3{}^2(\phi)=z\left(q^2x^2+\frac{q^2q'^2}{\kappa^2-q^2}y^2-q'^2z^2 + c^2q^2q'^2\sin^2\gamma\right) \quad . \quad (21),$$

where $\qquad q^2=\frac{2}{5}[1+\kappa^2\mp(1-\frac{7}{4}\kappa^2+\kappa^4)^{\frac{1}{2}}]$, and $\kappa^2=q^2\dfrac{4-5q^2}{3-4q^2}$,

so that $\qquad \dfrac{q^2q'^2}{\kappa^2-q^2}=3-4q^2$.

From (15)

$$\mathsf{P}_3{}^1(\nu)\,\mathsf{P}_3{}^1(\mu)\,\mathfrak{C}_3{}^1(\phi)\ \text{and}\ \mathsf{P}_3{}^3(\nu)\,\mathsf{P}_3{}^3(\mu)\,\mathfrak{C}_3{}^3(\phi)=x\left(q^2x^2+\frac{q^2q'^2}{\kappa^2-q^2}y^2-q'^2z^2 + c^2q^2q'^2\sin^2\gamma\right) \quad . \quad (22),$$

where $$q^2 = \tfrac{1}{5}\left(1 + 2\kappa^2 \mp (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}}\right), \quad \text{and} \quad \kappa^2 = q^2 \frac{2 - 5q^2}{1 - 4q^2},$$

so that $$\frac{q^2 q'^2}{\kappa^2 - q^2} = 1 - 4q^2.$$

From (16)

$$\mathfrak{P}_3^1(\nu)\,\mathfrak{P}_3^1(\mu)\,\mathfrak{S}_3^1(\phi) \text{ and } \mathfrak{P}_3^3(\nu)\,\mathfrak{P}_3^3(\mu)\,\mathfrak{S}_3^3(\phi) = y\left(q^2x^2 + \frac{q^2q'^2}{\kappa^2 - q^2}y^2 - q'^2z^2 + c^2q^2q'^2\sin^2\gamma\right) \quad . \quad (23),$$

where $$q^2 = \tfrac{1}{5}\left(2 + \kappa^2 \mp (4 - \kappa^2\kappa'^2)^{\frac{1}{2}}\right) \quad \text{and} \quad \kappa^2 = q^2\frac{4 - 5q^2}{1 - 2q^2},$$

so that $$\frac{q^2q'^2}{\kappa^2 - q^2} = \tfrac{1}{3}(1 - 2q^2).$$

Lastly, from (17),

$$\mathsf{P}_3^2(\nu)\,\mathsf{P}_3^2(\mu)\,\mathsf{S}_3^2(\phi) = xyz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

It is easy to verify that each of these expressions satisfies LAPLACE's equation.

§ 4. *The Expression for the Q-functions in Elliptic Integrals.*

In this paper I drop the factors $\mathfrak{C}$ and E which were found to be necessary when the Q-functions were expressed in series.

We make the following definition :—

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0) = [\mathfrak{P}_i^s(\nu_0)]^2 \int_{\nu_0}^{\infty} \frac{d\nu}{[\mathfrak{P}_i^s(\nu)^2](\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}},$$

and a similar formula holds for $\mathsf{P}_i^s\,\mathsf{Q}_i^s$.

It is clear that $\mathfrak{P}_i^s$ may be multiplied by any constant factor without changing the result ; hence we may use the forms which have been found in §§ 2, 3.

The notation must now be changed.

We have $\nu = \dfrac{1}{\kappa\sin\psi}$ and $\nu_0 = \dfrac{1}{\kappa\sin\gamma}$. Therefore, when ψ is adopted as variable, the limits are γ to 0, and the sign of the whole is changed.

But $$d\nu = -\frac{\cos\psi}{\kappa\sin^2\psi}\,d\psi,$$

and $$(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} = \frac{\cos\psi\,(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}{\kappa^2\sin^2\psi}.$$

Therefore $$\int_{\nu_0}^{\infty}\frac{d\nu}{(\nu^2 - 1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}} = \kappa\int_0^{\gamma}\frac{d\psi}{(1 - \kappa^2\sin^2\psi)^{\frac{1}{2}}}.$$

In accordance with the usage in elliptic integrals, I write

$$\Delta^2 = 1 - \kappa^2 \sin^2 \psi$$

under the integral sign, or $1 - \kappa^2 \sin^2 \gamma$ outside the integral.

I shall also for brevity write

$$\Delta_1^2 = 1 - q^2 \sin^2 \psi$$

under the integral, or $1 - q^2 \sin^2 \gamma$ outside the integral.

We have then

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0) = \kappa\,[\mathfrak{P}_i^s(\nu_0)]^2 \int_0^\gamma \frac{d\psi}{[\mathfrak{P}_i^s(\nu)]^2\,\Delta}\,.$$

I apply this formula successively to the several functions, as given in (9) to (17), and introduce the abridged notation just defined, but I do not reiterate the special meanings to be attached to the symbol q in each case.

Since $\mathfrak{P}_0(\nu) = 1$, we have (dropping the now unnecessary suffix 0),

$$\left.\begin{aligned}
&\mathfrak{P}_0(\nu)\,\mathfrak{Q}_0(\nu) = \kappa\int_0^\gamma \frac{d\psi}{\Delta}\,,\\
&\mathfrak{P}_1(\nu)\,\mathfrak{Q}_1(\nu) = \frac{\kappa}{\sin^2\gamma}\int_0^\gamma \frac{\sin^2\psi\, d\psi}{\Delta}\,,\\
&\mathsf{P}_1^1(\nu)\,\mathsf{Q}_1^1(\nu) = \kappa\cot^2\gamma\int_0^\gamma \frac{\tan^2\psi}{\Delta}\,d\psi,\\
&\mathfrak{P}_1^1(\nu)\,\mathfrak{Q}_1^1(\nu) = \frac{\kappa\Delta^2}{\sin^2\gamma}\int_0^\gamma \frac{\sin^2\psi}{\Delta^3}\,d\psi,\\
&\mathfrak{P}_2(\nu)\,\mathfrak{Q}_2(\nu) \text{ and } \mathfrak{P}_2^2(\nu)\,\mathfrak{Q}_2^2(\nu) = \frac{\kappa\Delta_1^4}{\sin^4\gamma}\int_0^\gamma \frac{\sin^4\psi}{\Delta_1^4\Delta}\,d\psi,\\
&\mathsf{P}_2^1(\nu)\,\mathsf{Q}_2^1(\nu) = \frac{\kappa\cos^2\gamma}{\sin^4\gamma}\int_0^\gamma \frac{\sin^4\psi}{\cos^2\psi\,\Delta}\,d\psi,\\
&\mathfrak{P}_2^1(\nu)\,\mathfrak{Q}_2^1(\nu) = \frac{\kappa\Delta^2}{\sin^4\gamma}\int_0^\gamma \frac{\sin^4\psi}{\Delta^3}\,d\psi,\\
&\mathsf{P}_2^2(\nu)\,\mathsf{Q}_2^2(\nu) = \frac{\kappa\cos^2\gamma\,\Delta^2}{\sin^4\gamma}\int_0^\gamma \frac{\sin^4\psi}{\cos^2\psi\,\Delta^3}\,d\psi,\\
&\mathfrak{P}_3(\nu)\,\mathfrak{Q}_3(\nu) \text{ and } \mathfrak{P}_3^2(\nu)\,\mathfrak{Q}_3^2(\nu) = \frac{\kappa\Delta_1^4}{\sin^6\gamma}\int_0^\gamma \frac{\sin^6\psi}{\Delta_1^4\Delta}\,d\psi,\\
&\mathsf{P}_3^1(\nu)\,\mathsf{Q}_3^1(\nu) \text{ and } \mathsf{P}_3^3(\nu)\,\mathsf{Q}_3^3(\nu) = \frac{\kappa\cos^2\gamma\,\Delta_1^4}{\sin^6\gamma}\int_0^\gamma \frac{\sin^6\psi}{\cos^2\psi\,\Delta_1^4\Delta}\,d\psi,\\
&\mathfrak{P}_3^1(\nu)\,\mathfrak{Q}_3^1(\nu) \text{ and } \mathfrak{P}_3^3(\nu)\,\mathfrak{Q}_3^3(\nu) = \frac{\kappa\Delta_1^4\Delta^2}{\sin^6\gamma}\int_0^\gamma \frac{\sin^6\psi}{\Delta_1^4\Delta^3}\,d\psi,\\
&\mathsf{P}_3^2(\nu)\,\mathfrak{Q}_3^2 = \frac{\kappa\cos^2\gamma\,\Delta^2}{\sin^6\gamma}\int_0^\gamma \frac{\sin^6\psi}{\cos^2\psi\,\Delta^3}\,d\psi
\end{aligned}\right\} \quad . \quad . \quad (25).$$

All these integrals are expressible in terms of the elliptic integrals

$$F=\int_0^\gamma\frac{d\psi}{\Delta},\qquad E=\int_0^\gamma\Delta d\psi,\qquad \Pi=\int_0^\gamma\frac{d\psi}{\Delta_1^2\Delta}.$$

It will, however, be found that in fact the coefficient of Π vanishes in every case. The cases of $i=0$ and $i=1$ are very simple, and we have

$$\mathfrak{P}_0\mathfrak{Q}_0=\kappa F,$$

$$\mathfrak{P}_1\mathfrak{Q}_1=\frac{\kappa}{\sin^2\gamma}\left(\frac{1}{\kappa^2}F-\frac{1}{\kappa^2}E\right),$$

$$\mathsf{P}_1{}^1\mathsf{Q}_1{}^1=\kappa\cot^2\gamma\left(\frac{1}{\kappa'^2}\Delta\tan\gamma-\frac{1}{\kappa'^2}E\right),$$

$$\mathfrak{P}_1{}^1\mathfrak{Q}_1{}^1=\frac{\kappa\Delta^2}{\sin^2\gamma}\left(-\frac{1}{\kappa^2}F+\frac{1}{\kappa^2\kappa'^2}E-\frac{\sin\gamma\cos\gamma}{\kappa'^2\Delta}\right).$$

It is possible by direct differentiation to verify the following results, although the verification will be found pretty tedious.

$$\int\frac{\sin^4\psi}{\Delta_1{}^4\Delta}d\psi=\frac{(2-3q^2)q^2-\kappa^2(1-2q^2)}{2q^4q'^2(\kappa^2-q^2)}\Pi+\frac{2q'^2-1}{2q^4q'^2}F-\frac{1}{2q^2q'^2(\kappa^2-q^2)}E+\frac{\Delta\sin\psi\cos\psi}{2q'^2(\kappa^2-q^2)\Delta_1{}^2},$$

$$\int\frac{\sin^4\psi}{\cos^2\psi\Delta}d\psi=-\frac{1}{\kappa^2}F+\frac{1-2\kappa^2}{\kappa^2\kappa'^2}E+\frac{1}{\kappa'^2}\Delta\tan\psi,$$

$$\int\frac{\sin^4\psi}{\Delta^3}d\psi=-\frac{2}{\kappa^4}F+\frac{1+\kappa'^2}{\kappa^4\kappa'^2}E-\frac{\sin\psi\cos\psi}{\kappa^2\kappa'^2\Delta},$$

$$\int\frac{\sin^4\psi}{\cos^2\psi\Delta^3}d\psi=\frac{1}{\kappa^2\kappa'^2}F-\frac{1+\kappa^2}{\kappa^2\kappa'^4}E+\frac{\tan\psi}{\kappa'^4\Delta}[2-(1+\kappa^2)\sin^2\psi].$$

These are all the integrals needed for the harmonics of the second degree. In the case of the first we have

$$\kappa^2=q^2\frac{2-3q^2}{1-2q^2}.$$

Thus the coefficient of Π vanishes and the results are

$$\mathfrak{P}_2(\nu)\mathfrak{Q}_2(\nu)\text{ and }\mathfrak{P}_2{}^2(\nu)\mathfrak{Q}_2{}^2(\nu)=\frac{\kappa\Delta_1{}^4}{\sin^4\gamma}\left[\frac{1-2q^2}{2q^4q'^2}F-\frac{1-2q^2}{2q^4q'^4}E+\frac{(1-2q^2)\Delta\sin\gamma\cos\gamma}{2q^2q'^4\Delta_1{}^2}\right],$$

$$\mathsf{P}_2{}^1(\nu)\mathsf{Q}_2{}^1(\nu)=\frac{\kappa\cos^2\gamma}{\sin^4\gamma}\left[-\frac{1}{\kappa^2}F+\frac{1-2\kappa^2}{\kappa^2\kappa'^2}E+\frac{1}{\kappa'^2}\Delta\tan\gamma\right],$$

$$\mathfrak{P}_2{}^1(\nu)\mathfrak{Q}_2{}^1(\nu)=\frac{\kappa\Delta^2}{\sin^4\gamma}\left[-\frac{2}{\kappa^4}F+\frac{1+\kappa'^2}{\kappa^4\kappa'^2}E-\frac{\sin\gamma\cos\gamma}{\kappa^2\kappa'^2\Delta}\right],$$

$$\mathsf{P}_2{}^2(\nu)\mathsf{Q}_2{}^2(\nu)=\frac{\kappa\cos^2\gamma\Delta^2}{\sin^4\gamma}\left[\frac{1}{\kappa^2\kappa'^2}F-\frac{1+\kappa^2}{\kappa^2\kappa'^4}E+\frac{\tan\gamma(2-(1+\kappa^2)\sin^2\gamma)}{\kappa'^4\Delta}\right].$$

In the first of these $q^2=\frac{1}{3}[1+\kappa^2\mp(1-\kappa^2\kappa'^2)^{\frac{1}{2}}]$ and $\kappa^2=q^2\dfrac{2-3q^2}{1-2q^2}$.

The following integrals may also be verified by differentiation :

$$\int \frac{\sin^6\psi}{\Delta_1^4\Delta}\,d\psi = \frac{q^2(4-5q^2)-\kappa^2(3-4q^2)}{2q^6q'^2(\kappa^2-q^2)}\,\Pi + \frac{2q^2q'^2+\kappa^2(3-4q^2)}{2\kappa^2q^6q'^2}\,F - \frac{\kappa^2(3-2q^2)-2q^2q'^2}{2\kappa^2q^4q'^2(\kappa^2-q^2)}\,E$$
$$+ \frac{\Delta\sin\psi\cos\psi}{2q^2q'^2(\kappa^2-q^2)\Delta_1^2} \quad . \quad (26),$$

$$\int \frac{\sin^6\psi}{\cos^2\psi\Delta_1^4\Delta}\,d\psi = \frac{\kappa^2(1-4q^2)-q^2(2-5q^2)}{2q^4q'^4(\kappa^2-q^2)}\,\Pi + \frac{4q^2-1}{2q^4q'^4}\,F + \frac{1+2q^4-\kappa^2(2q^2+1)}{2\kappa'^2q^2q'^4(\kappa^2-q^2)}\,E$$
$$+ \frac{\Delta\tan\psi}{\kappa'^2q'^4} - \frac{\Delta\sin\psi\cos\psi}{2q'^4(\kappa^2-q^2)\Delta_1^2} \quad . \quad (27),$$

$$\int \frac{\sin^6\psi}{\Delta_1^4\Delta^3}\,d\psi = \frac{\kappa^2(1-2q^2)-q^2(4-5q^2)}{2q^4q'^2(\kappa^2-q^2)^2}\,\Pi + \frac{2q^2q'^2-\kappa^2(1-2q^2)}{2\kappa^2q^4q'^2(\kappa^2-q^2)}\,F + \frac{2q^2q'^2+\kappa^2\kappa'^2}{2\kappa^2\kappa'^2q^2q'^2(\kappa^2-q^2)^2}\,E$$
$$- \frac{\sin\psi\cos\psi}{\kappa'^2(\kappa^2-q^2)^2\Delta} - \frac{\Delta\sin\psi\cos\psi}{2q'^2(\kappa^2-q^2)^2\Delta_1^2} \quad . \quad (28),$$

$$\int \frac{\sin^6\psi}{\cos^2\psi\Delta^3}\,d\psi = \frac{2-\kappa^2}{\kappa^4\kappa'^2}\,F - \frac{2(1-\kappa^2\kappa'^2)}{\kappa^4\kappa'^4}\,E + \frac{\sin\psi\cos\psi}{\kappa^2\kappa'^4\Delta} + \frac{\Delta\tan\psi}{\kappa'^4} \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Now in (26) we have to put

$$\kappa^2 = q^2\frac{4-5q^2}{3-4q^2};$$

in (27)

$$\kappa^2 = q^2\frac{2-5q^2}{1-4q^2};$$

and in (28)

$$\kappa^2 = q^2\frac{4-5q^2}{1-2q^2}.$$

Introducing these values, and taking the integrals between the limits γ and 0, we find :

$$\mathfrak{P}_3\mathfrak{Q}_3 \text{ and } \mathfrak{P}_3{}^2\mathfrak{Q}_3{}^2 = \frac{\kappa\Delta_1^4}{\sin^6\gamma}\left\{\frac{7q'^2-1}{2\kappa^2q^4q'^2}\,F - \frac{2q'^4+5q'^2-1}{2\kappa^2q^4q'^4}\,E + \frac{(4q'^2-1)}{2q^4q'^4}\,\frac{\Delta\sin\gamma\cos\gamma}{\Delta_1^2}\right\}$$
$$. \quad . \quad . \quad . \quad (30).$$

$$\mathsf{P}_3{}^1\mathsf{Q}_3{}^1 \text{ and } \mathsf{P}_3{}^3\mathsf{Q}_3{}^3 = \frac{\kappa\cos^2\gamma\Delta_1^4}{\sin^6\gamma}\left\{\frac{4q^2-1}{2q^4q'^4}\,F + \frac{1-5q^2-2q^4}{2\kappa'^2q^4q'^4}\,E\right.$$
$$\left. - \left(\frac{1-7q^2-(1-5q^2-2q^4)\sin^2\gamma}{2\kappa'^2q^2q'^4}\right)\frac{\Delta\tan\gamma}{\Delta_1^2}\right\} \quad . \quad (31).$$

$$\mathfrak{P}_3{}^1\mathfrak{Q}_3{}^1 \text{ and } \mathfrak{P}_3{}^3\mathfrak{Q}_3{}^3 = \frac{\kappa\Delta_1^4\Delta^2}{\sin^6\gamma}\left\{-\frac{(1-2q^2)(2-3q^2)}{6\kappa^2q^4q'^4}\,F + \frac{2-11q^2q'^2}{6\kappa^2\kappa'^2q^4q'^4}\,E\right.$$
$$\left. - \left(\frac{1-5q^2+6q^4-q^2(2-11q^2q'^2)\sin^2\gamma}{6\kappa'^2q^4q'^4}\right)\frac{\sin\gamma\cos\gamma}{\Delta\Delta_1^2}\right\} \quad . \quad (32).$$

$$\mathsf{P}_3{}^2\mathsf{Q}_3{}^2 = \frac{\kappa\cos^2\gamma\Delta^2}{\sin^6\gamma}\left\{\frac{1+\kappa'^2}{\kappa^4\kappa'^2}F - \frac{2(1-\kappa^2\kappa'^2)}{\kappa^4\kappa'^4}E \right.$$
$$\left. + \left(\frac{1+\kappa^2-(1+\kappa^4)\sin^2\gamma}{\kappa^2\kappa'^4}\right)\frac{\tan\gamma}{\Delta}\right\} \quad . \quad (33).$$

In (30) $q^2 = \frac{2}{5}[1+\kappa^2 \mp (1-\frac{7}{4}\kappa^2+\kappa^4)^{\frac{1}{2}}], \qquad \kappa^2 = q^2\frac{4-5q^2}{3-4q^2}.$

In (31) $q^2 = \frac{1}{5}[1+2\kappa^2 \mp (1-\kappa^2+4\kappa^4)^{\frac{1}{2}}], \qquad \kappa^2 = q^2\frac{2-5q^2}{1-4q^2}.$

In (32) $q^2 = \frac{1}{5}[2+\kappa^2 \mp (4-\kappa^2\kappa'^2)^{\frac{1}{2}}], \qquad \kappa^2 = q^2\frac{4-5q^2}{1-2q^2}.$

§ 5. *Bifurcation of Jacobi's Ellipsoid.*

If a mass of liquid be rotating like a rigid body about an axis, x, with uniform angular velocity ω, the determination of the figure of equilibrium may be treated as a statical problem, if the mass be subjected to a potential $\frac{1}{2}\omega^2(y^2+z^2)$.

The energy lost in the concentration of a body from a condition of infinite dispersion is equal to the potential of the body in its final configuration at the position of each molecule, multiplied by the mass of the molecule and summed throughout the body. In the proposed system, as rendered a statical one, it is necessary to add $\frac{1}{2}\omega^2(y^2+z^2)$ to the gravitation potential before making the summation. If A denotes the moment of inertia of the body about x, this latter portion of the sum is $\frac{1}{2}A\omega^2$, and is therefore the kinetic energy of the system.

If dm_1, dm_2 denote any pair of molecules and D_{12} the distance between them, and E the energy lost, we have

$$E = \frac{1}{2}\int\frac{dm_1\,dm_2}{D_{12}} + \frac{1}{2}A\omega^2.$$

If the system had been considered as a dynamical one, the expression for the energy of the system, say U, would have resembled that for E, but the former of these terms would have presented itself with a negative sign.

It is clear that the variation of $\frac{1}{2}A\omega^2$, when the moment of momentum is kept constant, is equal and opposite to the variation of the same function when the angular velocity is kept constant.

The condition for a figure of equilibrium is that U shall be stationary for constant moment of momentum, or E stationary for constant ω, in both cases subject to the condition of constancy of volume. The variations in question lead to identical results, and I shall proceed from the variation of E.

It
$$\Psi = \int_0^\infty \frac{du}{(u+a^2)^{\frac{1}{2}}(u+b^2)^{\frac{1}{2}}(u+c^2)^{\frac{1}{2}}},$$

the internal potential of an ellipsoid of mass M and semi-axes a, b, c is

$$\tfrac{3}{4}M\left[\Psi + \frac{x^2}{a}\frac{d\Psi}{da} + \frac{y^2}{b}\frac{d\Psi}{db} + \frac{z^2}{c}\frac{d\Psi}{dc}\right].$$

Hence
$$\tfrac{1}{2}\int\frac{dm_1 dm_2}{D_{12}} = \tfrac{3}{8}M\int_0^\infty\left[\Psi + \frac{x^2}{a}\frac{d\Psi}{da} + \ldots\right]dm.$$

Now if A, B, C denote the principal moments of inertia of the ellipsoid about x, y, z,

$$\int x^2 dm = \tfrac{1}{2}(C+B-A) = \tfrac{1}{5}Ma^2,$$

and similar formulæ hold for the two other axes.

Therefore

$$\tfrac{1}{2}\int\frac{dm_1 dm_2}{D_{12}} = \tfrac{3}{8}M^2\left[\Psi + \tfrac{1}{5}\left(a\frac{d\Psi}{da} + b\frac{d\Psi}{db} + c\frac{d\Psi}{dc}\right)\right].$$

But since Ψ is a homogeneous function of degree -1 in a, b, c, the sum of the three differential terms is equal to $-\Psi$. Hence this expression is equal to $\tfrac{3}{10}M^2\Psi$.

Since
$$\tfrac{1}{2}A\omega^2 = \tfrac{1}{10}M(b^2+c^2)\,\omega^2,$$

we have
$$E = \tfrac{3}{10}M^2\left[\Psi + \frac{b^2+c^2}{3M}\,\omega^2\right].$$

If E be varied, whilst abc and ω are constant, it is stationary if

$$\frac{d\Psi}{da}\,\delta a + \left(\frac{d\Psi}{db} + \frac{2b}{3M}\,\omega^2\right)\delta b + \left(\frac{d\Psi}{dc} + \frac{2c}{3M}\,\omega^2\right)\delta c = 0,$$

$$\frac{\delta a}{a} + \frac{\delta b}{b} + \frac{\delta c}{c} = 0.$$

Eliminating δa, δb, δc we have the well-known conditions for JACOBI's ellipsoid

$$\left.\begin{aligned}\frac{2\omega^2 b^2}{3M} &= a\frac{d\Psi}{da} - b\frac{d\Psi}{db},\\ \frac{2\omega^2 c^2}{3M} &= a\frac{d\Psi}{da} - c\frac{d\Psi}{dc},\\ \frac{1}{b^2}\left(a\frac{d\Psi}{da} - b\frac{d\Psi}{da}\right) &= \frac{1}{c^2}\left(a\frac{d\Psi}{da} - c\frac{d\Psi}{dc}\right).\end{aligned}\right\} \quad \ldots\ldots\ (34)$$

If we add together the first two of these, and avail ourselves of the property that Ψ is homogeneous of degree -1, we easily prove that the stationary value of E is

$$E = \tfrac{9}{20} M^2 \left[\Psi + a \frac{d\Psi}{da} \right].$$

Since the potential of the ellipsoid must satisfy POISSON's equation

$$\frac{d\Psi}{ada} + \frac{d\Psi}{bdb} + \frac{d\Psi}{cdc} = -\frac{2}{abc}.$$

Also

$$a\frac{d\Psi}{da} + b\frac{d\Psi}{dc} + c\frac{d\Psi}{dc} = -\Psi.$$

By means of these and two out of the three equations (34), we may eliminate the differentials of Ψ, and writing ρ for the density find

$$\frac{\omega^2}{2\pi\rho} = \frac{\Psi abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 6}{(b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 6} \quad \ldots\ldots\ldots \quad (35).$$

I do not happen to have seen this form for the angular velocity of JACOBI's ellipsoid in any book.

It is easy also to show that the stationary value of E may be written

$$E = \tfrac{9}{20} M^2 \frac{\left[(b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 4\right]\Psi - 2\frac{b^2 + c^2}{abc}}{(b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 6}.$$

We may now express the potential, say V, of the system entirely in terms of Ψ and $a\frac{d\Psi}{da}$, for

$$V = \tfrac{3}{4}M\left[\Psi + \frac{x^2}{a}\frac{d\Psi}{da} + \frac{y^2}{b^2}\left(a\frac{d\Psi}{da} - \frac{2\omega^2 b^2}{3M}\right) + \frac{z^2}{c^2}\left(a\frac{d\Psi}{da} - \frac{2\omega^2 c^2}{3M}\right)\right] + \tfrac{1}{2}\omega^2(y^2 + z^2),$$

$$= \tfrac{3}{4}M\left[\Psi + a\frac{d\Psi}{da}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\right].$$

We thus verify that V is constant over the surface of the ellipsoid.

Let g denote the value of gravity at the surface. Then if dn be an element of the outward normal, $g = -\frac{dV}{dn}$. Since

$$\frac{dx}{dn} = \frac{px}{a^2}, \quad \frac{dy}{dn} = \frac{py}{b^2}, \quad \frac{dz}{dn} = \frac{pz}{c^2},$$

where $$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4},$$

$$g = -\tfrac{3}{2} M a \frac{d\Psi}{da} p \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) = -\tfrac{3}{2} \frac{M}{p} a \frac{d\Psi}{da}.$$

Now change the notation and write

$$a^2 = k^2 \left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right), \quad b^2 = k^2(\nu_0^2 - 1), \quad c^2 = k^2 \nu_0^2,$$

$$u = k^2 (\nu^2 - \nu_0^2).$$

Then $$\Psi = \frac{2}{k} \int_{\nu_0}^{\infty} \frac{d\nu}{(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}} (\nu^2 - 1)^{\frac{1}{2}}},$$

$$a \frac{d\Psi}{da} = -\frac{2}{k} \left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right) \int_{\nu_0}^{\infty} \frac{d\nu}{(\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{3}{2}} (\nu^2 - 1)^{\frac{1}{2}}}.$$

Now $$\mathfrak{P}_0(\nu) = 1, \quad \mathsf{P}_1^1(\nu) = \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}},$$

and $$\mathfrak{P}_i^s(\nu_0) \mathfrak{Q}_i^s(\nu_0) = [\mathfrak{P}_i^s(\nu_0)]^2 \int_{\nu_0}^{\infty} \frac{d\nu}{[\mathfrak{P}_i^s]^2 (\nu^2 - \frac{1+\beta}{1-\beta})^{\frac{1}{2}} (\nu^2 - 1)^{\frac{1}{2}}};$$

so that $$\left.\begin{aligned} \Psi &= \frac{2}{k} \mathfrak{P}_0(\nu_0) \mathfrak{Q}_0(\nu_0), \\ a \frac{d\Psi}{da} &= -\frac{2}{k} \mathsf{P}_1^1(\nu_0) \mathsf{Q}_1^1(\nu_0), \\ \text{and} \quad g &= \frac{3M}{pk} \mathsf{P}_1^1(\nu_0) \mathsf{Q}_1^1(\nu_0) \end{aligned}\right\} \quad \ldots\ldots\ldots (36).$$

We may note in passing that the condition for a Jacobian ellipsoid (the last equation of (34)) is reducible to the form

$$\frac{\kappa\Delta^2}{\sin^4\gamma} \int_0^{\gamma} \frac{\sin^4\psi}{\Delta^3} d\psi = \kappa \cot^2\gamma \int_0^{\gamma} \frac{\tan^2\psi}{\Delta} d\psi.$$

On examining the series of functions given in (25), we see that it may be written

$$\mathfrak{P}_2^1(\nu_0) \mathfrak{Q}_2^1(\nu_0) = \mathsf{P}_1^1(\nu_0) \mathsf{Q}_1^1(\nu)_0.$$

This agrees with M. POINCARÉ's equation (1) on p. 341 of his memoir.

We will now suppose that the body, instead of being an ellipsoid, is an ellipsoidal harmonic deformation of an ellipsoid, which is itself a figure of equilibrium for rotation ω.

The addition to E will consist of three parts; first that due to the mutual energy of the layer of deformation; secondly that due to the ellipsoid and the layer; thirdly that due to the change in the moment of inertia.

If a subscript l denotes integration throughout the space occupied by the layer, U the potential of the ellipsoid, and dv an element of volume,

$$\delta E = \tfrac{1}{2}\int_l \frac{dm_1 dm_2}{D_{12}} + \int_l U\rho dv + \tfrac{1}{2}\omega^2 \int_l (y^2 + z^2)\rho dv.$$

If ζ denotes the thickness of the layer standing on the element $d\sigma$, the first of these terms is $\tfrac{1}{2}\rho^2 \iint \frac{\zeta_1\zeta_2 d\sigma_1 d\sigma_2}{D_{12}}$.

The value of $U + \tfrac{1}{2}\omega^2(y^2 + z^2)$ throughout the layer is equal to $V_0 - g\zeta'$, where V_0 is the constant value of $U + \tfrac{1}{2}\omega^2(y^2 + z^2)$ over the surface of the ellipsoid, and ζ' is the distance measured along the normal to the element $d\zeta' d\sigma$ of volume.

Hence
$$\int_l U\rho dv + \tfrac{1}{2}\omega^2 \int_l (y^2 + z^2)\rho dv = \iint_0^\zeta \rho\,(V_0 - g\zeta')\, d\zeta' d\sigma.$$

Since V_0 is constant and the total mass of the layer is zero, this is equal to $-\tfrac{1}{2}\rho\int g\zeta^2 d\sigma$.

It follows that

$$\delta E = \tfrac{1}{2}\rho^2 \iint \frac{\zeta_1\zeta_2 d\sigma_1 d\sigma_2}{D_{12}} - \tfrac{1}{2}\rho \int g\zeta^2 d\sigma.$$

The axes of the ellipsoid have been chosen so as to make our original E stationary, and the further condition to be satisfied is that δE shall be stationary.

Let us suppose that

$$\zeta = pe\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi),$$

which expression shall be deemed to include any one of the other types of harmonic.

Then it is shown in (51) of "Harmonics" that the potential of this layer at the surface of the ellipsoid is

$$\frac{3M}{k} e\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0)\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi).$$

Since the mass of an element is $pe\rho\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)\,d\sigma$, we have

$$\tfrac{1}{2}\rho^2 \int \frac{\zeta_1\zeta_2 d\sigma_1 d\sigma_2}{D_{12}} = \tfrac{3}{2}\frac{M\rho}{k} e^3\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0) \int [\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)]^2 p d\sigma.$$

With the value of g found in (36)

$$\tfrac{1}{2}\rho \int g\zeta^2 d\sigma = \tfrac{3}{2}\frac{M\rho}{k} e^2\,\mathsf{P}_1^1(\nu_0)\,\mathsf{Q}_1^1(\nu_0) \int [\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)]^2 p d\sigma.$$

Hence

$$\delta E = -\tfrac{3}{2}\frac{M\rho}{k}e^2\,\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)\left[1-\frac{\mathfrak{P}_i{}^s(\nu_0)\,\mathfrak{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)}\right]\int[\mathfrak{P}_i{}^s(\mu)\,\mathfrak{C}_i{}^s(\phi)]^2\,p\,d\sigma.$$

In order that the new figure may be one of equilibrium, this expression must be stationary for variations of e. It follows that we must either have $e = 0$, which leads back to JACOBI's ellipsoid, or else

$$1-\frac{\mathfrak{P}_i{}^s(\nu_0)\,\mathfrak{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)}=0.$$

This last condition is what M. POINCARÉ calls the vanishing of a coefficient of stability.* It shows that if ν_0 and β satisfy not only the condition for the Jacobian ellipsoid, namely, $\mathfrak{P}_2{}^1(\nu_0)\,\mathfrak{Q}_2{}^1(\nu_0) = \mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)$, but also this equation, we have arrived at a figure which belongs at the same time to two series, and there is a bifurcation at this point. The form of the figure is found by attributing to e any arbitrary but small value.

§ 6. *The Properties of the Successive Coefficients of Stability.*

Corresponding to each harmonic deformation of the ellipsoid, there is a coefficient of stability of one of the two forms

$$1-\frac{\mathfrak{P}_i{}^s(\nu_0)\,\mathfrak{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)}\quad\text{or}\quad 1-\frac{\mathsf{P}_i{}^s(\nu_0)\,\mathsf{Q}_i{}^s(\nu_0)}{\mathsf{P}_1{}^1(\nu_0)\,\mathsf{Q}_1{}^1(\nu_0)}.$$

These coefficients may be written $\mathfrak{K}_i{}^s$ or $\mathsf{K}_i{}^s$ according to an easily intelligible notation. The Jacobian ellipsoid is defined by ν_0, and the question arises as to the possibility of the vanishing of the several $\mathfrak{K}$'s as ν_0 gradually diminishes from infinity, that is to say, as the ellipsoid lengthens.

An harmonic of the first order merely denotes a shift of the centre of inertia along one of the three axes; one of the second order denotes a change of ellipticity of the ellipsoid. Since we must keep the centre of inertia at the origin, and since the ellipticity is determined by the consideration that the ellipsoid is a Jacobian, these harmonics need not be considered, and we may begin with those of the third order.

I shall not attempt to follow M. POINCARÉ in his masterly discussion of the properties of the coefficients of stability,† but will merely restate in my own notation the principal conclusions at which he has arrived.

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* 'Acta Math.,' vol. 7, 1885, p. 321. The factors $\frac{1}{3}$ and $1/2n + 1$ (or $1/2i + 1$, if i is the degree of the harmonic) which occur in his form of the condition are included in my functions.

† Sections 10 and 12 of his memoir. I have to thank him for saving me from making a serious mistake in this portion of my work.

1st. The equation

$$\mathbf{P}_1^{\,1}(\nu)\,\mathbf{Q}_1^{\,1}(\nu) - \mathfrak{P}_i^{\,s}(\nu)\,\mathfrak{Q}_i^{\,s}(\nu) \text{ or } \mathbf{P}_i^{\,s}(\nu)\,\mathbf{Q}_i^{\,s}(\nu) = 0, \ (i > 2)$$

is not satisfied by any value of ν between 1 and infinity, if $\mathfrak{P}_i^s$ or $\mathbf{P}_i^s$ is divisible by $\left(\nu^2 - \dfrac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$. It appears from the forms of the functions as given in § 4 of "Harmonics" that the $\mathbf{P}$ functions are so divisible. These functions appertain to the types EES, OOC, OES, EOC, and therefore the ellipsoid cannot bifurcate into deformations of these types.

2nd. The equation has no solution if $\mathfrak{P}_i^s$ is divisible by $(\nu^2 - 1)^{\frac{1}{2}}$. We again see from § 4 of "Harmonics" that $\mathfrak{P}_i^s$ is so divisible if it is of the types OOS, EOS. Hence the ellipsoid cannot bifurcate into these types. The only types remaining are EEC, OEC.

3rd. The equation has no solution if any of the roots of $\mathfrak{P}_i^s(\nu) = 0$ lie outside the limits $+1$ to -1. The only $\mathfrak{P}_i^s$ of the types EEC, OEC which has all its roots inside the limits $+1$ to -1 is the zonal harmonic for which $s = 0$.

Hence the ellipsoid can only bifurcate into a zonal harmonic.

4th. The equation

$$\mathbf{P}_1^{\,1}\,\mathbf{Q}_1^{\,1} - \mathfrak{P}_i\,\mathfrak{Q}_i = 0 \quad (i > 2)$$

must have a solution between 1 and infinity for all values of i.

It follows from these four propositions that the Jacobian ellipsoid is stable for all deformations except the zonal ones, and that as it lengthens it must at successive stages bifurcate into each and all the zonal deformations.

5th. As the ellipsoid lengthens, the first coefficient of stability to vanish is that of the third zonal harmonic. This stage is the end of the stability of the Jacobian ellipsoids, and there is almost certainly exchange of stability with the pear-shaped figure defined by this harmonic.

6th. It has not been rigorously proved that there is only one solution of the equation $\mathfrak{K}_i = 0$ even in the case where $i = 3$, but M. POINCARÉ believes that this is almost certainly the case.

7th. The functions

$$\left.\begin{matrix}\mathfrak{P}_i^{\,s}(\nu_0)\\ \text{or}\\ \mathbf{P}_i^{\,s}(\nu_0)\end{matrix}\right\} \times \left.\begin{matrix}\mathfrak{P}_i^{\,t}(\nu)\\ \text{or}\\ \mathbf{P}_i^{\,t}(\nu)\end{matrix}\right\} - \left.\begin{matrix}\mathfrak{P}_i^{\,s}(\nu)\\ \text{or}\\ \mathbf{P}_i^{\,s}(\nu)\end{matrix}\right\} \times \left.\begin{matrix}\mathfrak{P}_i^{\,t}(\nu_0)\\ \text{or}\\ \mathbf{P}_i^{\,t}(\nu_0)\end{matrix}\right\}$$

have always the same sign as ν increases from ν_0 to infinity, provided that s and t are both greater than zero, and i greater than 2.

The seventh of the preceding propositions renders it easy to determine the relative magnitudes of all the $\mathfrak{K}$'s belonging to a single degree i.

In what follows I may take the symbols $\mathfrak{P}$, $\mathfrak{Q}$ as including also $\mathbf{P}$, $\mathbf{Q}$.

Now $\mathfrak{K}_i^s > = < \mathfrak{K}_i^t$ as,

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{Q}_i^s(\nu_0) - \mathfrak{P}_i^t(\nu_0)\,\mathfrak{Q}_i^t(\nu_0) < = > 0\,.$$

If we express the $\mathfrak{Q}$'s in terms of integrals this becomes

$$\int_{\nu_0}^{\infty} \frac{[\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^t(\nu)]^2 - [\mathfrak{P}_i^s(\nu)\,\mathfrak{P}_i^t(\nu_0)]^2}{[\mathfrak{P}_i^s(\nu)\,\mathfrak{P}_i^t(\nu)]^2(\nu^2-1)^{\frac{1}{2}}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}}\,d\nu < = > 0\,.$$

The seventh proposition shows that when s and t are greater than zero, and i is greater than 2, all the elements of the integral have the same sign. Hence the question is whether.

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} < = > \frac{\mathfrak{P}_i^t(\nu_0)}{\mathfrak{P}_i^t(\nu)}\,.$$

Therefore we have to arrange all the $\dfrac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)}$ in descending order of magnitude, and shall thereby obtain the non-zonal $\mathfrak{K}$'s in ascending order.

I wish first to show that these coefficients may to a great extent be sorted by considering the inequality.

$$\frac{P_i^s(\nu_0)}{P_i^s(\nu)} < = > \frac{P_i^t(\nu_0)}{P_i^t(\nu)} \quad (s = 1, 2, 3\ \ldots,\ i\,;\ t = 1, 2, 3\ \ldots,\ i).$$

Suppose, if possible, that whereas, for the ellipsoids defined by β, ν, ν_0,

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} < \frac{\mathfrak{P}_i^t(\nu_0)}{\mathfrak{P}_i^t(\nu)}\,, \quad \text{yet} \quad \frac{P_i^s(\nu_0)}{P_i^s(\nu)} > \frac{P_i^t(\nu_0)}{P_i^t(\nu)}\,.$$

Then there must be some value of β for which

$$\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^t(\nu) = \mathfrak{P}_i^s(\nu)\,\mathfrak{P}_i^t(\nu_0)$$

for all values of ν greater than ν_0.

It is almost obvious that there is no one value of β which renders this equation possible; but consider for example the case of $s = 2$, $t = 0$.

Now

$$\mathfrak{P}_3^2(\nu) = -\beta q_0 P_3(\nu) + P_3^2(\nu), \qquad \mathfrak{P}_3(\nu) = P_3(\nu) + \beta q_2 P_3^2(\nu_0).$$

It we substitute this in the equation we find

$$P_3^2(\nu_0)\,P_3(\nu) = P_3^2(\nu)\,P_3(\nu_0).$$

This can only be satisfied by $\nu = \nu_0$, and hence the hypothesis is negatived. Similarly the assumption of other values of s and t leads to an impossibility.

Thus we may consider the P functions in place of the $\mathfrak{P}$ functions.

Consider the inequality

$$\frac{P_i^s(\nu_0)}{P_i^s(\nu)} > = < \frac{P_i^{s+1}(\nu_0)}{P_i^{s+1}(\nu)}, \text{ for } s = 1, 2\,.\,.\,.,\, i-1.$$

If the inequality is determined for any value of ν, it is determined for all values. Now when ν is very large

$$P_i^s(\nu) = \frac{2i\,!}{2^i\,i\,!\,\overline{i-s}\,!}\,\nu^i, \qquad P_i^{s+1}(\nu) = \frac{2i\,!}{2^i\,i\,!\,\overline{i-s-1}\,!}\,\nu^i.$$

Hence our inequality becomes

$$(i-s)\,P_i^s(\nu_0) > = < P_i^{s+1}(\nu_0).$$

This inequality is of the same kind for all values of ν_0. Now $P_i^s(\nu_0)$ involves the factor $(\nu_0^2-1)^{\frac{1}{2}s}$ and $P_i^{s+1}(\nu_0)$ involves $(\nu_0^2-1)^{\frac{1}{2}(s+1)}$. Putting therefore $\nu_0^2 = 1+\epsilon$, the left-hand side involves $\epsilon^{\frac{1}{2}s}$ and the right $\epsilon^{\frac{1}{2}(s+1)}$. It follows that unless s is equal to i the left-hand side is greater than the right; but s is necessarily equal to $i-1$ at greatest.

Therefore
$$\frac{P_i^s(\nu_0)}{P_i^s(\nu)} > \frac{P_i^{s+1}(\nu_0)}{P_i^{s+1}(\nu)}.$$

Hence K's with smaller s are less than those with greater s.

It remains to discriminate between the two sorts of P-functions which occur in ellipsoidal harmonic analysis; that is to say we must determine

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} > = < \frac{\mathsf{P}_i^s(\nu_0)}{\mathsf{P}_i^s(\nu)}.$$

Since the β of "Harmonics" is equal to $\dfrac{\kappa'^2}{2-\kappa'^2}$ in the present notation, when β and κ' are small we have by the formulæ of that paper

$$\mathfrak{P}_i^s(\nu) = P_i^s(\nu) + \tfrac{1}{2}\kappa'^2 q_{s+2} P_i^{s+2}(\nu) + \tfrac{1}{2}\kappa'^2 q_{s-2} P_i^{s-2}(\nu) + \,.\,.\,.\,,$$

$$\mathsf{P}_i^s(\nu) = \frac{(\nu^2-1/\kappa^2)^{\frac{1}{2}}}{(\nu^2-1)^{\frac{1}{2}}}\left[P_i^s(\nu) + \frac{s+2}{2s}\kappa'^2 q_{s+2} P_i^{s+2}(\nu) + \frac{s-2}{2s}\kappa'^2 q_{s-2} P_i^{s-2}(\nu) + \,.\,.\,.\right].$$

When ν is very great and κ' very small $\mathfrak{P}_i^s = \mathsf{P}_i^s$, so it suffices to determine the inequality

$$\mathfrak{P}_i^s(\nu_0) > = < \mathsf{P}_i^s(\nu_0);$$

and this may be considered for any value of ν_0 greater than unity. By taking ν_0 very large and κ' very small the inequality becomes

$$(\nu^2-1)^{\frac{1}{2}} > = < \left(\nu_0^2 - \frac{1}{\kappa^2}\right)^{\frac{1}{2}},$$

or
$$1 > = < \kappa.$$

But $\kappa < 1$, hence the first sign holds true and

$$\frac{\mathfrak{P}_i^s(\nu_0)}{\mathfrak{P}_i^s(\nu)} > \frac{\mathsf{P}_i^s(\nu_0)}{\mathsf{P}_i^s(\nu)},$$

whence $$\mathfrak{K}_i^s < \mathsf{K}_i^s.$$

Thus it follows that for order i

$$\mathfrak{K}_i^1 < \mathsf{K}_i^1 < \mathfrak{K}_i^2 < \mathsf{K}_i^2 \; . \; . \; . \; . \; < \mathfrak{K}_i^i < \mathsf{K}_i^i.$$

The order of magnitude of these coefficients is therefore completely determined.

As confirmatory of the correctness of this result it may be mentioned that I find that when $\gamma = 69° \, 50'$ and $\kappa = \sin 73° \, 56'$,

$$\mathfrak{K}_3^1 = \cdot 1765, \quad \mathsf{K}_3^1 = \cdot 2990, \quad \mathfrak{K}_3^2 = \cdot 4467, \quad \mathsf{K}_3^2 = \cdot 4550, \quad \mathfrak{K}_3^3 = \cdot 5719, \quad \mathsf{K}_3^3 = \cdot 5876.$$

When $\gamma = 75°$ and $\kappa = \sin 81° \, 4'$ (another Jacobian ellipsoid) the numbers run ·130, ·224, ·460, ·465, ·604, ·614.

We see that for the harmonics of higher order the ellipsoid is more stable than it was and for those of lower order less stable.

§ 7. *The critical Jacobian Ellipsoid.*

From a number of preliminary calculations I saw reason to believe that the critical ellipsoid would be found within the region comprised between $\gamma = 69° \, 48'$ and $69° \, 50'$, and $\sin^{-1} \kappa = 73° \, 52'$ and $73° \, 56'$.

If we write

$$f(\gamma, \sin^{-1} \kappa) = \frac{\mathrm{E}}{\kappa'^2} \left(1 + \frac{\kappa^4 \sin^2 \gamma \cos^2 \gamma}{1 - \kappa^2 \sin^3 \gamma}\right) - (2\mathrm{F} - \mathrm{E}) - \frac{\kappa^2 \sin \gamma \cos \gamma \, (1 + \kappa^2 \sin^2 \gamma)}{\kappa'^2 (1 - \kappa^2 \sin^2 \gamma)^{\frac{1}{2}}},$$

where the amplitudes of E and F are γ and their moduli κ, the existence of the Jacobian ellipsoid is determined by

$$f(\gamma, \sin^{-1} \kappa) = 0.^*$$

The coefficient of stability is

$$\mathfrak{K}_3 (\gamma, \sin^{-1} \kappa) = 1 - \frac{\mathfrak{P}_3 (\nu_0) \, \mathfrak{Q}_3 (\nu_0)}{\mathsf{P}_1^1 (\nu_0) \, \mathsf{Q}_1^1 (\nu_0)}.$$

The formulæ for computing $\mathfrak{K}_3$ are given in § 4.

The values of E and F are from LEGENDRE'S tables.

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[*] See 'Roy. Soc. Proc.,' vol. 41, p. 323, where the formula is reduced to a form convenient for computation.

Now I find

$$f(69^\circ\,48',\ 73^\circ\,52') = +\cdot 000191\,; \quad f(69^\circ\,50',\ 73^\circ\,52') = +\cdot 001319.$$

$$f(69^\circ\,48',\ 73^\circ\,56') = -\cdot 001186\,; \quad f(69^\circ\,50',\ 73^\circ\,56') = -\cdot 000031.$$

$$\mathfrak{K}_3(69^\circ\,48',\ 73^\circ\,52') = +\cdot 001058\,; \quad \mathfrak{K}_3(69^\circ\,50',\ 73^\circ\,52') = -\cdot 000885.$$

$$\mathfrak{K}_3(69^\circ\,48',\ 73^\circ\,56') = +\cdot 000655\,; \quad \mathfrak{K}_3(69^\circ\,50',\ 73^\circ\,56') = -\cdot 000765.$$

By interpolation we get the following results:—

The Jacobian ellipsoid is given by

$$(\gamma - 69^\circ\,48') - \cdot 59642\,(\sin^{-1}\kappa - 73^\circ\,52') + \cdot 33091 = 0.$$

The vanishing of the coefficient of stability is given by

$$(\gamma - 69^\circ\,48') + \cdot 041625\,(\sin^{-1}\kappa - 73^\circ\,52') - 1\cdot 0890 = 0.$$

In these equations the minute of arc is the unit.

Solving them I find

$$\gamma = 69^\circ\,48'\cdot 997 = 62^\circ\,49'\cdot 0,$$

$$\sin^{-1}\kappa = 73^\circ\,54'\cdot 225 = 73^\circ\,54'\cdot 2.$$

With these values I find that the three axes a, b, c, where $abc = \mathrm{a}^3$ are

$$\frac{a}{\mathrm{a}} = \cdot 650659,$$

$$\frac{b}{\mathrm{a}} = \cdot 814975,$$

$$\frac{c}{\mathrm{a}} = 1\cdot 885827.$$

The last place of decimals in these is certainly doubtful.

The formula for ω^2 is given in (35).

Now $\Psi = \frac{2}{k}\,\mathfrak{P}_0(\nu_0)\,\mathfrak{Q}_0(\nu_0)$, $\quad k = c\kappa\sin\gamma$, $\quad \mathfrak{P}_0(\nu_0)\,\mathfrak{Q}_0(\nu_0) = \kappa\mathrm{F}$.

Then since $a = c\cos\gamma$, $b = c\Delta$,

$$\frac{\omega^2}{2\pi\rho} = \frac{2\mathrm{F}\Delta\cot\gamma - \dfrac{6}{1 + \Delta^{-2} + \sec^2\gamma}}{1 + \Delta^2 - \dfrac{6}{1 + \Delta^{-2} + \sec^2\gamma}}.$$

In this formula, F, γ, Δ must correspond with values interpolated amongst those used in obtaining the solution.

From this I find

$$\frac{\omega^2}{2\pi\rho} = \cdot 1419990 = \cdot 14200.$$

In the paper on the Jacobian ellipsoid referred to above the moment of momentum is tabulated by means of μ, where the moment of momentum is $(\frac{4}{3}\pi\rho)^{\frac{1}{2}}a^5\mu$. The formula for μ is given in (25) of that paper, and, modified to suit the present notation, is

$$\mu = \frac{3^{\frac{1}{2}}}{5}(\Delta\cos\gamma)^{-\frac{2}{3}}(1+\Delta^2)\left(\frac{\omega^2}{4\pi\rho}\right)^{\frac{1}{2}}.$$

For the critical ellipsoid I find $\mu = \cdot 389570$.

The following table gives the numerical values for a number of Jacobian ellipsoids, beginning with the initial one and terminating just beyond instability. The last line gives the corresponding values for the critical ellipsoid.

JACOBI'S Ellipsoids.*

γ.	$\sin^{-1}\kappa$.	$\cos^{-1}\Delta$.	$a/$a.	$b/$a.	$c/$a.	$\omega^2/2\pi\rho$.	μ.
° ′ ″	° ′	° ′					
54 21 27 . .	0 0	0 0	·6977	1·1972	1·1972	·18712	30375
55	$17\frac{3}{4}$	$14\frac{1}{4}$	·697	1·179	1·216	·18706	·304
57	$34\frac{3}{4}$	$28\frac{1}{2}$	·696	1·123	1·279	·186	·306
60	49 7	40 54	·6916	1·0454	1·3831	·1812	·3134
65	64 19	54 46	·6765	·9235	1·6007	·1659	·3407
70	74 12	64 43	·6494	·8111	1·899	·1409	·3920
69 49 . .	73 54	64 24	65066	·81498	1·88583	·14200	·38957

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* I have been criticised with respect to my paper on JACOBI'S ellipsoid, from which these results are extracted, by M. S. KRÜGER (Nieuw Archief voor Wiskunde, Tweede Reeks, Derde Deel and 'Ellipsoidale Evenwichtsvormen,' &c., Thesis for Degree of Doctor, Leiden, J. W. van Leeuwen, Hoogewoerd 89, 1896), because I wrote it in ignorance of certain previous work, especially of a paper by PLANA ('Ast. Nachr.,' 36, n. 851, c. 169). But I cannot but congratulate myself on my ignorance, since it appears that PLANA gave a number of numerical results which were wholly wrong. A knowledge of that paper would no doubt have caused me much further trouble.

My paper gives a number of solutions of the problem which I believe to be correct. Unfortunately the methods of the paper are clumsy, and there are several mistakes. The formula for ω^2 used in this present paper, is much better than that given there.

The complicated formula on p. 325 is susceptible of reduction to a simple form, for on substituting for γ its approximate form (i) we have simply

$$\gamma - \delta = \tfrac{1}{4}\kappa^2 \sin\delta\cos\delta,$$

where

$$\delta = 54^\circ\, 21'\, 27''.$$

The final numerical result was, however, nearly right, for I now find

$$\sin^2\alpha = 10^{\cdot 9266821}\sin(\gamma-\delta),$$

whereas I had ·9266528. The sin α is the same as the κ used here.

The formula at the top of p. 326 which is reproduced as (22) on p. 328 is, I think, illusory, for if in the

In order to determine the question as to whether or not it is possible that $\mathfrak{K}_3 = 0$ should have another solution than that found in the next section, I have computed the value of this coefficient for the Jacobian ellipsoid $\gamma = 75^\circ$, $\kappa = \sin 81^\circ\ 4'{\cdot}4$, and find it to be $-6{\cdot}627$. From the manner in which the numbers in the computation present themselves, it is obvious that for more elongated ellipsoids $\mathfrak{K}_3$ will always remain negative, and will become numerically greater. I have therefore not thought it necessary to seek for an algebraic proof that there is no second root of the equation.

Very long Jacobian ellipsoids tend to become figures of revolution, and the coefficients of stability tend to assume the forms

$$1 - \frac{P_i(\nu)\,Q_i(\nu)}{P_1{}^1(\nu)\,Q_1{}^1(\nu)}\,.$$

The forms of these functions are well known, and I think that fair approximations to the incidences of the successive figures of bifurcation might be derived from the vanishing of this expression.

For example

$$P_1{}^1(\nu)\,Q_1{}^1(\nu) = \tfrac{1}{2}\left[\nu - (\nu^2 - 1)\log\left(\frac{\nu+1}{\nu-1}\right)^{\frac{1}{2}}\right]$$

$$P_4(\nu)\,Q_4(\nu) = \tfrac{1}{64}\left[(35\nu^4 - 30\nu^2 + 3)\log\left(\frac{\nu+1}{\nu-1}\right)^{\frac{1}{2}} - \tfrac{5}{3}\nu\,(21\nu^2 - 11)(35\nu^4 - 30\nu^2 + 3)\right].$$

I have not, however, attempted to solve the equation found by equating these two expressions to one another.

Even when $i = 3$ and $\gamma = 69^\circ\ 49'$ (the critical Jacobian) this rough approximation makes the coefficient of stability very small, but it is to be admitted that $P_1{}^1\,Q_1{}^1$ and P_3Q_3 differ very sensibly from $\mathbf{P}_1{}^1(\nu)\,\mathbf{Q}_1{}^1(\nu)$ and $\mathfrak{P}_3(\nu)\,\mathfrak{Q}_3(\nu)$, although in such a way that the errors compensate one another.

---

first term we put $\gamma = \delta + \frac{1}{4}\kappa^2 \sin\delta\cos\delta$ (as is clearly allowable in approximation) the term with coefficient κ^2 or $\sin^2\alpha$ disappears. This shows that it was necessary to proceed in the approximation as far as κ^4 in order to obtain a result.

The methods of approximation adopted on pp. 326–7 are of doubtful propriety, but will, I think, lead to nearly correct results. There is, however, a mistake towards the bottom of p. 327 which runs on to the end. M. KRÜGER correctly points out that the second line of formula (24) p. 329 should run

$$\tfrac{1}{2}\cos^2\alpha\left[\frac{1}{\sin\gamma}\log_e\cot\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\gamma\right).\left(\tfrac{15}{8} + 3\tan^2\gamma + \tan^4\gamma\right) - \tfrac{15}{8} - \tfrac{29}{8}\tan^2\gamma - \tfrac{7}{4}\tan^4\gamma\right].$$

Lastly, on p. 335, line 13, for $C = 0{\cdot}3573$, read $C = 0{\cdot}5379$; and on p. 336, line 7, for $1{\cdot}3573$, read $1{\cdot}5379$; and for $\frac{\mathfrak{a}}{\mathfrak{c}} = 1{\cdot}696$, read $\frac{\mathfrak{c}}{\mathfrak{a}} = 4{\cdot}65$.

§ 8. *The pear-shaped Figure of Equilibrium.*

By (21) the normal displacement δn for the third zonal harmonic deformation may be written

$$\delta n = e\frac{z\left[q'^2z^2 - q^2x^2 - (3-4q^2)y^2 - c^2q^2q'^2\sin^2\gamma\right]}{c^3q'^2(1-q^2\sin^2\gamma)\,(x^2/\cos^4\gamma + y^2/\Delta^4 + z^2)^{\frac{1}{2}}},$$

subject to the condition

$$\frac{x^2}{\cos^2\gamma} + \frac{y^2}{\Delta^2} + z^2 = c^2.$$

The expression has been arranged so that when $x = y = 0$, $z = c$, we have $\delta n = e$. Hence $+e$ and $-e$ are the normal displacements at the stalk and blunt end of the pear respectively.

In the section $y = 0$, this may be written

$$\delta n = \frac{e\cos\gamma}{q'^2}\cdot\frac{z(z^2 - c^2q^2)}{c^2(c^2 - z^2\sin^2\gamma)^{\frac{1}{2}}}.$$

The nodal points are given by $\frac{z}{c} = \pm q = \pm\ {\cdot}758056$.

In the section $x = 0$, since $\kappa^2 = q^2\frac{4-5q^2}{3-4q^2}$, it may be written

$$\delta n = e\frac{\Delta(4-5q^2)}{q'^2}\cdot\frac{z(\kappa^2z^2 - c^2q^2)}{c^3\kappa^2(c^2 - \kappa^2z^2\sin^2\gamma)^{\frac{1}{2}}}.$$

The nodal points are given by $\frac{z}{c} = \pm\frac{q}{\kappa} = \pm\ {\cdot}788986$.

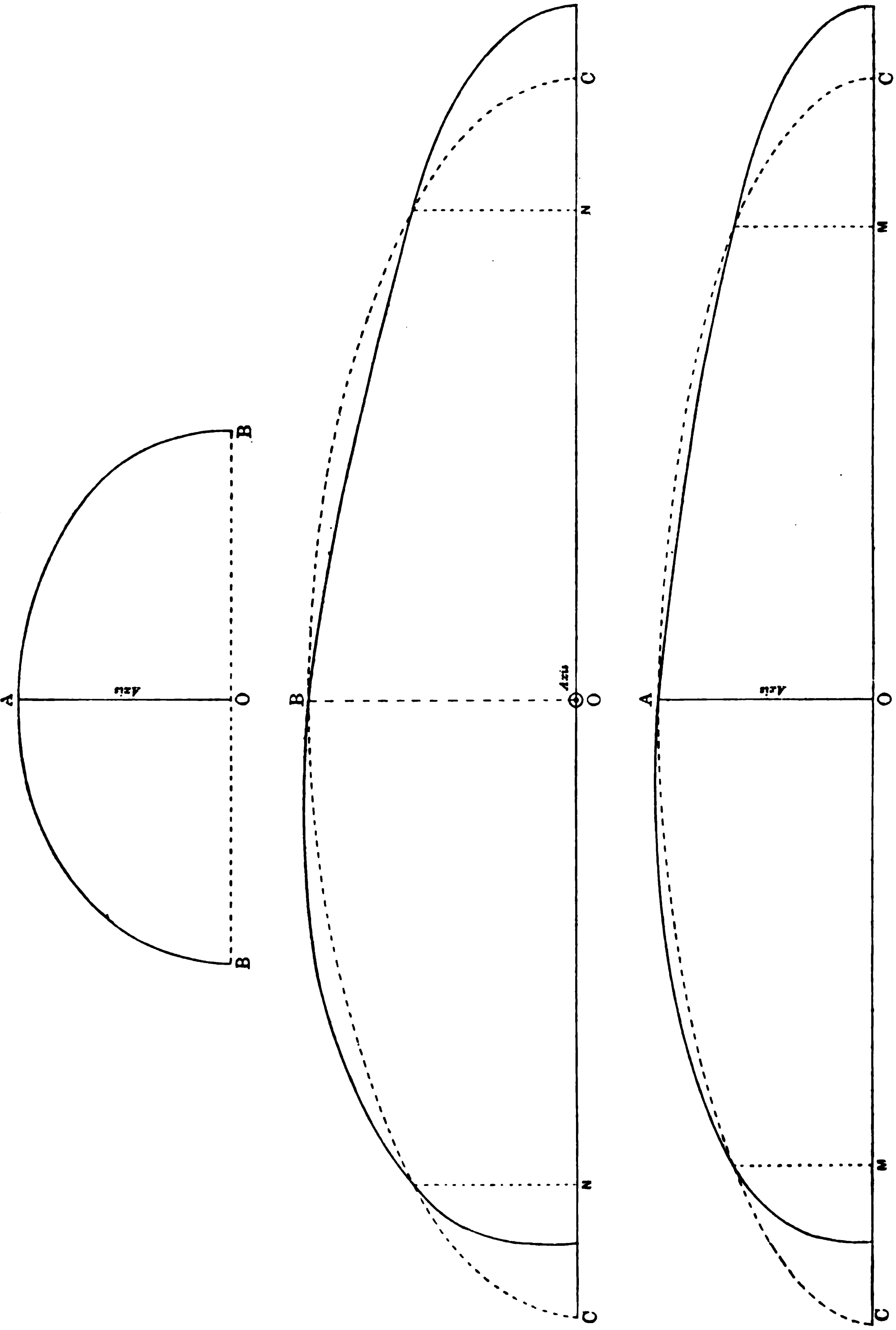
The section $z = 0$ is obviously another nodal line for all sections.

By means of these formulæ it is easy to compute the normal displacements from the surface of the critical Jacobian.

The figure opposite showing the three sections $x = 0$, $y = 0$, $z = 0$, is drawn from these formulæ, the dotted line being the critical Jacobian and the firm line the pear. The scale of the normal displacements is, of course, arbitrary.

Comparison with M. POINCARÉ's sketch shows that the figure is considerably longer than he supposed.

In this first approximation the positions of the nodal lines are independent of the magnitude of e, and they lie so near the ends that it is impossible to construct an exaggerated figure, for if we do so the blunt end acquires a dimple, which is absurd. It might have been hoped that such an exaggeration would afford us some idea of the mode of development of the pear.



PEAR-SHAPED FIGURE OF EQUILIBRIUM.

OA = ·65066, OB = ·81498, OC = 1·88583; $\frac{\omega^2}{2\pi\rho}$ = ·14200, $\frac{OM}{OC}$ = ·75806, $\frac{ON}{OC}$ = ·78899.

M. SCHWARZSCHILD has remarked[*] that it is not absolutely certain that the principle of exchange of stability holds with reference to this figure, and that we cannot feel absolutely certain that the pear is stable unless we can prove that the moment of momentum is greater than in the critical Jacobian.

With reference to this objection, M. POINCARÉ writes to me as follows :—

" Faisons croître le moment de rotation, que j'appellerai M. Deux hypothèses sont possibles.

" Ou bien pour $M < M_0$ (the moment of momentum of the Jacobian), nous aurons une seule figure, *stable*, à savoir l'ellipsoïde de JACOBI, et pour $M > M_0$ trois figures, une instable, l'ellipsoïde, et deux stables (d'ailleurs égales entre elles), les deux figures pyriformes.

" Ou bien pour $M < M_0$, nous aurons trois figures d'équilibre, deux pyriformes instables, une stable, l'ellipsoïde, et pour $M < M_0$ une seule figure instable, l'ellipsoïde—auquel cas la masse fluide devrait se dissoudre par un cataclysme subit.

" Il y a donc à vérifier si pour les figures pyriformes, $M >$ ou $< M_0$."

It seems very improbable that the latter can be the case; but this opinion is not a proof.

Since ω^2 is stationary for the initial pear, a small change in the angular velocity will certainly produce a great change in the figure of the pear. If this investigation has, in fact, its counterpart in the genesis of satellites and planets, it seems clear that the birth of a new body, although not cataclysmal, is rapid.

§ 9. *Summary.*

It is possible by the methods explained in my previous paper on " Harmonics " to form rigorous expressions for the ellipsoidal harmonics of the third degree. Accordingly in § 1 I proceed to form those functions. In § 2 the notation is changed with a view to convenience in subsequent work, and for the sake of completeness the harmonics of the first and second degrees are also given. In § 3 the corresponding solid harmonics are expressed in rectangular co-ordinates x, y, z. In § 4 I find the Q-functions, the harmonic functions of the second kind, and express the results in terms of the elliptic integrals E and F. It appears that both the P- and Q-functions of the third degree of harmonics occur in three pairs which have the same algebraic forms, and that in each pair one of them only differs from the other in the value of a certain parameter. There is, lastly, a seventh function which stands by itself; this last corresponds to the solid harmonic xyz.

In § 5 the equations for JACOBI's ellipsoid are determined by the consideration that the energy must be stationary, and the superficial value of gravity is found in terms of the appropriate P- and Q-functions. I then proceed to find the additional terms in the energy when the mass of fluid is subject to an ellipsoidal harmonic deformation. This section is a paraphrase of M. POINCARÉ'S work, but the notation and manner of presentation are somewhat different. The additional terms in the energy are shown to involve a certain coefficient, which is called by M. POINCARÉ a coefficient of stability. It is clear that when any coefficient vanishes we are at a point of bifurcation, and the particular Jacobian ellipsoid for which it vanishes is also a member of another series of figures of equilibrium.

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[*] "Die Poincarésche Theorie des Gleichgewichts," 'Annalen der K. Sternwarte, München,' Bd. III.

In § 6 the principal properties of these coefficients, as established by M. POINCARÉ, are enumerated. He has shown that the ellipsoid can bifurcate only into figures defined by zonal harmonics; that it must do so for all degrees, and that the first bifurcation occurs with the third zonal harmonic. The order of magnitude of the coefficients of the several orders and of the same degree is determined. A numerical result seems to indicate that as the ellipsoid lengthens, it becomes more stable as regards deformations of the third degree and of higher orders, and less stable as regards the lower orders of the same degree.

In § 7 the numerical solution of the vanishing of the coefficient corresponding to the third zonal harmonic is found, and it is shown that the critical ellipsoid has its three axes proportional to ·65066, ·81498, 1·88583, and that the square of the angular velocity is given by $\frac{\omega^2}{2\pi\rho} = \cdot 14200$. A short table is also given showing the march of the axes of the Jacobian ellipsoids from their beginning on to instability at this critical stage. The nature of the formula for the third zonal coefficient of stability seems to show that it can only vanish once—a point which it appears that M. POINCARÉ found himself unable to prove rigorously.

A suggestion is made for the approximate determination of the bifurcations into the successive zonal deformations, but no numerical results are given.

In § 8 the nature of the pear-shaped figure is determined numerically, and the reader may refer to the figure above, where it is delineated. It will be seen to be longer than was shown in M. POINCARÉ'S conjectural sketch.

If, as M. POINCARÉ suggests, the bifurcation into the pear-shaped body leads onward stably and continuously to a planet attended by a satellite, the bifurcation into the fourth zonal harmonic probably leads unstably to a planet with a satellite on each side, that into the fifth to a planet with two satellites on one side and one on the other, and so on.

The pear-shaped bodies are almost certainly stable, but a rigorous and conclusive proof is wanting until the angular velocity and moment of momentum corresponding to a given pear are determined. To do this further approximation is needed.

VII. *Sur la Stabilité de l'Équilibre des Figures Pyriformes affectées par une Masse Fluide en Rotation.*

By H. Poincaré, *Foreign Member R.S.*

Received October 29,—Read November 21, 1901.

Introduction.

J'ai publié autrefois dans le Tome 7 des 'Acta Mathematica' un mémoire sur l'équilibre d'une masse fluide en rotation. C'est à ce mémoire que je renverrai souvent dans la suite en écrivant simplement 'Acta.' Dans ce mémoire je décris en particulier une figure d'équilibre particulière qui est pyriforme, et que pour cette raison on peut appeler la poire (pear-shaped figure).

Cette figure, est-elle stable ? La question ne peut pas être regardée comme entièrement résolue. En effet, comme l'a fait remarquer M. Schwarzschild, le principe de l'échange des stabilités ne peut pas être appliqué à ce cas sans modification.

La condition de stabilité peut être présentée sous deux formes différentes. Soit U l'énergie potentielle de la masse fluide (ou plutôt ce que M. Darwin appelle l'énergie perdue), ω la vitesse de rotation, J le moment d'inertie. La quantité

$$U + \tfrac{1}{2}\omega^2 J$$

doit être minimum, ω *étant regardé comme donné.*

La condition est nécessaire et suffisante pour la stabilité séculaire, si on suppose que la masse est entraînée par frottement par un axe de rotation qui la traverse de part en part comme dans les expériences de Plateau. Elle est suffisante, mais elle n'est plus nécessaire, si on suppose que la masse est isolée dans l'espace (*cf.* 'Acta,' pp. 293, 295, 367, 369).

Voici la seconde forme. Soit $\mu = \omega J$, le moment de rotation de la masse fluide, la quantité

$$U - \frac{\mu^2}{2J}$$

devra être minimum, μ *étant regardé comme donné.*

La condition aussi énoncée est nécessaire et suffisante, si on suppose la masse isolée dans l'espace.

Cela posé, considérons la série des ellipsoïdes de Jacobi, et d'autre part la série des figures pyriformes. Nous aurons une figure de bifurcation qui appartiendra à la fois aux deux séries, et que nous appellerons le Jacobien critique.

Les figures pyriformes n'admettent pas le plan des xy pour plan de symétrie; on doit donc regarder comme distinctes deux de ces figures, symétriques l'une de l'autre par rapport à ce plan. De sorte que les figures de la série pyriforme sont symétriques deux à deux, à l'exception bien entendu du Jacobien critique, qui admet le plan des xy pour plan de symétrie. Il est clair que pour deux figures symétriques les valeurs de ω, de J, et de U sont les mêmes.

L'ensemble des deux séries peut être représenté schématiquement par une droite représentant les ellipsoïdes de JACOBI, et par une courbe ayant cette droite pour axe de symétrie, et représentant les figures pyriformes. Le point d'intersection de la droite et de la courbe représente alors le Jacobien critique, et deux points symétriques de la courbe représentent deux figures symétriques.

Cela posé, si nous suivons la série des Jacobiens en allant du moins allongé au plus allongé, nous savons que ω va en décroissant, tandis que $\omega J = \mu$ va en croissant.

Si nous suivons la série pyriforme, il est évident que quand nous atteindrons le Jacobien critique, ω atteindra, soit un minimum, soit un maximum, et il en est de même de ωJ.

Si nous adoptons le premier critère de la stabilité fondé sur les minima de la fonction $U + \frac{1}{2}\omega^2 J$, le principe de l'échange des stabilités entendu comme il doit l'être, nous enseigne ceci.

La condition nécessaire et suffisante pour la stabilité séculaire, si l'on supposait la masse entraînée par la rotation d'un axe fixe comme dans les expériences de PLATEAU, serait que ω soit *maximum*, c'est-à-dire plus grand pour le Jacobien critique que pour les autres figures pyriformes.

Si ω est maximum, on aurait pour une valeur donnée de ω supérieure au maximum une seule figure d'équilibre, un Jacobien stable; pour une valeur donnée de ω inférieure au maximum on en aura trois, un Jacobien instable et deux figures pyriformes stables.

Si au contraire ω est minimum, on aurait pour une valeur donnée de ω supérieure au minimum trois figures d'équilibre, deux pyriformes et instables, et une ellipsoïdale et stable; pour une valeur inférieure au minimum on n'aurait plus qu'une figure d'équilibre ellipsoïdale et instable.

Si maintenant nous supposons la masse isolée dans l'espace, la condition reste suffisante, mais elle n'est plus nécessaire. Pour avoir une condition nécessaire et suffisante, il faut avoir recours au second critère fondé sur les minima de $U - \frac{\mu^2}{2J}$. Le principe de l'échange des stabilités nous apprend alors que la condition nécessaire et suffisante de la stabilité séculaire, c'est que ωJ soit *minimum*, c'est-à-dire, plus petit pour le Jacobien critique que pour les autres figures pyriformes.

Si ωJ est minimum on aura pour une valeur donnée de ωJ

inférieure au minimum : 1 Jacobien *stable*,
supérieure au minimum : 1 Jacobien *instable*, 2 figures pyriformes, *stables* et symétriques l'une de l'autre.

Si ωJ est maximum on aura pour une valeur donnée de ωJ

inférieure au maximum : 1 Jacobien *stable* et 2 figures pyriformes, *instables* et symétriques l'une de l'autre,
supérieure au maximum : 1 Jacobien instable.

La question à résoudre est donc de savoir si ωJ est maximum ou minimum ; mais elle ne peut être résolue que par un calcul compliqué. Supposons qu'une masse fluide homogène en rotation se refroidisse lentement, elle prendra successivement (dans la première hypothèse ωJ minimum) la forme d'un ellipsoïde de révolution de plus en plus aplati, puis celle d'un ellipsoïde de Jacobi, puis celle d'une poire.

Si au contraire on venait à reconnaître que ωJ est maximum et non minimum, on devrait conclure que cette masse après avoir pris la forme de divers ellipsoïdes de révolution, puis de divers ellipsoïdes de JACOBI, et avoir atteint finalement celle du Jacobien critique, subira tout à coup une déformation énorme et une série d'oscillations, par une sorte de catastrophe subite.

Diverses raisons contribuent à rendre la première hypothèse beaucoup vraisemblable ; néanmoins jusqu'ici la preuve n'est pas faite, et je déclare tout de suite que je ne l'apporte pas encore dans le présent travail.

Mais quelle que soit l'hypothèse qui doive triompher un jour, je tiens à mettre tout de suite en garde contre les conséquences cosmogoniques qu'on pourrait en tirer. Les masses de la nature ne sont pas homogènes, et si on reconnaissait que les figures pyriformes sont instables, il pourrait néanmoins arriver qu'une masse hétérogène fût susceptible de prendre une forme d'équilibre analogue aux figures pyriformes, et qui serait stable. Le contraire pourrait d'ailleurs arriver également.

À la suite d'une correspondance que j'ai eue avec M. DARWIN, nous nous sommes mis l'un et l'autre à étudier la question, et pendant qu'il écrivait deux mémoires sur ce sujet, et que dans l'un de ces mémoires il déterminait les axes du Jacobien critique, j'obtenais des résultats qui sont l'objet du présent travail. J'ai formé l'inégalité qui doit être satisfaite pour qu'il y ait stabilité, mais je ne l'ai pas traduite en chiffres, parce que je me défie de mon habileté arithmétique, et que je ne suis pas un calculateur assez sûr.

Les notations dont je fais usage diffèrent, malheureusement, beaucoup de celles de M. DARWIN ; elles se rapprochent de celles de mon mémoire des 'Acta' sans être tout à fait identiques, parce que je rapporte ici, pour plus de symétrie, les coordonnées elliptiques à l'ellipsoïde :

$$\frac{x^2}{\rho^2 - a^2} + \frac{y^2}{\rho^2 - b^2} + \frac{z^2}{\rho^2 - c^2} = 1,$$

et non plus à l'ellipsoïde

$$\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - b^2} + \frac{z^2}{\rho^2 - c^2} = 1,$$

comme le faisaient LIOUVILLE et LAMÉ, et comme je l'ai fait moi-même dans le mémoire des 'Acta.' (Je suppose de plus $a^2 > b^2 > c^2$, au lieu de supposer $b^2 < c^2$.)

Les indices, que j'attribue aux fonctions de LAMÉ, ne sont pas non plus les mêmes que dans les 'Acta.' Les fonctions que j'appelle ici *

$$R_1,\ R_2,\ R_3,\ R_4,\ R_5,$$

s'appelaient dans les 'Acta':

$$R_0,\ R_1,\ R'_{2,0},\ R'_{2,2},\ R_{3,0}.$$

C'est la fonction $R'_{3,0} = R_5$, qui est la plus importante, parce que c'est elle qui sert à définir la figure pyriforme; on la désigne quelquefois sous le nom de "third zonal harmonic."

Les fonctions R sont toutes égales, soit à un polynôme entier en ρ^2, soit à un pareil polynôme multiplié par l'un des trois radicaux

$$\sqrt{(\rho^2 - a^2)},\quad \sqrt{(\rho^2 - b^2)},\quad \sqrt{(\rho^2 - c^2)}$$

soit à un pareil polynôme multiplié par deux de ces radicaux, soit à un pareil polynôme multiplié par ces trois radicaux.

Celles de ces fonctions qui sont égales à un polynôme en ρ^2 seront ce que nous appellerons plus loin des *fonctions de* LAMÉ *paires et uniformes.*

Calculs Préliminaires.

Soit

$$\frac{x^2}{\rho_0^2 - a^2} + \frac{y^2}{\rho_0^2 - b^2} + \frac{z^2}{\rho_0^2 - c^2} = 1$$

l'équation de l'ellipsoïde de JACOBI de bifurcation que j'appelle E_0; soient ρ, μ, ν les coordonnées elliptiques déduites de cet ellipsoïde; de sorte que $\rho = \rho_0$ est l'équation de E_0 en coordonnées elliptiques.

[* In the paper in the 'Acta' there is a slight inconsistency in the notation adopted, for in one part of the paper the first of the double suffixes to the R's denotes the degree of the harmonic, while in another part it is the second which has that meaning. Thus, for example, $R'_{2,0}$ is sometimes written $R'_{0,2}$. Further I do not find that R_0 is used explicitly in that paper.

It may be convenient to point out the identities of the R's used here with my notation, as used in "Ellipsoidal Harmonics" and "The pear-shaped Figure of Equilibrium." They are as follows:—

$$R_1 = \mathfrak{P}_0(\nu), \text{ a constant}; \ R_2 = \mathsf{P}_1^1(\nu); \ R_3 = \mathfrak{P}_2(\nu); \ R_4 = \mathfrak{P}_2^2(\nu); \ R_5 = \mathfrak{P}_3(\nu).$$

The identities of the S functions which occur below are:—

$S_1 = \mathfrak{Q}_0(\nu)$, $S_2 = 3\mathsf{Q}_1^1(\nu)$, $S_3 = 5\mathfrak{Q}_2(\nu)$, $S_4 = 5\mathfrak{Q}_2^2(\nu)$, $S_5 = 7\mathfrak{Q}_3(\nu)$.—G. H. DARWIN.]

On aura :

$$x^2 = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{(a^2 - b^2)(a^2 - c^2)},$$

de même pour y^2 et z^2.

On en déduit :

$$\frac{dx}{d\rho} = \rho \sqrt{\frac{(\mu^2 - a^2)(\nu^2 - a^2)}{(\rho^2 - a^2)(a^2 - b^2)(a^2 - c^2)}},$$

d'où :

$$A = \left(\frac{dx}{d\rho}\right)^2 + \left(\frac{dy}{d\rho}\right)^2 + \left(\frac{dz}{d\rho}\right)^2 = \frac{\rho^2(\mu^2 - \rho^2)(\nu^2 - \rho^2)}{(\rho^2 - a^2)(\rho^2 - b^2)(\rho - c^2)},$$

d'où pour le carré d'un élément d'arc quelconque :

$$ds^2 = A^2 d\rho^2 + B^2 d\mu^2 + C^2 d\nu^2,$$

où B et C sont formés avec μ et ν comme A avec ρ.

Il vient alors pour ΔV l'expression suivante :

$$ABC\Delta V = \frac{d}{d\rho}\left(\frac{BC}{A}\frac{dV}{d\rho}\right) + \frac{d}{d\mu}\left(\frac{AC}{B}\frac{dV}{d\mu}\right) + \frac{d}{d\nu}\left(\frac{AB}{C}\frac{dV}{d\nu}\right).$$

Nous désignerons les fonctions de LAMÉ par des indices.

R_1 se réduit à une constante ; R_2 à $\sqrt{(\rho^2 - a^2)}$; R_3 et R_4 sont les deux polynômes du première degré en ρ^2 ; R_5 est la troisième "zonal harmonic." *Il faut remarquer que les indices choisis ne sont pas les mêmes que dans le mémoire des 'Acta.'*

Les fonctions correspondantes S, M, N porteront les mêmes indices.

R_i^0 et S_i^0 seront les valeurs de R_i et S_i pour $\rho = \rho_0$.

Nous introduirons les variables elliptiques θ, θ_1, θ_2 par les équations

$$d\theta = \frac{\rho d\rho}{(\rho^2 - a^2)^{\frac{1}{2}}(\rho^2 - b^2)^{\frac{1}{2}}(\rho^2 - c^2)^{\frac{1}{2}}}$$

θ_1 et θ_2 étant formés avec μ et ν comme θ avec ρ.

Il vient alors :

$$ds^2 = \frac{1}{l^2} d\theta^2 + \frac{1}{l_1^2} d\theta_1^2 + \frac{1}{l_2^2} d\theta_2^2,$$

où :

$$l^2 = \frac{1}{(\mu^2 - \rho^2)(\nu^2 - \rho^2)}, \quad l_1^2 = \frac{1}{(\rho^2 - \mu^2)(\nu^2 - \mu^2)}, \quad l_2^2 = \frac{1}{(\rho^2 - \nu^2)(\mu^2 - \nu^2)}.$$

Formules relatives au Potentiel d'une Simple Couche.

Le potentiel à l'extérieur aura pour expression

$$V = \Sigma H R^0 S M N,$$

les H étant des coefficients quelconques, et à l'intérieur

$$V = \Sigma H R S^0 M N.$$

Si nous considérons maintenant les dérivées dV/dn estimées suivant la normale ; si nous convenons de représenter par des lettres accentuées les derivées prises par rapport à θ, il viendra, puisque $l = d\theta/dn$:—

A l'extérieur

$$\frac{dV}{dn} = \Sigma HR^0S'lMN$$

et à l'intérieur

$$\frac{dV}{dn} = \Sigma HS^0R'lMN.$$

La différence

$$\Sigma H(RS' - SR')lMN$$

représente $4\pi\delta$, δ étant la densité de la simple couche, et comme

$$SR' - RS' = 2n + 1$$

on aura

$$4\pi\delta = \Sigma H(2n + 1)lMN.$$

Si donc la densité a pour expression

$$\Sigma BlMN$$

le potentiel à la surface aura pour expression

$$\Sigma B\frac{4\pi}{2n + 1}R^0S^0MN.$$

Formules relatives au Potentiel d'une Double Couche.

Le potentiel à l'extérieur aura pour expression

$$V = \Sigma HSMN$$

et à l'intérieur

$$V = \Sigma H_1RMN$$

les coefficients étant différents. Comme la dérivée dV/dn devra être continue on aura :

$$\Sigma HS_0'MN = \Sigma H_1R'_0MN.$$

Posons $H = KR'_0$, $H_1 = KS'_0$; la différence entre les deux valeurs de V pour $\rho = \rho_0$ sera :

$$- \Sigma KMN(R_0'S_0 - S_0'R_0) = + \Sigma KMN(2n + 1).$$

Ce sera $- 4\pi\delta$, δ étant la densité de la double couche.

Energie de la Simple Couche.

Cette énergie est

$$\int \tfrac{1}{2} V S d\sigma$$

$d\sigma$ étant l'élément de surface.

Si

$$\delta = \Sigma BlMN, \quad V = \Sigma B \frac{4\pi}{2n+1} RSMN,$$

cela fera

$$2\pi \Sigma \frac{1}{2n+1} B^2 RS \int l M^2 N^2 d\sigma.$$

Energie de la Double Couche.

Je ne calculerai ici qu'une portion de cette énergie, à savoir l'intégrale

$$\tfrac{1}{2} \int d\tau \left[\left(\frac{dV}{dx} \right)^2 + \left(\frac{dV}{dy} \right)^2 + \left(\frac{dV}{dz} \right)^2 \right]$$

étendue à tous les éléments $d\tau$ de l'espace *sauf ceux qui sont compris entre les deux surfaces infiniment voisines qui constituent la double couche.* Cette portion est égale à $\int \frac{1}{2} \frac{dV}{dn} \delta d\sigma$.

Si

$$\delta = \Sigma BMN, \quad \frac{dV}{dn} = - \Sigma B \frac{4\pi}{2n+1} R'SMNl,$$

d'où

$$- 2\pi \Sigma \frac{1}{2n+1} B^2 R'S \int l M^2 N^2 d\sigma.$$

Définition de la Poire.

Par les différents points de E_0, je mène des lignes normales aux ellipsoïdes homofocaux à E_0, c'est-à-dire des lignes $\mu =$ const., $\nu =$ const., et je les prolonge jusqu'à la rencontre avec la surface de la poire. Soit $d\sigma_0$ un élément de surface de E_0 et dv le volume engendré par les lignes ainsi menées par les différents points de $d\sigma_0$, le rapport

$$dv/d\sigma_0$$

sera une fonction de μ et de ν que je pourrai développer en série de **Lamé** sous la forme :

$$dv/d\sigma_0 = \Sigma l \xi_i M_i N_i.$$

Ce sont les coefficients ξ_i qui définissent la forme de la poire.

Parmi ces coefficients, je remarque :—

1. ξ_1, qui est nul, parce que le volume de la poire est égal à celui de E_0, ce qui s'écrit $\int dv = 0$.
2. ξ_5, qui est du premier ordre.
3. ξ_3 et ξ_4, qui sont du second ordre.
4. Les autres ξ_i, qui sont du second ordre, si la fonction M_i est paire, uniforme, et d'ordre supérieur ; et négligeables dans le cas contraire.

Assez fréquemment, et quand il n'en pourra résulter aucune confusion, je supprimerai l'indice 0 et *j'écrirai simplement* $d\sigma$ *et* $dv/d\sigma$ *au lieu de* $d\sigma_0$ et $dv/d\sigma_0$.

Définition de la Simple Couche C.

Je considère la couche attirante formée par tous les petits volumes dv, et située par conséquent entre la surface de la poire et celle de E_0. Je suppose que l'on concentre toute la masse attirante située dans cette couche sur la surface de E_0 ; nous aurons ainsi une simple couche attirante, la densité de la matière sur l'élément $d\sigma$ étant précisément $dv/d\sigma$.

L'attraction de cette simple couche, que j'appelle Σ, est à très peu près égale à celle de la couche C.

Je puis considérer l'attraction due à la couche C moins l'attraction due à la couche Σ ; elle peut être considérée comme due à une matière attirante en partie positive et en partie négative ; c'est ce que j'appellerai la matière $\mathfrak{M}$, comprenant la couche C prise positivement et la couche Σ changée de signe.

Calcul du Potentiel.

Le potentiel V pourra être décomposée en trois parties :

$$V = V_1 + V_2 + V_3.$$

V_1 potentiel de E_0 ; V_2 de Σ ; V_3 de $\mathfrak{M}$.

Voici quelle est l'expression analytique de V_1 et de V_2.

Pour V_1 :—

à l'extérieur de E_0

$$V_1 = A_1 S_1 M_1 N_1 + A_3 S_3 M_3 N_3 + A_4 S_4 M_4 N_4,$$

à l'intérieur de E_0

$$V_1 = A'_1 R_1 M_1 N_1 + A'_3 R_3 M_3 N_3 + A'_4 R_4 M_4 N_4 + A'_0 \Pi,$$

ette énergie, où $d\tau$ représente l'élément de volume de $\mathfrak{M}$, de sorte que

$$d\tau = d\theta\, d\theta_1\, d\theta_2 \sqrt{-1}\,.\,(\mu^2 - \rho^2)(\rho^2 - \nu^2)(\nu^2 - \mu^2).$$

Soit $d\sigma_0$ un élément de la surface de E_0, $d\sigma$ l'élément correspondant de la surface d'un ellipsoïde E homofocal à E_0 (je considère deux éléments comme correspondants quand ils ont mêmes coordonnées elliptiques μ et ν).

Soit $d\lambda$ un élément de la courbe $\mu = \text{const.}$, $\nu = \text{const.}$, compris entre deux ellipsoïdes E et E' infiniment voisins et homofocaux à E_0 ; nous aurons

$$d\tau = d\lambda d\sigma = \frac{d\theta d\sigma}{l}.$$

D'autre part $l d\sigma = l_0\, d\sigma_0$, où

$$l = \frac{1}{(\rho^2 - \mu^2)^{\frac12} (\rho^2 - \nu^2)^{\frac12}}, \quad l_0 = \frac{1}{(\rho_0{}^2 - \mu^2)^{\frac12} (\rho_0{}^2 - \nu^2)^{\frac12}}.$$

Si nous posons $\qquad [V_1 + \tfrac12 \omega_0{}^2 (y^2 + z^2)] = P,$

$$d\theta = du \frac{(\rho_0{}^2 - \mu^2)^{\frac12} (\rho_0{}^2 - \nu^2)^{\frac12}}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)} = \frac{l^2 du}{l_0},$$

et que nous considérions un instant P comme fonction de u ; nous remarquerons d'abord que $d\tau = du\, d\sigma_0$.

Développons P par la formule de MacLaurin :

$$P = P_0 + u \frac{dP}{du} + \tfrac12 u^2 \frac{d^2P}{du^2} + \tfrac16 u^3 \frac{d^3P}{du^3}.$$

Il va sans dire que la variable auxiliaire u a été définie de façon à s'annuler pour $\rho = \rho_0$, et que dans P et ses dérivées on a fait $\rho = \rho_0$. Notre variable u variera donc de 0 à $dv/d\sigma$ quand on passera de la surface de E_0 à celle de la poire.

Alors nous avons pour la portion de l'énergie envisagée :

$$\int P du d\sigma_0 = \int P_0 \frac{dv}{d\sigma_0} d\sigma_0 + \tfrac12 \int \frac{dP}{du} \left(\frac{dv}{d\sigma_0}\right)^2 d\sigma_0 + \tfrac16 \int \frac{d^2P}{du^2} \left(\frac{dv}{d\sigma_0}\right)^3 d\sigma_0 + \tfrac{1}{24} \int \frac{d^3P}{du^3} \left(\frac{dv}{d\sigma_0}\right)^4 d\sigma_0.$$

Le premier terme est nul (puisque $\int dv = 0$), le second nous donnerait le terme du second ordre

$$-2\pi \Sigma \tfrac13 \xi_i{}^2 R_2{}^0 S_2{}^0 \Omega_i,$$

dont j'ai déjà donné l'expression d'après le mémoire des 'Acta.'

Le troisième va nous donner des termes en $\xi_3^2\, \xi_i$, et le quatrième un terme en ξ_3^4.

Je commence par rechercher les dérivées de P par rapport à u, en fonction des dérivées $d\rho/d\theta$, $d^2\rho/d\theta^2$.

On a

$$\left.\begin{aligned}
&\frac{dP}{du}=\frac{dP}{d\theta}\frac{d\theta}{du}=\frac{dP}{d\theta}l,\ \text{car}\ \frac{d\theta}{du}=l,\ \text{pour}\ \rho=\rho_0\,;\\
&\frac{d^2P}{du^2}=\frac{dP}{d\theta}\frac{d^2\theta}{du^2}+\frac{d^2P}{d\theta^2}\left(\frac{d\theta}{du}\right)^2\\
&\frac{d^3P}{du^2}=\frac{dP}{d\theta}\frac{d^3\theta}{du^3}+3\frac{d^2P}{d\theta^2}\frac{d\theta}{du}\frac{d^2\theta}{du^2}+\frac{d^3P}{d\theta^3}\left(\frac{d\theta}{du}\right)^3
\end{aligned}\right\}\quad(1.)$$

Tout ce que je veux retenir pour le moment c'est que $d\theta/du$, $d^2\theta/du^2$, $d^3\theta/du^3$ sont des fonctions de μ^2 et de ν^2 symétriques, paires et *uniformes* (je veux dire par ce dernier mot qu'elles ne changent pas quand μ^2 ou ν^2 tournent autour des valeurs singulières a^2, b^2, ou c^2). Leur développement en séries de Lamé ne contiendra donc que des fonctions de Lamé paires et uniformes. Il en sera de même pour $d\rho/du$, $d^2\rho/du^2$, $d^3\rho/du^3$ qui entrent dans les formules :

$$\frac{dP}{du}=\frac{dP}{d\rho}\frac{d\rho}{du}\,;\quad\frac{d^2P}{du^2}=\frac{dP}{d\rho}\frac{d^2\rho}{du^2}+\frac{d^2P}{d\rho^2}\left(\frac{d\rho}{du}\right)^2,\ \&\text{c.}\ .\ .\ .\ .\ (1\ \text{bis}).$$

Nous supposons bien entendu dans les formules (1) et (1 *bis*) qu'on remplace partout ρ par ρ_0 à la fin du calcul.

Une autre difficulté provient de ce que P n'a pas la même expression analytique à l'intérieur ou à l'extérieur de E_0.

Si la couche était tout entière à l'intérieur de E_0 (ce qui ne pourrait avoir lieu que si on renonçait à l'hypothèse $\int dv=0$) nous pourrions réduire P à Π, à un facteur constant près, nous aurions en effet :

$$P=(A_1'+\tfrac{1}{2}\omega_0^2B_1)\,R_1M_1N_1+(A_0'+\tfrac{1}{2}\omega_0^2B_0)\,\Pi$$

le premier terme se réduit à une constante qui n'a pas de dérivées.

Il est aisé de voir que $d\Pi/d\rho$ se réduit à un terme en ρ ; de sorte que

$$\frac{d\Pi}{d\rho}=\rho\frac{d^2\Pi}{d\rho^2}\,;\qquad\frac{d^3\Pi}{d\rho^3}=0.$$

Nos intégrales se réduiraient donc, à un facteur constant près, à

$$\tfrac{1}{6}\int\frac{d^2\Pi}{d\rho^2}\left(\frac{dv}{d\sigma}\right)^3d\sigma\left[\rho_0\frac{d^2\rho}{du^2}+\left(\frac{d\rho}{du}\right)^2\right]+\tfrac{1}{24}\int\frac{d^2\Pi}{d\rho^2}\left(\frac{dv}{d\sigma}\right)^4d\sigma\left[\rho_0\frac{d^3\rho}{du^3}+3\frac{d\rho}{du}\frac{d^2\rho}{du^2}\right].$$

J'écris $d\sigma$ au lieu de $d\sigma_0$, en supprimant l'indice 0, devenu inutile.

Remarquons que les quantités entre crochets sont les dérivées première et seconde de $\rho\ d\rho/du$

Remarquons en outre que

$$\frac{dP}{d\rho}\frac{d\rho}{du}=-\frac{4\pi}{3l}R_2^0S_2^0,$$

que $d\rho/du$ est proportionnel à l, et par conséquent $dP/d\rho$ et $d^2\Pi/d\rho^2$ à $1/l^2$.

Les termes qui nous intéressent sont :—

1. Dans $(dv/d\sigma)^3$ les termes $3\xi_5^2\Sigma\xi_i\, M_5^2N_5^2\, M_iN_i\, l^3$ qui donnent

$$\Sigma\, \tfrac{1}{2}\xi_5^2\xi_i \int \frac{d^2P}{d\rho^2}\frac{d}{du}\left(\rho\frac{d\rho}{du}\right) M_5^2N_5^2M_iN_il^3d\sigma.$$

Il suffira de prendre les fonctions M_iN_i qui sont paires et uniformes, puisque les développements de $M_5^2N_5^2$, $l^3d^2P/d\rho^2$ n'en contiennent pas d'autres. Si on se borne aux formules précédentes, où P est regardé comme proportionnel à Π, nous pourrons poser

$$\frac{d^2P}{d\rho^2} = \Pi_0\frac{1}{l^2},$$

M_0 étant une constante ; si nous développons alors :

$$\Pi_0\frac{d}{du}\left(\rho\frac{d\rho}{du}\right) M_5^2N_5^2 = \Sigma\eta_iM_iN,$$

en série de LAMÉ, on aura, pour les termes en question :

$$\Sigma\, \tfrac{1}{2}\xi_5^2\xi_i\eta_i\Omega_i.$$

Malheureusement toute la masse $\mathfrak{M}$ n'est pas à l'intérieur de E_0, c'est donc

$$l^2\frac{d^2P}{du^2}M_5^2N_5^2$$

qu'il faudrait développer ; et cette expression n'est égale à $M_0\frac{d}{du}\left(\rho\frac{d\rho}{du}\right)M_5^2N_5^2$ que pour les points intérieurs à E_0.

2. Dans $(dv/d\sigma)^4$ le terme $\xi_5^4l^4M_5^4N_5^4$ ce qui donne :

$$\tfrac{1}{24}\xi_5^4\int\frac{d^2P}{d\rho^2}l^4\frac{d^2}{du^2}\left(\rho\frac{d\rho}{du}\right)M_5^4N_5^4d\sigma.$$

Nous reviendrons sur tous ces points plus en détail.

6. *L'énergie de* Σ *par rapport à* $\mathfrak{M}$.

C'est

$$\int V_2d\tau$$

étendu à $\mathfrak{M}$, c'est-à-dire étendu à C en supprimant ensuite les termes du second degré ; cela fait, puisque

$$\int V_2d\tau = \int V_2\,du\,d\sigma = \int V_2\frac{dv}{d\sigma}d\sigma + \tfrac{1}{2}\int\frac{dV_2}{du}\left(\frac{dv}{d\sigma}\right)^2d\sigma + \tfrac{1}{6}\int\frac{d^2V_2}{du^2}\left(\frac{dv}{d\sigma}\right)^3d\sigma,$$

cela fait, dis-je, puisqu'on doit supprimer les termes du second degré :

$$\tfrac{1}{2}\int\frac{dV_2}{du}\left(\frac{dv}{d\sigma}\right)^2d\sigma + \tfrac{1}{6}\int\frac{d^2V_2}{du^2}\left(\frac{dv}{d\sigma}\right)^3d\sigma.$$

Les dérivées dV_2/du, etc., se calculeraient comme dP/du, etc., et on y ferait à la fin du calcul $\rho = \rho_0$.

Les termes qui nous intéressent sont :

dans $(dv/d\sigma)^2$ $\quad 2l^2\Sigma\xi_i\xi_5 M_5 N_5 M_i N_i$ et $l^2\xi_5{}^2 M_5{}^2 N_5{}^2$;

dans $(dv/d\sigma)^3$ $\quad l^3\xi_5{}^3 M_5{}^3 N_5{}^3$.

Quant à V_2 nous devons distinguer le cas où $\mathfrak{M}$ est intérieur à E_0, et celui où $\mathfrak{M}$ est extérieur à E_0.

Dans le premier cas les termes intéressants sont :

$$\int l^2 \frac{4\pi}{7} \Sigma\xi_i\xi_5{}^2 M_5{}^2 N_5{}^2 M_i N_i R'_5 S_5 \frac{d\theta}{du}\, d\sigma$$

$$+ \Sigma \int l^2 \frac{2\pi}{2n+1}\, \xi_i\xi_5{}^2 M_5{}^2 N_5{}^2 M_i N_i R'_i S_i \frac{d\theta}{du}\, d\sigma$$

$$+ \tfrac{1}{6} \int l^3 \xi_5{}^4 M_5{}^4 N_5{}^4 \frac{4\pi}{7} \left[R_5{}' S_5 \frac{d^2\theta}{du^2} + R_5{}'' S_5 \left(\frac{d\theta}{du}\right)^2 \right] d\sigma.$$

Dans le second cas ils deviennent :

$$\int l^2 \frac{4\pi}{7} \Sigma\xi_i\xi_5{}^2 M_5{}^2 N_5{}^2 M_i N_i R_5 S'_5 \frac{d\theta}{du}\, d\sigma$$

$$+ \Sigma \int l^2 \frac{2\pi}{n+1}\, \xi_i\xi_5{}^2 M_5{}^2 N_5{}^2 M_i N_i R_i S'_i \frac{d\theta}{du}\, d\sigma$$

$$+ \tfrac{1}{6} \int l^3 \xi_5{}^4 M_5{}^4 N_5{}^4 \frac{4\pi}{7} \left[R_5 S'_5 \frac{d^2\theta}{du^2} + R_5 S''_5 \left(\frac{d\theta}{du}\right)^2 \right] d\sigma.$$

Si $\mathfrak{M}$ est en partie extérieure et en partie intérieure à E_0 il faudra employer une formule mixte.

7. L'énergie de $\mathfrak{M}$ par rapport à $\mathfrak{M}$.

Pour l'obtenir il faut calculer le potentiel de $\mathfrak{M}$ et d'abord revenir sur l'étude du potentiel d'une double couche.*

Considérons une double couche très mince, mais non infiniment mince. Elle se compose de deux surfaces attirantes, Σ et Σ', très peu différentes l'une de l'autre. Je considère une série de courbes que j'appelle C, de façon que par chaque point de l'espace passe une courbe C et une seule. Ordinairement on prend pour les courbes C les normales à Σ. Dans les applications qui vont suivre, je prendrai les courbes $\mu =$ const., $\nu =$ const.

Deux points de Σ et de Σ', se trouvant sur une même courbe C, sont dits *corre-*

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* Je prie le lecteur de bien remarquer que pendant quelques pages, et jusqu'à nouvel avertissement, beaucoup de lettres n'ont plus la même signification que dans ce qui précède et dans ce qui suit.

spondants. L'hypothèse qui sert de définition à la double couche, c'est que les masses attirantes qui se trouvent sur un élément de Σ et sur l'élément correspondant de Σ' sont égales et de signe contraire.

Cela posé soit M un point de Σ, M' le point correspondant de Σ', soient V et V' le potentiel de la double couche en M et en M'. Il s'agit d'évaluer la différence $V - V'$.

1. Dans le cas où la double couche est infiniment mince, on a par un théorème bien connu :

$$V - V' = 4\pi\delta \, . \, MM' \, . \cos\gamma$$

δ étant la densité de la matière au point M, MM' la distance des deux points correspondants M et M', γ l'angle de la courbe C avec la normale à Σ.

Je rappelle d'ailleurs que si la courbe C est normale aux deux surfaces, c'est-à-dire si $\gamma = 0$, la dérivée normale dV/dn est continue, même quand on franchit la double couche ; et que par conséquent cette dérivée a même valeur à des infiniment petits près en deçà des deux surfaces, et au delà des deux surfaces.

2. Supposons maintenant que la double couche soit très mince, mais non infiniment mince. Nous la décomposerons en une infinité de doubles couches infiniment minces. Pour cela entre Σ et Σ' nous ferons passer une infinité de surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ (n très grand). Soit E un élément de Σ, et $E_1, E_2, \ldots, E_n, E'$ les éléments correspondants de $\Sigma_1, \Sigma_2, \ldots, \Sigma_n, \Sigma'$. Nous avons sur E une masse μ et sur E' une masse $-\mu$. Plaçons sur E_1 une masse $-\mu$ et une masse μ qui se détruiront ; faisons de même pour $E_2, E_3, \ldots, E_n$. Associons la masse $-\mu$ de E_1 avec la masse μ de E, faisons de même pour tous les autres éléments de E, nous obtiendrons une double couche formée par les deux surfaces Σ et Σ_1 ; je l'appelle K_1. De même en combinant la masse $-\mu$ de E_2 avec la masse μ de E_1, j'aurai une seconde double couche que j'appelle K_2, et ainsi de suite jusqu'à la double couche K_n due aux masses $-\mu$ de E_n et μ de E_{n-1}, et à la double couche K_{n+1} due aux masses $-\mu$ de E' et μ de E_n.

La double couche proposée est donc remplacée par $n + 1$ *doubles couches élémentaires*.

Soit alors v et v' les potentiels aux points M et M' d'une double couche élémentaire K ; soient P et P' les points où la courbe C qui joint M à M' perce les deux surfaces de cette double couche élémentaire K. Soient w et w' les potentiels de K aux points P et P'. Soit dl un élément de la ligne C, et dv/dl la dérivée du potentiel de K le long de cette ligne, nous aurons :

$$w - w' = 4\pi\delta_1 \, . \, PP' \cos\gamma_1,$$

δ_1 étant la densité au point P et γ_1 l'angle de C avec la normale

$$v - w = -\int_M^P \frac{dv}{dl} dl, \qquad w' - v' = -\int_{P'}^{M'} \frac{dv}{dl} dl.$$

Je puis donc écrire :

$$v - v' = 4\pi\delta_1 PP' \cos\gamma_1 - \int_M^{M'} \frac{dv}{dl}\, dl$$

parce que l'arc PP' est très petit par rapport à MM', à la condition d'attribuer sur cet arc à dv/dl la même valeur qu'au point P en dehors de la double couche. On peut observer que si $d\sigma$ est un élément de la surface Σ, $d\sigma_1$ l'élément correspondant de la surface à laquelle appartient P, on aura :

$$\delta d\sigma = \delta_1 d\sigma_1.$$

Nous pouvons donc écrire

$$V = \Sigma v, \qquad V' = \Sigma v'$$

d'où :

$$V - V' = 4\pi\delta \Sigma PP' \cos\gamma_1 \frac{d\sigma}{d\sigma_1} - \int_M^{M'} \Sigma \frac{dv}{dl}\, dl.$$

Le premier terme est de l'ordre de $\Sigma PP' = MM'$. Le second est de l'ordre de $(MM')^2$; car dv/dl est de l'ordre de PP' et $\Sigma \frac{dv}{dl}$ de l'ordre de $\Sigma PP' = MM'$. Nous pousserons l'approximation jusqu'à $(MM')^2$. Si comme nous le supposons la courbe C est normale à Σ, l'angle γ_1 sera de l'ordre de MM' ; nous pourrons donc remplacer $\cos\gamma_1$ par 1, l'erreur commise sur $V - V'$ sera de l'ordre de $(MM')^3$. Maintenant $\Sigma \frac{dv}{dl}$ sera sensiblement constant et égal à dV/dn, c'est-à-dire à la dérivée de V estimée suivant la normale au point M et du côté extérieur à la double couche comprise entre les deux surfaces Σ et Σ'.

Appliquons cela à l'évaluation du potentiel de $\mathfrak{M}$, et pour cela revenons encore à notre double couche $\Sigma\Sigma'$; soit M'' un point de C compris entre M et M' ; soit v'' le potentiel de la double couche élémentaire K au point M'', $V'' = \Sigma v''$ le potentiel de la double couche totale.

Supposons d'abord que M'' soit entre P' et M', nous aurons :

$$v - w = -\int_M^P \frac{dv}{dl}\, dl; \qquad w' - v'' = -\int_{P'}^{M''} \frac{dv}{dl}\, dl$$

d'où :

$$v - v'' = 4\pi\delta_1 PP' \cos\gamma_1 - \int_M^{M''} \frac{dv}{dl}\, dl.$$

Si M'' est entre M et P, nous aurons simplement :

$$v - v'' = -\int_M^{M''} \frac{dv}{dl}\, dl.$$

Nous aurons donc encore

$$V - V'' = 4\pi\delta \Sigma PP' \cos\gamma_1 \frac{d\sigma}{d\sigma_1} - \int_M^{M''} \Sigma \frac{dv}{dl}\, dl,$$

mais avec cette condition que dans le premier terme du second membre la sommation ne doit être étendue qu'aux doubles couches élémentaires comprises entre $\boldsymbol{M}$ et $\boldsymbol{M''}$. On aura comme plus haut :

$$\Sigma\frac{dv}{dl} = \frac{dV}{dn}, \qquad \cos\gamma_1 = 1.$$

D'autre part :

$$\frac{d\sigma}{d\sigma_1} = 1 + \frac{MP}{MM'}\left[\frac{d\sigma}{d\sigma'} - 1\right]$$

d'où nous tirerons :

$$V - V'' = 4\pi\delta\,.\,MM'' + 2\pi\delta\frac{MM''^2}{MM'}\left[\frac{d\sigma}{d\sigma'} - 1\right] - \frac{dV}{dn}MM''.$$

Nous poserons d'ailleurs

$$\frac{d\sigma}{d\sigma'} - 1 = kMM'$$

de façon que k soit fini, et que

$$V - V'' = 4\pi\delta\,.\,MM'' + 2\pi k\delta(MM'')^2 - \frac{dV}{dn}MM''.$$

Si le point M'' est au delà de M', on aura

$$V - V'' = 4\pi\delta MM' + 2\pi k\delta(MM')^2 - \frac{dV}{dn}MM''$$

et s'il est en deçà de M :

$$V - V'' = -\frac{dV}{dn}MM''.$$

Cela posé, partageons la couche C, qui est comprise entre la surface de E_0 et celle de la poire, en couches infiniment minces par une série de surfaces très rapprochées, que j'appelle A_0, A_1, A_2, . . . , A_n ; A_0 coincidera avec E_0 et A_n avec la surface de la poire. J'appelle C_p la couche comprise entre A_{p-1} et A_p. Je suppose que l'on concentre la masse de C_p sur E'_0 en suivant les lignes μ, $\nu =$ const., qui jouent ici le rôle que jouaient tout à l'heure les courbes C. J'appelle Σ_p la simple couche ainsi obtenue. Alors Σ est la somme de toutes les simples couches Σ_p. L'attraction de C_p, moins celle de Σ_p, est l'attraction d'une double couche D_p, et il est clair que $\mathfrak{M}$ est équivalent à l'ensemble de ces doubles couches.

Soit v le potentiel dûe à l'une des doubles couches D_p, soit δ_p la densité de Σ_p en un point M de E_0 ; soit P un point de A_p et M'' un point quelconque de $\mathfrak{M}$, M' un point de la surface de la poire. Les quatre points M, P, M' et M'' sont supposé situés sur une même courbe μ, $\nu =$ const. Si alors v et v'' sont les valeurs de v en M et en M'', nous aurons :

$$v - v'' = 4\pi\delta_p MM'' + 2\pi k\delta_p MM''^2 - \frac{dv}{dn}MM''$$

si M'' est entre M et P ; et

$$v - v'' = 4\pi\delta_p MP + 2\pi v\delta_p(MP)^2 - \frac{dv}{dn}MM''$$

si P est entre M et M''.

Soit $V = \Sigma v$ et $V'' = \Sigma v''$ les valeurs du potentiel de $\mathfrak{M}$ en M et en M'', nous aurons :

$$V - V'' = 4\pi[\Sigma'\delta_p MM'' + \Sigma''\delta_p MP] + 2\pi k[\Sigma'\delta_p(MM'')^2 + \Sigma''\delta_p(MP)^2] - \frac{dV}{dn}MM''.$$

Nous remarquerons dans les parenthèses du second membre deux signes de sommation différents Σ' et Σ'' ; le premier Σ' s'étendra à toutes les doubles couches situées entre M'' et la poire, le second Σ'' à toutes les doubles couches situées entre M'' et l'ellipsoïde E_0. Nous conserverons le signe Σ pour les sommations étendues à toutes les doubles couches.

Posons alors $MP = l$, de sorte que $-\delta_p d\sigma$ qui représente la masse de la partie de la couche C_p qui correspond à l'élément $d\sigma$ sera $d\sigma_p dl$, $d\sigma_p$ étant l'élément de A_p qui correspond à $d\sigma$; or

$$\frac{d\sigma}{d\sigma_p} - 1 = kl, \quad \text{d'où :} \quad \frac{d\sigma}{d\sigma_p} = \frac{1}{1 + kl} = 1 - kl,$$

d'où enfin

$$\delta_p = -(1 - kl)\,dl.$$

$$\Sigma'\delta_p \,.\, MM'' = MM'' \,.\, [(\tfrac{1}{2}k(MM)'^2 - MM') - (\tfrac{1}{3}k(MM'')^2 - MM'')]$$

$$\Sigma''\delta_p MP = \tfrac{1}{3}k(MM'')^3 - \tfrac{1}{2}(MM'')^2$$

$$\Sigma''\delta(MP)^2 = \tfrac{1}{4}k(MM'')^4 - \tfrac{1}{3}(MM'')^3$$

et si nous posons un instant pour abréger

$$MM' = \epsilon, \quad MM'' = \zeta$$

il viendra :

$$V - V'' = 4\pi\zeta(\tfrac{1}{2}k\epsilon^2 - \epsilon - \tfrac{1}{2}k\zeta^2 + \zeta) + 4\pi(\tfrac{1}{3}k\zeta^3 - \tfrac{1}{2}\zeta^2) + 2\pi k\zeta^2(\zeta - \epsilon) - \tfrac{2}{3}\pi k\zeta^3 - \zeta\frac{dV}{dn},$$

ou

$$V - V'' = 4\pi(\tfrac{1}{2}\zeta^2 - \zeta\epsilon) + 2\pi k(\tfrac{1}{3}\zeta^3 - \zeta^2\epsilon + \zeta\epsilon^2) - \zeta\frac{dV}{dn}.$$

L'énergie de $\mathfrak{M}$ sur $\mathfrak{M}$ sera représentée par l'intégrale :

$$\tfrac{1}{2}\int(V'' - V)\,d\tau,$$

étendue à tous les éléments $d\tau$ de C.

Or si par M″ je fais passer l'une de ces surfaces très peu différentes de Σ, et qui me servaient tout à l'heure à définir mes doubles couches, si j'appelle $d\sigma''$ l'élément de cette surface correspondant à $d\sigma$, nous aurons :

$$d\tau = d\sigma'' dMM'' = d\sigma'' d\zeta$$

et

$$d\sigma'' = d\sigma (1 - k\zeta).$$

L'intégrale à chercher est donc :

$$\int d\sigma d\zeta \left[2\pi (\zeta\epsilon - \tfrac{1}{2}\zeta^2) + \pi k (\zeta^2\epsilon - \zeta\epsilon^2 - \tfrac{1}{3}\zeta^3) - 2\pi k\zeta (\zeta\epsilon - \tfrac{1}{2}\zeta^2) + \tfrac{1}{2}\zeta \frac{dV}{dn} \right],$$

et elle doit être prise par rapport à ζ entre 0 et ϵ ; on trouve ainsi :

$$\int \left(\tfrac{2}{3}\pi\epsilon^3 - \tfrac{2}{3}\pi k\epsilon^4 + \tfrac{1}{4}\epsilon^2 \frac{dV}{dn} \right) d\sigma .$$

Pour rendre la formule comparable à celles qui précèdent il faut exprimer ϵ et k en fonctions de $dv/d\sigma$ et de $d\sigma/du$, $d\theta/du$, etc.

Nous avons d'abord

$$dv = \int d\tau = d\sigma \int_0^\epsilon (1 - k\zeta)\, d\zeta = \epsilon - \tfrac{1}{2}k\epsilon^2 ,$$

d'où :

$$\epsilon = \frac{dv}{d\sigma} + \tfrac{1}{2}k \left(\frac{dv}{d\sigma}\right)^2, \qquad \epsilon^3 = \left(\frac{dv}{d\sigma}\right)^3 + \tfrac{3}{2}k \left(\frac{dv}{d\sigma}\right)^4 ,$$

et pour l'intégrale de l'énergie :

$$\int d\sigma \left[\tfrac{2}{3}\pi \left(\frac{dv}{d\sigma}\right)^3 + \tfrac{1}{3}\pi k \left(\frac{dv}{d\sigma}\right)^4 + \tfrac{1}{4} \frac{dV}{dn} \left(\frac{dv}{d\sigma}\right)^2 \right].$$

Observons que le calcul a été fait dans l'hypothèse où la surface de la poire est extérieure à celle de E_0. Dans l'hypothèse contraire, il faudrait changer le signe des deux premiers termes et écrire

$$\int d\sigma \left[-\tfrac{2}{3}\pi \left(\frac{dv}{d\sigma}\right)^3 - \tfrac{1}{3}\pi k \left(\frac{dv}{d\sigma}\right)^4 + \tfrac{1}{4} \frac{dV}{dn} \left(\frac{dv}{d\sigma}\right)^2 \right].$$

Il reste à calculer k. Reprenons la lettre l dans son sens primitif, de sorte que

$$l = \frac{1}{(\rho^2 - \mu^2)^{\frac{1}{2}} (\rho^2 - \nu^2)^{\frac{1}{2}}} \qquad l_0 = \frac{1}{(\rho_0{}^2 - \mu^2)^{\frac{1}{2}} (\rho_0{}^2 - \nu^2)^{\frac{1}{2}}},$$

$$\frac{d\theta}{du} = \frac{l^2}{l_0}.$$

Nous savons que k est défini par la relation

$$\frac{d\sigma''}{d\sigma} = 1 - k\zeta.$$

si nous reprenons les notations employées plus haut, nous devrons écrire u au lieu de ζ, $d\sigma$ au lieu de $d\sigma''$ et $d\sigma_0$ au lieu de $d\sigma$, et notre relation deviendra

$$\frac{d\sigma}{d\sigma_0} = 1 - ku; \qquad \text{or} \quad \frac{d\sigma}{d\sigma_0} = \frac{l_0}{l},$$

on a donc :

$$k = + \frac{l_0}{\rho^2}\frac{dl}{du} \qquad \frac{d^2\theta}{du^2} = \frac{2l}{l_0}\frac{dl}{du},$$

et enfin, puisque sur E_0 on a $l = l_0$

$$k = + \frac{1}{2l}\frac{d^2\theta}{du^2}\text{*}.$$

Cela posé, dans notre intégrale les termes qui nous intéressent sont :—

1. Dans $(dv/d\sigma)^3$ les termes $3\,\xi_5^{\,2}\,\Sigma\,\xi_i\,M_5^{\,2}N_5^{\,2}\,M_iN_i\,l^3$, qui donnent :

$$\Sigma 2\pi\xi_5^{\,2}\xi_i\int M_5^{\,2}N_5^{\,2}M_iN_i\,.\,l^3d\sigma$$

(j'écris $d\sigma$ en supprimant l'indice 0 devenu inutile).

2. Dans $(dv/d\sigma)^4$ le terme $\xi_5^{\,4}\,l^4\,M_5^{\,4}\,N_5^{\,4}$ qui donne :

$$+ \tfrac{1}{6}\pi\xi_5^{\,4}\int\frac{d^2\theta}{du^2}\,l^3M_5^{\,4}N_5^{\,4}d\sigma.$$

3. Enfin dans $(dv/d\sigma)^2\,dV/du$ le terme intéressant se calculera en supposant tous les ξ nuls sauf ξ^5.

C'est la portion de l'énergie de la double couche que nous avons calculée au début de ce travail ; nous n'avons donc qu'à appliquer la formule établie au début.

D'après cette formule, si $\delta = \Sigma\,B\,M\,N$ est la densité de la double couche, cette portion de l'énergie sera :

$$- 2\pi\Sigma\frac{\mathrm{B}^2\mathrm{R}'\mathrm{S}'}{2n+1}\int l^2MN^2d\sigma.$$

Mais ici

$$\delta = \tfrac{1}{2}\left(\frac{dv}{d\sigma}\right)^2 = \tfrac{1}{2}l^2\xi_5^{\,2}M_5^{\,2}N_5^{\,2}.$$

Si donc

$$l^2M_5^{\,2}N_5^{\,2} = \Sigma\beta MN$$

alors la portion cherchée de l'énergie sera :

$$\tfrac{1}{2}\pi\xi_5^{\,4}\Sigma\frac{\beta^2\mathrm{R}'\mathrm{S}'}{2n+1}\int lM^2N^2d\sigma.$$

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* Je puis donc ici reprendre toutes les notations du début de ce travail, et que j'avais abandonnées momentairement, ainsi qu'il est expliqué dans la note de la page 345. On observera que k est une constante généralement négative.

Unification des Formules.

Une difficulté provient de ce que quelques-unes des formules précédentes ont une forme analytique différente suivant que l'élément $d\sigma$ de la surface de E_0 est au-dessous ou au-dessus de la poire. Il est permis toutefois de prévoir qu'il doit y avoir compensation, et que dans la formule finale nous retomberons toujours sur la même forme analytique. Il reste à voir comment se fait cette compensation.

Les termes d'où provient la difficulté sont (outre ceux dus à l'action de $\mathfrak{M}$ sur $\mathfrak{M}$) :—

1. L'énergie de E_0 sur $\mathfrak{M}$ dont l'expression est :

$$\int P\frac{dv}{d\sigma}d\sigma + \tfrac{1}{2}\int\frac{dP}{du}\left(\frac{dv}{d\sigma}\right)^2 d\sigma + \tfrac{1}{6}\int\frac{d^2P}{du^2}\left(\frac{dv}{d\sigma}\right)^3 d\sigma + \tfrac{1}{24}\int\frac{d^3P}{du^3}\left(\frac{dv}{d\sigma}\right)^4 d\sigma.$$

2. L'énergie de Σ sur $\mathfrak{M}$ dont l'expression est :

$$\tfrac{1}{2}\int\frac{dV_2}{du}\left(\frac{dv}{d\sigma}\right)^2 d\sigma + \tfrac{1}{6}\int\frac{d^2V_2}{du^2}\left(\frac{dv}{d\sigma}\right)^3 d\sigma.$$

J'observe que V_1, dV_1/du, V_2 sont continus quand on franchit la surface de E_0 et qu'il en est de même de

$$P - V_1 = \tfrac{1}{2}\omega_0^{\,2}(y^2 + z^2)$$

et de toutes ses dérivées. Si donc j'appelle

$$D\frac{dV_1}{du^2},\quad D\frac{d^3V_1}{du^3},\quad D\frac{dV_2}{du},\quad D\frac{d^2V_2}{du^2}$$

les sauts brusques subis par d^2V_1/du^2 quand on franchit cette surface, du dedans au dehors, la différence entre les deux formules qu'il s'agit de comparer sera :

$$\tfrac{1}{6}\int\left(\frac{dv}{d\sigma}\right)^3 D\frac{d^2V_1}{du^2}d\sigma + \tfrac{1}{24}\int\left(\frac{dv}{d\sigma}\right)^4 D\frac{d^3V_1}{du^3}d\sigma$$

pour l'action de E_0 sur $\mathfrak{M}$, et

$$\tfrac{1}{2}\int\left(\frac{dv}{d\sigma}\right)^2 D\frac{dV_2}{du}d\sigma + \tfrac{1}{6}\int\left(\frac{dv}{d\sigma}\right)^3 D\frac{d^2V_2}{du^2}d\sigma$$

pour l'action de Σ sur $\mathfrak{M}$.

Calcul de $D\dfrac{d^2V_1}{du^2}$.

Nous nous servirons pour ce calcul de l'équation suivante :—

$$D\Delta V_1 = 4\pi,$$

puisque $\Delta V_1 = 0$ à l'extérieur, et -4π à l'intérieur.

Or si nous nous rappelons l'expression de ΔV_1 et que le D de $dV_1/d\mu$, $d^2V_1/d\mu^2$, $dV_1/d\nu$, $d^2V_1/d\nu^2$ est nul ; ainsi que celui de $dV_1/d\rho$, il vient :

$$ABC.\,D\Delta V_1 = D\frac{d}{d\rho}\left(\frac{BC}{A}\frac{dV_1}{d\rho}\right) = \frac{BC}{A}D\frac{d^2V_1}{d\rho^2},$$

d'où

$$D\frac{d^2V_1}{d\rho^2} = A^2D\Delta V_1 = 4\pi A^2.$$

Donc

$$D\frac{d^2V_1}{du^2} = \frac{d^2\rho}{du^2}D\frac{dV_1}{d\rho} = \left(\frac{d\rho}{du}\right)^2 D\frac{d^2V_1}{d\rho^2} = 4\pi A^2\left(\frac{d\rho}{du}\right)^2 = 4\pi.$$

Calcul de $D\frac{d^3V_1}{du^3}$.

Calculons d'abord $D\frac{d^3V_1}{d\rho^3}$; pour cela nous nous servirons de :—

$$D\frac{d\Delta V_1}{d\rho} = 0.$$

Si nous observons que :

$$D\frac{d^2V_1}{d\mu^2} = D\frac{dV_1}{d\mu} = D\frac{d^2V_1}{d\mu d\rho} = D\frac{d^3V_1}{d\mu^2 d\rho} = 0,$$

et de même pour les dérivées correspondantes par rapport à ν, je puis écrire :

$$D\frac{d\Delta V_1}{d\rho} = D\frac{d}{d\rho}\left[\frac{1}{ABC}\frac{d}{d\rho}\left(\frac{BC}{A}\frac{dV_1}{d\rho}\right)\right]$$

Mais d'ailleurs on a :

$$Ad\rho = \frac{d\theta}{l} = du\frac{l}{l_0},$$

$$BC = H\frac{1}{l},$$

H étant indépendant de ρ ; nous pouvons donc écrire :

$$\frac{BC}{A}\frac{dV_1}{d\rho} = \frac{Hl_0}{l^2}\frac{dV_1}{du};\ \frac{1}{ABC}\frac{d}{d\rho}\left(\frac{BC}{A}\frac{dV_1}{d\rho}\right) = \frac{l_0}{H}\frac{d}{du}\left(\frac{Hl_0}{l^2}\frac{dV_1}{du}\right) = l_0^2\frac{d}{du}\left(\frac{dV_1}{l^2du}\right)$$

$$0 = D\frac{d\Delta V_1}{du} = l_0^2 D\frac{d^2}{du^2}\left(\frac{dV_1}{l^2du}\right);\ \frac{1}{l^2}D\frac{d^3V_1}{du^3} - \frac{4dl}{l^3du}D\frac{d^2V_1}{du^2} = 0,$$

d'où enfin :

$$D\frac{d^3V_1}{du^3} = \frac{4dl}{ldu}4\pi = +16\pi k.$$

Calcul de $D\frac{dV_2}{du}$.

D'après la propriété fondamentale des surfaces attirantes, on a :

$$D\frac{dV_2}{du} = -4\pi\frac{dv}{d\sigma}.$$

Calcul de $D\frac{d^2V_2}{du^2}$.

Pour ce calcul nous nous servirons de

$$0 = D\Delta V_2 = D\left[\frac{1}{ABC}\frac{d}{d\rho}\left(\frac{BC}{A}\frac{dV_2}{d\rho}\right)\right] = l_0{}^2 D\frac{d}{du}\left(\frac{dV_2}{l^2du}\right)$$

$$\frac{1}{l^2}D\frac{d^2V_2}{du^2} - \frac{2dl}{l^3du}D\frac{dV_2}{du} = 0$$

d'où enfin,

$$D\frac{d^2V_2}{du^2} = -4\pi\frac{dv}{d\sigma}\frac{2dl}{ldu} = -8\pi k\frac{dv}{d\sigma}.$$

En résumé la différence entre les deux formules qu'il s'agit d'identifier sera :—

1. Pour l'action de E sur $\mathfrak{M}$

$$\tfrac{1}{6}\int\left(\frac{dv}{d\sigma}\right)^3 D\frac{d^2V_1}{du^2}d\sigma + \tfrac{1}{24}\int\left(\frac{dv}{d\sigma}\right)^4 D\frac{d^3V_1}{du^3}d\sigma = \tfrac{2}{3}\pi\int\left(\frac{dv}{d\sigma}\right)^3 d\sigma + \tfrac{2}{3}\pi\int k\left(\frac{dv}{d\sigma}\right)^4 d\sigma.$$

2. Pour l'action de Σ sur $\mathfrak{M}$

$$\tfrac{1}{2}\int\left(\frac{dv}{d\sigma}\right)^2 D\frac{V}{du}\,a\sigma + \tfrac{1}{6}\int\left(\frac{dv}{d\sigma}\right)^3 D\frac{d^2V_2}{du^2}d\sigma = -2\pi\int\left(\frac{dv}{d\sigma}\right)^3 d\sigma - \tfrac{4}{3}\pi\int k\left(\frac{dv}{d\sigma}\right)^4 d\sigma.$$

3. Pour l'action de $\mathfrak{M}$ sur $\mathfrak{M}$

$$\int d\sigma\left[\tfrac{2}{3}\pi\left(\frac{dv}{d\sigma}\right)^3 + \tfrac{1}{3}\pi k\left(\frac{dv}{d\sigma}\right)^4 + \tfrac{1}{4}\frac{dV}{du}\left(\frac{dv}{d\sigma}\right)^2\right] - \int d\sigma\left[-\tfrac{2}{3}\pi\left(\frac{dv}{d\sigma}\right)^3 - \tfrac{1}{3}\pi k\left(\frac{dv}{d\sigma}\right)^4 + \tfrac{1}{4}\frac{dV}{du}\left(\frac{dv}{d\sigma}\right)^2\right]$$
$$= \tfrac{4}{3}\pi\int\left(\frac{dv}{d\sigma}\right)^3 d\sigma + \tfrac{2}{3}\pi\int k\left(\frac{dv}{d\sigma}\right)^4 d\sigma.$$

soit au total zéro.

Nous pourrons donc employer indifféremment l'une ou l'autre formule sans nous inquiéter de savoir si la poire est au dessus ou au dessous de l'ellipsoïde, pourvu que l'on se serve des formules correspondantes pour le calcul de tous les termes.

Nous choisirons désormais l'hypothèse intérieure.

Groupement des Formules.

Nous allons maintenant grouper ensemble les termes de même forme, afin d'additionner leurs coefficients.

Nous avons d'abord à envisager les termes en $\xi_i \xi_5^2$; le premier que nous trouvons est :—

$$\tfrac{1}{2}\xi_5^2\xi_i \int \frac{d^2P}{d\rho^2}\frac{d}{du}\left(\rho\frac{d\rho}{du}\right) M_5^2 N_5^2 M_i N_i l^3 d\sigma.$$

Nous avons posé

$$\frac{d^2P}{d\rho^2} = \frac{\Pi_0}{l^2},$$

Π_0 étant une constante. Nous avons d'autre part :—

$$\frac{d\rho}{du} = \frac{d\rho}{d\theta}\frac{d\theta}{du} = \frac{(\rho^2-a^2)^{\frac{1}{2}}(\rho^2-b^2)^{\frac{1}{2}}(\rho^2-c^2)^{\frac{1}{2}}}{\rho}\frac{(\rho_0^2-\mu^2)^{\frac{1}{2}}(\rho_0^2-\nu^2)^{\frac{1}{2}}}{(\rho^2-\mu^2)(\rho^2-\nu^2)} = \frac{f(\rho)}{\rho}\frac{l^2}{l_0}$$

en désignant par $f(\rho)$ une fonction de ρ et par l_0 ce que devient l pour $\rho = \rho_0$. Nous déduisons de là :—

$$\frac{d}{du}\left(\rho\frac{d\rho}{du}\right) = f'(\rho)\frac{d\rho}{du}\frac{l^2}{l_0} + f(\rho)\frac{2l\, dl/du}{l_0} = \frac{ff'}{l}\frac{l^4}{l_0^2} + 2f(\rho)\frac{l\, dl/du}{l_0}$$

et pour $\rho = \rho_0$:

$$\frac{d}{du}\left(\rho\frac{d\rho}{du}\right) = \frac{ff'}{\rho}l^2 + 2f(\rho)\frac{dl}{du}.$$

Qu'est-ce maintenant que dl/du pour $\rho = \rho_0$?

On trouve

$$\frac{dl}{du} = \frac{dl}{d\rho}\frac{d\rho}{du} = \frac{f(\rho)}{\rho}\frac{dl}{d\rho}\frac{l^2}{l_0} = \frac{f(\rho)}{\rho}l\frac{dl}{d\rho}.$$

Calculons encore la dérivée seconde :

$$\frac{d^2}{du^2}\left(\rho\frac{d\rho}{du}\right).$$

Nous venons de trouver :

$$\frac{d}{du}\left(\rho\frac{d\rho}{du}\right) = \frac{ff'}{\rho}\frac{l^4}{l_0^2} + \frac{2f^2}{\rho}\frac{l^3\frac{dl}{d\rho}}{l_0^2}.$$

On trouve de même :

$$\frac{d^2}{du^2}\left(\rho\frac{d\rho}{du}\right) = \frac{d}{d\rho}\left(\frac{ff'}{\rho}\right)\frac{f}{\rho}\frac{l^6}{l_0^3} + \left(\frac{8f^2f'}{\rho^2} - \frac{2f^3}{\rho^3}\right)\frac{l^5\frac{dl}{d\rho}}{l_0^3} + \frac{6f^3}{\rho^2}\frac{l^4\left(\frac{dl}{d\rho}\right)^2}{l_0^3} + \frac{2f^3}{\rho^2}\frac{l^5\frac{d^2l}{d\rho^2}}{l_0^3}$$

ou pour $\rho = \rho_0$:

$$\frac{d^2}{du^2}\left(\rho\frac{d\rho}{du}\right) = \frac{f}{\rho}\frac{d}{d\rho}\left(\frac{ff'}{\rho}\right)l^3 + \left(\frac{8f^2f'}{\rho^2} - \frac{2f^3}{\rho^3}\right)l^2\frac{dl}{d\rho} + \frac{2f^3}{\rho^2}\left[3l\left(\frac{dl}{d\rho}\right)^2 + l^2\frac{d^2l}{d\rho^2}\right]$$

Posons

$$\Pi_0\frac{ff'}{\rho} = \mathfrak{A}_1\,; \quad \Pi_0\frac{f^2}{\rho} = \mathfrak{B}.$$

$\mathfrak{A}_1$ et $\mathfrak{B}$ seront des constantes dépendant seulement de ρ et faciles à calculer. On aura, pour $\rho = \rho_0$:

$$\Pi_0\frac{d}{du}\left(\rho\frac{d\rho}{du}\right) = \mathfrak{A}_1 l^2 + 2\mathfrak{B}l\frac{dl}{d\rho}.$$

Or nous avons posé plus haut :

$$\Pi_0\frac{d}{du}\left(\rho\frac{d\rho}{du}\right)M_5^2N_5^2 = \Sigma\eta_iM_iN_i$$

et nous avons trouvé que le coëfficient cherché était égal à $\frac{1}{2}\eta_i\Omega_i$, nous avons posé un peu plus loin :—

$$l^2M_5^2N_5^2 = \Sigma\beta_iM_iN_i.$$

En rapprochant toutes ces formules, nous trouvons :—

$$\eta_i = \mathfrak{A}_1\beta_i + \mathfrak{B}\frac{d\beta_i}{d\rho}.$$

Nous avons trouvé ensuite comme terme en $\xi_i\,\xi_5^2$:

$$\frac{4\pi}{7}\int l^2M_5{}^2N_5{}^2M_iN_i\frac{d\theta}{du}R_5'S_5d\sigma + \frac{2\pi}{2n+1}R_i'S_i\int l^2M_5{}^2N_5{}^2M_iN_i\frac{d\theta}{du}d\sigma.$$

Si nous observons que $d\theta/du$ se réduit à l pour $\rho = \rho_0$, nous trouverons pour le coëfficient en question :

$$\pi\beta_i\Omega_i\left(\tfrac{4}{7}R'_5S_5 + \frac{2}{2n+1}R_i'S_i\right)$$

Enfin dans l'énergie de $\mathfrak{M}$ sur $\mathfrak{M}$ nous avons encore un terme en $\xi_i\,\xi_5^2$, qui a pour coëfficient :

$$-\,2\pi\int M_5{}^2N_5{}^2M_iN_il^3d\sigma = -\,2\pi\beta_i\Omega_i.$$

Je prends le signe $-$ parce que j'ai adopté l'hypothèse d'après laquelle $\mathfrak{M}$ est intérieur à E_0.

En réunissant tous ces termes, je trouve que le coëfficient définitif de $\xi_i\,\xi_5^2$ est

$$\beta_i\Omega_i\left(\mathfrak{A}_1 + \frac{4}{7}\pi\,R'_5S_5 + \frac{2}{2n+1}\pi\,R_i'S_i + 2\pi\right) + \mathfrak{B}\Omega_i\frac{d\beta_i}{d\rho}.$$

Remarquons encore que nous avons trouvé :—

$$\frac{dP}{d\rho}\frac{d\rho}{du} = \frac{\rho}{l^2}\Pi_0\frac{d\rho}{du} = -\frac{4\pi}{3l}R_2S_2.$$

Nous pouvons en déduire :

$$\mathfrak{A}_1 = -\tfrac{4}{3}\pi R_2S_2\frac{f'}{\rho}; \quad \mathfrak{B} = -\tfrac{4}{3}\pi R_2S_2\frac{f}{\rho}.$$

Je rappelle que $\frac{4}{3}\pi f$ n'est autre chose que le volume de E_0.

Passons maintenant aux termes en ξ_5^4 ; le premier que nous rencontrons a pour coëfficient l'intégrale :

$$\frac{1}{24}\int\frac{d^2P}{d\rho^2}l^4\frac{d^2}{du^2}\left(\rho\frac{d\rho}{du}\right)M_5{}^4N_5{}^4\,d\sigma = \frac{\Pi_0}{24}\int l^2\frac{d^2}{du^2}\left(\rho\frac{d\rho}{du}\right)M_5{}^4N_5{}^4d\sigma.$$

Supposons que l'on veuille développer en série de Lamé la fonction

$$l^4M_5{}^3N_5{}^3$$

et soit

$$l^4M_5{}^3N_5{}^3 = \Sigma\Gamma_iM_iN_i$$

les Γ_i étant certains coëfficients, qui naturellement dépendront de ρ, nous en déduirons :—

$$4l^3\frac{dl}{d\rho}M_5{}^3N_5{}^3 = \Sigma\frac{d\Gamma_i}{d\rho}M_iN_i$$

$$4\left[3l^2\left(\frac{dl}{d\rho}\right)^2 + l^3\frac{d^2l}{d\rho^2}\right]M_5{}^3N_5{}^3 = \Sigma\frac{d^2\Gamma_i}{d\rho^2}M_iN_i.$$

Il est clair alors que nous aurons pour le seul coëfficient qui nous intéresse, qui est Γ_5 et que je désignerai simplement par Γ :—

$$\Omega_5\Gamma_5 = \Omega_5\Gamma = \int l^5M_5{}^4N_5{}^4d\sigma;$$

$$\Omega_5\frac{d\Gamma}{d\rho} = 4\int l^4\frac{dl}{d\rho}M_5{}^4N_5{}^4d\sigma; \quad \Omega_5\frac{d^2\Gamma}{d\rho^2} = 4\int\left[3l^3\left(\frac{dl}{d\rho}\right)^2 + l^4\frac{d^2l}{d\rho^2}\right]M_5{}^4N_5{}^4d\sigma.$$

Cela nous montre, en rapprochant de l'expression de $\frac{d^2}{du^2}\left(\rho\frac{d\rho}{du}\right)$, que si nous posons pour abréger :

$$\mathfrak{C}_1 = \frac{\Pi_0f}{24\rho}\frac{d}{d\rho}\left(\frac{ff'}{\rho}\right); \quad \mathfrak{D}_1 = \frac{\Pi_0}{48}\left(\frac{4f}{\rho^2} - \frac{f^3}{\rho^3}\right); \quad \mathfrak{E}_1 = \frac{\Pi_0}{48}\frac{f^3}{\rho^2}.$$

le coëfficient du terme envisagé en ξ_5^4 sera :

$$\mathfrak{C}_1\Gamma\Omega_5 + \mathfrak{D}_1\frac{d\Gamma}{d\rho}\Omega_5 + \mathfrak{E}_1\frac{d^2\Gamma}{d\rho^2}\Omega_5.$$

Vient ensuite comme terme en $\xi_5{}^4$

$$\frac{4\pi}{7}\frac{1}{6}\int l^3 M_5{}^4 N_5{}^4 \left[R'_5 S_5 \frac{d^2\theta}{du^2} + R''_5 S_5 \left(\frac{d\theta}{du}\right)^2\right] d\sigma.$$

Pour le calcul de ce coëfficient, il nous faut $d^2\theta/du^2$, quant à $(d\theta/du)^2$ c'est l^2. Or nous avons :

$$\frac{d\theta}{du} = \frac{l^2}{l_0};\quad \frac{d^2\theta}{du^2} = \frac{2l}{l_0}\frac{dl}{d\rho}\frac{d\rho}{du} = 2l\frac{dl}{d\rho}\frac{f}{\rho},$$

ce qui nous donne en définitive pour le coëfficient cherché :

$$\frac{2\pi}{21} R''_5 S_5 \Gamma \Omega_5 + \frac{\pi}{21} R'_5 S_5 \frac{f}{\rho}\frac{d\Gamma}{d\rho}\Omega_5.$$

Enfin dans l'énergie de $\mathfrak{M}$ sur $\mathfrak{M}$, nous avons deux termes en $\xi_5{}^4$, le premier a pour coëfficient

$$-\tfrac{1}{6}\pi\int\frac{d^2\theta}{du^2} l^3 M_5{}^4 N_5{}^4 d\sigma = -\tfrac{1}{12}\pi\frac{f}{\rho}\frac{d\Gamma}{d\rho}\Omega_5,$$

(je prends le signe — à cause de l'hypothèse intérieure) et le second

$$-\tfrac{1}{2}\pi\Sigma\frac{\beta_i^2 R'_i S'_i}{2n+1}\Omega_i.$$

En réunissant tous ces termes, nous trouvons finalement pour le coëfficient de ξ_5^4 :—

$$\left[\mathfrak{C}_1 + \tfrac{2}{21}\pi R''_5 S_5\right]\Gamma\Omega_5 + \left[\mathfrak{D}_1 + \tfrac{1}{21}\pi\frac{f}{\rho} R'_5 S_5 - \tfrac{1}{12}\pi\frac{f}{\rho}\right]\frac{d\Gamma}{d\rho}\Omega_5$$
$$+ \mathfrak{E}_1\frac{d^2\Gamma}{d\rho^2}\Omega_5 - \tfrac{1}{2}\pi\Sigma\frac{1}{2\pi+1}B_i^2 R'_i S'_i \Omega_i.$$

Observons que $R''_5 S_5$ est égal à $R_5 S_5$ au facteur constant près R''_5/R_5, qui est égal d'après l'équation de LAMÉ à un polynôme connu du second ordre en ρ^2. . D'ailleurs $R_5 S_5$ est égal au facteur $\frac{7}{3}$ près à $R_2 S_2$, qui figure dans l'expression de Π_0, et cela parce que le coëfficient de stabilité correspondant à R_5 doit s'annuler pour l'ellipsoïde E_0.

Calcul du Moment d'Inertie.

Le calcul de J est plus facile ; nous avons en effet

$$J = \int (y^2 + z^2)\, d\tau$$

l'intégrale étant étendue à tous les éléments $d\tau$ de la poire ; le moment d'inertie de l'ellipsoïde E_0 sera

$$J_0 = \int (y^2 + z^2)\, d\tau$$

l'intégrale étant étendue à tous les éléments $d\tau$ de l'ellipsoïde, et la différence $J - J_0$ sera la même intégrale étendue à tous les éléments de la couche comprise entre l'ellipsoïde et la poire. Posons

$$Q = y^2 + z^2 = Q_0 + u \frac{dQ}{du} + \ldots$$

Nous aurons

$$J - J_0 = \int Q\, du\, d\sigma = \int \left(Q_0 + u \frac{dQ}{du} \right) du\, d\sigma = \int Q_0 \frac{dv}{d\sigma}\, d\sigma + \tfrac{1}{2} \int \frac{dQ}{du} \left(\frac{dv}{d\sigma} \right)^2 d\sigma.$$

Il nous faut donc calculer Q_0 et dQ/du, nous avons posé plus haut

$$Q = B_1 R_1 M_1 N_1 + B_3 R_3 M_3 N_3 + B_4 R_4 M_4 N_4 + B_0\, \Pi.$$

Pour $\rho = \rho_0$, Π est nul ; de sorte que :

$$Q_0 = B_1 R_1^0 M_1 N_1 + B_3 R_3^0 M_3 N_3 + B_4 R_4^0 M_4 N_4.$$

Comme les fonctions R ne sont définies qu'à un facteur constant près, nous pouvons supposer que $R_1 = 1$, et que le coëfficient de ρ^2 dans R_3 et dans R_4 est égal à 1.

On trouve d'autre part :

$$\frac{dQ}{d\rho} = 2\rho \left[\frac{(\mu^2 - b^2)(\nu^2 - b^2)}{(b^2 - a^2)(b^2 - c^2)} + \frac{(\mu^2 - c^2)(\nu^2 - c^2)}{(c^2 - a^2)(c^2 - b^2)} \right]$$

d'où :

$$\frac{dQ}{du} = 2fl \left[\frac{(\mu^2 - b^2)(\nu^2 - b^2)}{(b^2 - a^2)(b^2 - c^2)} + \frac{(\mu^2 - c^2)(\nu^2 - c^2)}{(c^2 - a^2)(c^2 - b^2)} \right].$$

Cette expression peut facilement se mettre sous la forme :

$$\frac{dQ}{du} = C_1 l M_1 N_1 + C_3 l M_3 N_3 + C_4 l M_4 N_4,$$

où les C sont des coëfficients numériques faciles à déterminer.

Nous trouvons d'abord :

$$\int Q_0 \frac{dv}{d\sigma}\, d\sigma = \int l\, d\sigma \left(B_1 R_1^0 M_1 N_1 + B_3 R_3^0 M_3 N_3 + B_4 R_4^0 M_4 N_4 \right) \Sigma \xi_i M_i N_i\, .$$

d'où :

$$\int Q_0 \frac{dv}{d\sigma}\, d\sigma = \xi_1 B_1 R_1^0 \Omega_1 + \xi_3 B^3 R_3^0 \Omega_3 + \xi_4 B_4 R_4^0 \Omega_4 = \xi_3 B_3 R_3^0 \Omega_3 + \xi_4 B_4 R_4^0 \Omega$$

(car nous savons que ξ_1 est nul).

Passons au calcul de

$$\frac{1}{2}\int \frac{dQ}{du}\left(\frac{dv}{d\sigma}\right)^2 d\sigma .$$

Nous pouvons réduire $(dv/d\sigma)^2$ au terme unique :

$$\xi_5^2 l^2 M_5^2 N_5^2 .$$

Le terme cherché se réduit donc à :

$$\frac{1}{2}\xi_5^2 \int l^3 M_5^2 N_5^2 (C_1 M_1 N_1 + C_3 M_3 N_3 + C_4 M_4 N_4)\, d\sigma$$

c'est-à-dire à :

$$\frac{1}{2}\xi_5^2 (C_1\beta_1\Omega_1 + C_3\beta_3\Omega_3 + C_4\beta_4\Omega_4)$$

de sorte que finalement :

$$J = J_0 + \xi_3 B_3 R_3^0\Omega_3 + \xi_4 B_4 R_4^0\Omega_4 + \frac{1}{2}\xi_5^2 (C_1\beta_1\Omega_1 + C_3\beta_3\Omega_3 + C_4\beta_4\Omega_4) .$$

Le calcul des coëfficients $B_3R_4^0\Omega_3$ et $B_4R_4^0\Omega_4$ est aisé.

Si en effet a_1, b_1, c_1, sont les trois axes d'un ellipsoïde, on sait que son moment d'inertie est :

$$J = \frac{4\pi}{15} a_1 b_1 c_1 (b_1^2 + c_1^2)$$

d'où :

$$\frac{dJ}{da_1} = \frac{4\pi}{15} b_1 c_1 (b_1^2 + c_1^2),\ \frac{dJ}{db_1} = \frac{4\pi}{15} a_1 c_1 (3b_1^2 + c_1^2),\ \frac{dJ}{dc_1} = \frac{4\pi}{15} b_1 c_1 (b_1^2 + 3c_1^2) .$$

Si nous ajoutons à l'ellipsoïde une couche infiniment mince d'épaisseur,

$$l\xi_3 M_3 N_3 ,$$

la figure reste ellipsoïdale, mais les trois axes subissent des accroissements

$$l^{(1)}\xi_3 M_3^{(1)} N_3^{(1)},\ l^{(2)}\xi_3 M_3^{(2)} N_3^{(2)},\ l^{(3)}\xi_3 M_3^{(3)} N_3^{(3)}$$

où $l^{(i)}$, $M_3^{(i)}$, $N_3^{(i)}$ $(i = 1, 2, 3)$ sont les fonctions l, M_3, N_3, où on a fait respectivement :—

Pour $i = 1$ $\mu = b$ $\nu = c$ $\rho = \rho_0$
$i = 2$ $\mu = a$ $\nu = c$ $\rho = \rho_0$
$i = 3$ $\mu = a$ $\nu = b$ $\rho = \rho_0$

On voit alors que

$$B_3 R_3^0 \Omega_3 = \frac{dJ}{da_1} l^{(1)} M_3^{(1)} N_3^{(1)} + \frac{dJ}{db_1} l^{(2)} M_3^{(2)} N_3^{(2)} + \frac{dJ}{dc_1} l^{(3)} M_3^{(3)} N_3^{(3)} .$$

Dans les dérivées dJ/da_1, etc., on a fait bien entendu

$$a_1 = \sqrt{(\rho_0^2 - a^2)}, \quad b_1 = \sqrt{(\rho_0^2 - b^2)}, \quad c_1 = \sqrt{(\rho_0^2 - c^2)}.$$

On trouverait de même l'expression de $B_4 R_4^0 \Omega_4$.

Conditions de la Stabilité.

Soit U l'énergie de gravitation de la masse envisagée, J le moment d'inertie, ω la vitesse angulaire ; l'énergie totale sera :—

$$U + \tfrac{1}{2}\omega^2 J$$

Soit ω_0 la vitesse angulaire de l'ellipsoïde critique E_0 et posons :

$$W = U + \tfrac{1}{2}\omega_0^2 J$$

$$\omega^2 = \omega_0^2 + 2\epsilon$$

notre énergie totale sera

$$W + \epsilon J.$$

Nous avons trouvé plus haut le développement de W et celui de J jusqu'à l'approximation qui nous convient ; nous avons d'abord :

$$W = W_0 + \Sigma G_i \xi_i^2 + H_0 \xi_5^4 + \Sigma Q_i \xi_5 \xi_i.$$

Nous avons appris à calculer les coëfficients G_i, H_0, et Q_i ; nous remarquerons : (1) que les G_i ne sont autre chose que les coëfficients de stabilité ; (2) que G_5 est nul, et qu'il en est de même de Q_5 ainsi que de tous les coëfficients Q_i qui ne se rapportent pas à une fonction de **Lamé** paire et uniforme. Comme H_0 se compose de deux parties qui joueront un rôle assez différent, j'écrirai :

$$H_0 = H + H',$$

$$H = [\mathfrak{C}_1 + \tfrac{2}{21}\pi R''_5 S_5]\Gamma\Omega_5 + \left[\mathfrak{D}_1 + \tfrac{1}{21}\pi \frac{f}{\rho} R'_5 S_5 - \tfrac{1}{12}\pi \frac{f}{\rho}\right]\frac{d\Gamma}{d\rho}\Omega_5 + \mathfrak{E}_1 \frac{d^2\Gamma}{d\;^2}\Omega_5$$

$$H' = -\tfrac{1}{2}\pi\Sigma \frac{\beta_i^2 R'_i S'_i}{2n+1}\Omega_i.$$

Nous avons d'autre part :

$$J = J_0 + \gamma_0 \xi_5^2 + \gamma_3 \xi_3 + \gamma_4 \xi_4$$

et nous avons appris plus haut à calculer les coëfficients γ.

On obtiendra les équations qui définissent la poire, en écrivant que les dérivées de l'énergie sont nulles ; on trouve ainsi :

1. Par la dérivée par rapport à ξ_5 :

$$2H_0\xi_5^2 + \epsilon\gamma_0 + \Sigma Q_i\xi_i = 0$$

2. Par la dérivée par rapport à ξ_3 :

$$2G_3\xi_3 + Q_3\xi_5^2 + \epsilon\gamma_3 = 0$$

3. Par la dérivée par rapport à ξ_4 :

$$2G_4\xi_4 + Q_4\xi_5^2 + \epsilon\gamma_4 = 0$$

4. Par la dérivée par rapport aux autres ξ_i :

$$2G_i\xi_i + Q_i\xi_5^2 = 0$$

Le rapprochement de ces diverses équations donne :

$$\xi_5^2\left[2H_0 - \Sigma\frac{Q_i^2}{2G_i}\right] + \epsilon\left(\gamma_0 - \frac{Q_3\gamma_3}{2G_3} - \frac{Q_4\gamma_4}{2G_4}\right) = 0.$$

La quantité dont il faut déterminer le signe, c'est :

$$\omega J - \omega_0 J_0 = \omega_0(J - J_0) + \frac{\epsilon}{\omega_0}J_0.$$

Or

$$J - J_0 = \xi_5^2\left(\gamma_0 - \frac{Q_3\gamma_3}{2G_3} - \frac{Q_4\gamma_4}{2G_4}\right) - \epsilon\left(\frac{\gamma_3^2}{2G_3} + \frac{\gamma_4^2}{2G_4}\right).$$

Posons alors

$$\gamma_0 - \frac{Q_3\gamma_3}{2G_3} - \frac{Q_4\gamma_4}{2G_4} = T$$

d'où

$$\frac{\epsilon}{\xi_5^2} = \frac{\Sigma\frac{Q_i^2}{2G_i} - 2H - 2H'}{T}.$$

On voit que la quantité dont il faut déterminer le signe sera :

$$\text{(A)} \qquad \frac{\omega J - \omega_0 J_0}{\omega_0\xi_5^2} = T + \frac{1}{T}\left(\frac{J_0}{\omega_0^2} - \frac{\gamma_3^2}{2G_3} - \frac{\gamma_4^2}{2G_4}\right)\left(\Sigma\frac{Q_i^2}{2G_i} - 2H - 2H'\right).$$

Il est aisé de vérifier que cette formule (A) est homogène ; voici ce que j'entends par là. Les fonctions de LAMÉ M_i ne sont définies qu'à un facteur constant près, et nos formules, pour avoir un sens, doivent être homogènes par rapport à chacun de ces facteurs constants arbitraires. L'intégrale

$$\Omega_i = \int lM_i^2N_i^2d\sigma$$

est évidemment proportionnelle à la quatrième puissance de ce facteur, puisque ce facteur entre également dans M_i et dans N_i.

Nous devons donc vérifier que la formule (A) est homogène par rapport à Ω_i, et en particulier par rapport à Ω_5. Pour écrire, par exemple, qu'une quantité (B) est proportionnelle à la α^e puissance de Ω_i et à la β^e puissance de Ω_5, j'écrirai :

$$B \propto \Omega_i^{\alpha}\,\Omega_5^{\beta}.$$

Je trouve ainsi :

$$\beta_i\Omega_i = \int l^3 M_5^2 N_5^2 M_i N_i \, d\sigma \propto \Omega_5 \sqrt{\Omega_i}$$

d'où

$$\beta_i \propto \frac{d\beta_i}{d\rho} \propto \Omega_5 \Omega_i^{-\frac{1}{2}}$$

De même :

$$\Gamma_i\Omega_i = \int l^5 M_5^3 N_5^3 M_i N_i \, d\sigma \propto \Omega_5^{\frac{3}{2}}\Omega_i^{\frac{1}{2}}$$

d'où

$$\Gamma_i \propto \Omega_5^{\frac{3}{2}}\Omega_i^{\frac{1}{2}}$$

et

$$\Gamma = \Gamma_5 \propto \frac{d\Gamma}{d\rho} \propto \frac{d^2\Gamma}{d\rho^2} \propto \Omega_5.$$

On trouve ensuite :

$$Q_i \propto \beta_i\Omega_i \propto \Omega_5\sqrt{\Omega_i}$$

$$H \propto \Gamma\Omega_5 \propto \Omega_5^2 \,; \;\; H' \propto \beta_i^2\Omega_i \propto \Omega_5^2 \,; \;\; G_i \propto \Omega_i$$

$$\Sigma \frac{Q_i^2}{G_i} \propto \Omega_5^2$$

et enfin

$$\Sigma \frac{Q_i^2}{G_i} - 2H - 2H' \propto \Omega_5^2 .$$

Les coëfficients appelés plus haut B_i et C_i dans le calul de J sont proportionels à $\Omega_i^{-\frac{1}{2}}$, ce qui donne :

$$\gamma_0 \propto C_i\beta_i\Omega_i \propto \beta_i\sqrt{\Omega_i} \propto \Omega_5 \,;$$

$$\gamma_3 \propto B_3\Omega_3 \propto \sqrt{\Omega_3} \,; \;\; \gamma_4 \propto B_4\Omega_4 \propto \sqrt{\Omega_4}$$

$$\frac{Q_i\gamma_i}{G_i} \propto \frac{\beta_i\Omega_i\sqrt{\Omega_i}}{\Omega_i} \propto \sqrt{\Omega_i} \propto \Omega_5$$

et enfin

$$T \propto \Omega_5.$$

D'autre part :

$$\frac{J_0}{\omega_0^2} \propto \frac{\gamma_i^2}{G_i} \propto \frac{\Omega_i}{\Omega_i} \propto 1$$

et

$$\frac{J_0}{\omega_0^2} - \frac{\gamma_3^2}{2G_3} - \frac{\gamma_4^2}{2G_4} \propto 1$$

de sorte que finalement le second membre de notre formule (A) est homogène et de degré 1 par rapport à Ω_5 et ne contient pas les autres Ω_i.

Détermination des Intégrales.

Dans les coëfficients et les formules qui précèdent entrent diverses intégrales, et nous devons chercher à les calculer.

1. Les axes de l'ellipsoïde E_0 étant supposés connus, on formera aisément les diverses fonctions R_i, ce n'est qu'une affaire de calcul algébrique ; on a à résoudre diverses équations algébriques, et ces équations sont du second degré pour toutes les fonctions de Lamé d'ordre 0, 1, 2, 3, et pour quelques unes de celles d'ordre 4 et 5.

Les fonctions R_i étant formées, on aura immédiatement les valeurs R_i^0, qui correspondent à $\rho = \rho_0$ et aussi celles des dérivées successives R'_i, R''_i, etc.

2. Dans nos équations figurent les intégrales S_i ; or le calcul de ces intégrales se ramène à celui des intégrales définies :

$$\int_{\rho_0}^{\infty} d\rho \left(\frac{d\theta}{d\rho}\right) \frac{1}{R_i^2}.$$

Quelle est la forme de la fonction $\frac{1}{R_i^2}$, qui figure sous le signe $\int$? Nous allons l'exprimer en fonction de l'argument elliptique θ, et nous emploierons la notation $\wp$ et σ de Weierstrass. Soit

$$R_i = \Pi_i \Pi'_i$$

Π_i étant le produit de 0, 1, 2, ou 3 des facteurs $\surd(\rho^2 - a^2)$, $\surd(\rho^2 - b^2)$, $\surd(\rho^2 - c^2)$ et Π'_i un produit de facteurs de la forme $\rho^2 - \lambda_k^2$. Nous poserons :

$$\rho^2 - a^2 = \wp(\theta) - e_1 ; \quad \rho^2 - b^2 = \wp(\theta) - e_2 ; \quad \rho^2 - c^2 = \wp(\theta) - e_3 ;$$

$$e_1 + e_2 + e_3 = 0 ;$$

d'où

$$\rho^2 - \wp(\theta) = a^2 - e_1 = b^2 - e_2 = c^2 - e_3 = \tfrac{1}{3}(a^2 + b^2 + c^2).$$

Nous avons d'ailleurs comme on sait :

$$\wp(\omega_1) = e_1, \quad \wp(\omega_2) = e_2, \quad \wp(\omega_3) = e_3.$$

La valeur zéro de l'argument θ correspond à $\rho = \infty$, et nous appellerons θ_0 et E_k, les valeurs qui correspondent à $\rho = \rho_0$ ou à $\rho = \lambda_k$.

Considérons alors la fonction $1/R_i^2$ comme une fonction doublement périodique, et décomposons-la en éléments simples. Les éléments simples seront :—

1. Un terme constant.
2. Des termes en

$$\wp(\theta - \omega_1), \quad \wp(\theta - \omega_2), \quad \wp(\theta - \omega_3)$$

provenant des facteurs $\surd(\rho^2 - a^2)$, $\surd(\rho^2 - b^2)$, $\surd(\rho^2 - c^2)$, qui peuvent exister dans R_i.

3. Des termes en

$$\zeta(\theta+\epsilon_\kappa)-\zeta(\theta-\epsilon_\kappa), \qquad \wp(\theta-\epsilon_\kappa)+\wp(\theta+\epsilon_\kappa)$$

provenant des facteurs $\rho^2 - \lambda_k^2$.

Les coëfficients de ces divers termes, sauf le terme constant, peuvent se déterminer par un calcul purement algébrique.

Quant au terme constant, c'est une fonction linéaire non homogène des $\zeta(\epsilon_k)$, fonction linéaire dont les coëfficients peuvent se calculer algébriquement.

L'intégrale indéfinie contiendra donc des termes en

$$\theta, \quad \theta\zeta(\epsilon_\kappa), \quad \zeta(\theta-\omega_1), \quad \zeta(\theta-\omega_2), \quad \zeta(\theta-\omega_3), \quad \log\frac{\mathcal{G}(\theta+\epsilon_\kappa)}{\mathcal{G}(\theta-\epsilon_\kappa)},$$
$$\zeta(\theta-\epsilon_\kappa)+\zeta(\theta+\epsilon_\kappa)$$

ce qui donnera dans l'intégrale définie des termes en

$$\theta_0; \quad \theta_0\zeta(\epsilon_\kappa); \qquad \zeta(\theta_0-\omega_i)+\zeta(\omega_i)=\zeta(\theta_0-\omega_i)+\eta_i;$$
$$\log\frac{\mathcal{G}(\theta_0+\epsilon_\kappa)}{\mathcal{G}(\theta_0-\epsilon_\kappa)}; \qquad \zeta(\theta_0-\epsilon_\kappa)+\zeta(\theta_0+\epsilon_\kappa).$$

Le calcul de S_i se ramène donc au calcul de ces diverses quantités. Connaissant S_i, on aura immédiatement S'_i et S''_i par les formules

$$R'_iS_i - R_iS'_i = 2n+1; \qquad R''_iS_i - R_iS''_i = 0.$$

4. Nous avons ensuite les intégrales doubles :

$$\Omega_i = \int lM_i^2N_i^2 d\sigma.$$

M_i^2 est un polynôme entier connu en $\wp(\theta_1)$; N_i^2 est le même polynôme en $\wp(\theta_2)$; nous avons d'ailleurs :

$$ld\sigma = d\theta_1 d\theta_2 \sqrt{-1}\,(\nu^2-\mu^2) = d\theta_1 d\theta^2\sqrt{-1}\,[\wp(\theta_2)-\wp(\theta_1)].$$

Quant aux limites d'intégration, elles sont données par les équations

$$a^2>\mu^2>b^2>\nu^2>c^2,$$

d'où

$$e_1>\wp(\theta_1)>e_2>\wp(\theta_2)>e_3$$

ce qui montre qu'il faut faire varier θ_1 depuis $\omega_1-\omega_3$ jusqu'à $\omega_1+\omega_3$ et θ_2 depuis $\omega_3-\omega_1$ jusqu'à $\omega_3+\omega_1$ le long des côtés convenables du rectangle des périodes.

Les limites étant constantes, l'intégrale double se ramène à une combinaison d'intégrales simples :

$$\Omega_i = \sqrt{-1}\,.\left[\int M_i^2 d\theta_1 \int N_i^2\wp(\theta_2)\,d\theta_2 - \int N_i^2 d\theta_2 \int M_i^2\wp(\theta_1)\,d\theta_1\right].$$

Ces intégrales simples se calculent d'une façon très simple. On peut par un calcul algébrique décomposer M_i^2 et $M_i^2\wp(\theta_1)$ en éléments simples, c'est-à-dire, en polynôme dont les termes sont des multiples de $\wp(\theta_1)$ et de ses dérivées. Parmi ces termes nous retiendrons seulement le terme constant et le terme en $\wp(\theta_1)$. Le premier donnera comme intégrales ω_1 et ω_3, le second η_1 et η_3 à un facteur numérique près.

Le calcul de Ω_i est ainsi ramené à celui des périodes ω et η.

5. Nous avons ensuite les intégrales

$$\Omega_i\beta_i = \int l^3 M_5^2 N_5^2 M_i N_i d\sigma = \sqrt{-1}\,.\int l^2 M_5^2 N_5^2 M_i N_i [\wp(\theta_2) - \wp(\theta_1)] d\theta_1 d\theta_2.$$

Ici encore $M_5^2\ M_i$ est un polynôme entier en $\wp(\theta_1)$ et $N_5^2\ N_i$ est le même polynôme en $\wp(\theta_2)$. On a d'ailleurs :

$$l^2 = \frac{1}{(\rho_0^2 - \mu^2)(\rho_0^2 - \nu^2)} = \frac{1}{[\wp(\theta_1) - \wp(\theta_0)][\wp(\theta_2) - \wp(\theta_0)]}.$$

Notre intégrale double se ramène encore à une combinaison d'intégrales simples.

$$\Omega_i\beta_i = \sqrt{-1}\,.\left[\int \frac{M_5^2 M_i d\theta_1}{\wp(\theta_1) - \wp(\theta_0)} \int \frac{N_5^2 N_i \wp(\theta_2)\, d\theta_2}{\wp(\theta_2) - \wp(\theta_0)} - \int \frac{N_5^2 N_i d\theta_2}{\wp(\theta_2) - \wp(\theta_0)} \int \frac{M_5^2 M_i \wp(\theta_1)\, d\theta_1}{\wp(\theta_1) - \wp(\theta_0)}\right].$$

Le calcul se fait de la même manière. Chacune des fonctions sous le signe ∫ doit être décomposée en éléments simples, ces éléments sont une constante ; $\wp(\theta_1)$ ou ses dérivées, et enfin :

$$\zeta(\theta_1 + \theta_0) - \zeta(\theta_1 - \theta_0) - 2\zeta(\theta_0) = \frac{\wp'(\theta_0)}{\wp(\theta_1) - \wp(\theta_0)}.$$

Les coëfficients de cette décomposition pouvant se calculer algébriquement, l'intégration introduira, outre les périodes ω et η, deux transcendantes nouvelles, qui seront

$$-4\zeta(\theta_0)\,\omega_3 + \log\frac{\sigma(\omega_1 - \omega_3 - \theta_0)\,\sigma(\omega_1 + \omega_3 + \theta_0)}{\sigma(\omega_1 + \omega_3 - \theta_0)\,\sigma(\omega_1 - \omega_3 + \theta_0)} = 4\eta_3\theta_0 - 4\omega_3\zeta(\theta_0)$$

$$-4\zeta(\theta_0)\,\omega_1 + \log\frac{\sigma(\omega_3 - \omega_1 - \theta_0)\,\sigma(\omega_3 + \omega_1 + \theta_0)}{\sigma(\omega_3 + \omega_1 - \theta_0)\,\sigma(\omega_3 - \omega_1 + \theta_0)} = 4\eta_1\theta_0 - 4\omega_1\zeta(\theta_0)$$

qui se ramènent d'ailleurs toutes deux à θ_0 et à $\zeta(\theta_0)$.

6. Nous avons ensuite l'intégrale

$$\Omega_i \frac{d\beta_i}{d\rho}$$

qui est la dérivée de la précédente par rapport à ρ. (Ici ρ est, bien entendu, pris égal à ρ_0.)

Elle dépend des dérivées par rapport à ρ des quatre intégrales simples qui figurent dans l'expression ci-dessus de $\Omega_i\beta_i$.

La dérivée de chacune de ces intégrales simples se calcule d'ailleurs aisément. Chacune de ces intégrales est une somme de produits où l'un des facteurs est

$$1,\ \omega_i,\ \eta_i,\ \text{ou}\ \eta_i\theta_0 - \omega_i\zeta(\theta_0)$$

et où l'autre facteur est un coëfficient calculable algébriquement.

La dérivée de ce coëfficient par rapport à ρ_0 sera ainsi calculable algébriquement, et quant à la dérivée du premier facteur elle sera :

$$0,\ 0,\ 0,\ \text{ou}\ \eta_i\frac{d\theta_0}{d\rho_0} - \omega_i\frac{d\zeta(\theta_0)}{d\rho_0} = \frac{\rho_0[\eta_i - \omega_i\wp(\theta_0)]}{\sqrt{\{(\rho_0^2 - a^2)(\rho_0^2 - b^2)(\rho_0^2 - c^2)\}}}.$$

Il ne s'introduit donc aucune transcendante nouvelle.

7. Considérons maintenant l'intégrale :

$$\Omega_5\Gamma = \int l^5 M_5^4 N_5^4 d\sigma.$$

Nous aurons :

$$l^4 = \frac{1}{[\wp(\theta_1) - \wp(\theta_0)]^2[\wp(\theta_2) - \wp(\theta_0)]^2}$$

d'où :

$$\Omega_5\Gamma = \sqrt{-1}\,.\left[\int\frac{M_5^4 d\theta_1}{[\wp(\theta_1) - \wp(\theta_0)]^2}\int\frac{N_5^4 d(\theta_2)\,d\theta_2}{[\wp(\theta_2) - \wp(\theta_0)]^2} - \int\frac{N_5^4 d\theta_2}{[\wp(\theta_2) - \wp(\theta_0)]^2}\int\frac{M_5^4\wp(\theta_1)\,d\theta_1}{[\wp(\theta_1) - \wp(\theta_0)]^2}\right].$$

On opérerait toujours de la même manière en décomposant chaque fonction sous le signe ∫ en éléments simples. Les éléments simples seront ici, outre une constante $\wp(\theta_1)$ et ses dérivées :

$$\zeta(\theta_1 + \theta_0) - \zeta(\theta_1 - \theta_0) - 2\zeta(\theta_0)$$

et

$$\wp(\theta_1 + \theta_0) + \wp(\theta_1 - \theta_0) - 2\wp(\theta_0).$$

L'intégration introduira donc les mêmes transcendantes que dans le cas de $\Omega_i\beta_i$ et en outre (par l'intégration du dernier élément simple que je viens de citer) :

$$4\eta_i - 4\omega_i\wp(\theta_0),$$

ce qui n'est pas une transcendante nouvelle.

7. Il ne nous reste plus que les intégrales

$$\Omega_5\frac{d\Gamma}{d\rho},\ \Omega_5\frac{d^2\Gamma}{d\rho^2}$$

qui sont les dérivées de la précédente.

En raisonnant comme dans le cas de $d\beta_i/d\rho$, on verrait que ces intégrales n'introduisent pas de transcendante nouvelle.

Le calcul de ces transcendantes ne peut présenter de difficulté si l'on emploie les formules de WEIERSTRASS réunies et mises sous une forme si commode par les soins de

M. **Schwarz**. On n'a qu'à employer des séries très convergentes procédant suivant les puissances de la quantité que **Jacobi** appelle q et **Weierstrass** h. D'ailleurs dans le cas de l'ellipsoïde E_0, la valeur de q est si petite que l'on pourrait s'arrêter au premier terme. Ainsi dans le calcul de β_i, de Γ, de leurs dérivées et de Ω_i, on ne rencontrera aucun obstacle ; car il ne s'introduit qu'un petit nombre de transcendantes :

$$\omega_1,\ \omega_3,\ \eta_1,\ \eta_3,\ \zeta(\theta_0),\ \theta_0.$$

Le calcul ne serait pas tout à fait aussi facile pour S_i de sorte que dans ce cas, on pourrait recourir avec avantage aux développements donnés par M. **Darwin** à la fin de son mémoire "Ellipsoidal Harmonic Analysis," et qui procèdent suivant la quantité qu'il appelle β.

Si les axes de l'ellipsoïde jacobien critique sont comme l'a calculé M. **Darwin** dans son second mémoire

$$0{\cdot}65066\ ;\ 0{\cdot}81498\ ;\ 1{\cdot}88583$$

on trouve, sauf erreur de calcul de ma part :

$$h_1 = e^{-\frac{\omega_1 \pi i}{\omega_3}} = \tfrac{1}{200}.$$

$$\omega_1 = 0{\cdot}53790\ ; \qquad \omega_3 = i \times 0{\cdot}90528\ ;$$

$$\eta_1 = 1{\cdot}1956\ ; \qquad \eta_3 = -\, i \times 0{\cdot}9080\ ;$$

$$\theta_0 = 0{\cdot}27501\ ;$$

$$\zeta(\theta_0) = 0{\cdot}71640.$$

Nouvelle Expression des Conditions de Stabilité.

La détermination de *chacune* des intégrales ne présente donc aucune difficulté, et le calcul serait en somme facile si ces intégrales n'étaient en nombre infini.

Rappelons le résultat obtenu plus haut. *La poire sera stable ou instable, suivant que l'expression*

$$T + \frac{1}{T}\left(\frac{J_0}{\omega_0^2} - \frac{\gamma_3^2}{2G_3} - \frac{\gamma_4^2}{2G_4}\right)\left(\Sigma \frac{Q_i^2}{2G_i} - 2H - 2H'\right)$$

sera positive ou négative.

Or nous pouvons tout de suite remarquer que parmi les quantités qui figurent dans cette expression

$$T,\ J_0,\ \omega_0^2,\ \gamma_3,\ \gamma_4,\ G_3,\ G_4,\ H$$

ne dépendent que d'un nombre fini d'intégrales, tandis que

$$\Sigma \frac{Q_i^2}{2G_i} \quad \text{et} \quad H' = \tfrac{1}{2}\pi \Sigma \frac{\beta_i^2 R'_i S'_i}{2n+1} \Omega_i$$

dépendent d'une infinité d'intégrales. Toute la difficulté provient donc du calcul de la quantité

$$Z = \Sigma \frac{Q_i^2}{2G_i} - 2H'.$$

Heureusement il ne s'agit pas de calculer la valeur exacte de cette quantité, mais de reconnaître si elle satisfait à une certaine inégalité. Pour étudier cette inégalité, il faut que nous mettions en évidence le signe de plusieurs de nos quantités.

Commençons par les coëfficients de stabilité G_i. Si nous suivons la série des ellipsoïdes de MacLaurin, tous ces coëfficients sont d'abord négatifs. Le coëfficient G_3 changera de signe, tous les autres restant négatifs, quand nous arriverons à l'ellipsoïde de bifurcation, qui est en même temps un ellipsoïde de MacLaurin et un ellipsoïde de Jacobi. Mais à partir de cet ellipsoïde de bifurcation, on abandonne la série des ellipsoïdes de MacLaurin pour suivre celle des Jacobiens.

Pour cette série le coëfficient G_3 est également négatif, en vertu du principe de l'échange des stabilités convenablement interprété. Pour les premiers Jacobiens jusqu'au Jacobien critique, tous les coëfficients G_i seront donc négatifs sauf G_3. Pour le Jacobien critique, tous les G_i sont negatifs sauf G_5, qui est nul, et G_3, qui est positif.

Déterminons ensuite le signe de

$$Y = \frac{J_0}{\omega_0^2} - \frac{\gamma_3^2}{2G_3} - \frac{\gamma_4^2}{2G_4}.$$

Je renverrai à mon mémoire du Tome 7 des 'Acta,' et au paragraphe intitulé Stabilité des Ellipsoïdes. J'ai expliqué dans ce paragraphe que tous les Jacobiens sont stables si l'on assujettit (à titre de liaison) la figure de la masse fluide à rester ellipsoïdale, c'est-à-dire si l'on assujettit tous les ξ_i à êtres nuls sauf ξ_3 et ξ_4.

J'ai exposé en même temps la condition de la stabilité, qui avec notre notation actuelle s'écrit

$$W - W_0 - \frac{\omega_0^2}{2J}(J - J_0)^2 < 0,$$

ou comme il s'agit de petites déformations

$$W - W_0 - \frac{\omega_0^2}{2J_0}(J - J_0)^2 < 0.$$

En supposant tous les ξ_i nuls sauf ξ_3 et ξ_4, et remplaçant $W - W_0$ et $J - J_0$ par leurs valeurs, nous trouvons

$$G_3\xi_3^2 + G_4\xi_4^2 - \frac{\omega_0^2}{2J_0}(\gamma_3\xi_3 + \gamma_4\xi_4)^2 < 0,$$

ce qui entraîne l'inégalité

$$\left(G_3 - \frac{\omega_0^2}{2J_0}\gamma_3^2\right)\left(G_4 - \frac{\omega_0^2}{2J_0}\gamma_4^2\right) > \frac{\omega_0^4}{4J_0^2}\gamma_3^2\gamma_4^2.$$

Comme ω_0^2, J_0, et G_3 sont positifs et G_4 négatif, l'inégalité change de sens quand on la divise par $\frac{\omega_0^4}{J_0^2} G_3\, G_4$, ce qui donne

$$\left(\frac{J_0}{\omega_0^2} - \frac{\gamma_3^2}{2G_3}\right)\left(\frac{J_0}{\omega_0^2} - \frac{\gamma_4^2}{2G_4}\right) < \frac{\gamma_3^2\gamma_4^2}{4G_3G_4}$$

ou

$$\frac{J_0}{\omega_0^2}\left(\frac{J_0}{\omega_0^2} - \frac{\gamma_3^2}{2G_3} - \frac{\gamma_4^2}{2G_4}\right) < 0$$

ou enfin

$$Y < 0.$$

Passons à la détermination de signe de T. Pour cela nous allons envisager le coëfficient G_5 pour un ellipsoïde de JACOBI très peu différent du Jacobien critique.

Soit E'_0 cet ellipsoïde, et $\frac{1}{2}\omega_0^2 + \epsilon$ la valeur de $\frac{1}{2}\omega^2$ correspondante. Nous pourrons considérer ϵ comme définissant l'ellipsoïde E'_0 ; nous supposons ϵ très petit.

Soit ensuite S une surface peu différente de E_0 et de E_0'. Soit $d\sigma'$ un élément de la surface de E'_0, et l' la quantité qui joue par rapport à E'_0 et à $d\sigma'$ le même rôle que l par rapport à E_0 et à $d\sigma$. Par les différents points de $d\sigma'$ je mène des courbes normales aux ellipsoïdes homofocaux à E'_0, et je les prolonge jusqu'à leur rencontre avec S. Soit dv' le petit volume ainsi formé. Je supposerai que la surface S ait été définie de telle sorte que l'on ait

$$dv'/d\sigma' = \eta l' M_5^* N_5^* ;$$

η étant un coëfficient constant très petit, M_5^* et N_5^* les fonctions qui jouent par rapport à E'_0 le même rôle que M_5 et N_5 par rapport à E_0.

Soit maintenant $d\sigma$ un élément de E_0, par $d\sigma$ menons des courbes $\mu = \text{const.}$ $\nu = \text{const.}$ prolongées jusqu'à S, et soit $d\sigma$ le petit volume ainsi engendré. Nous poserons

$$dv/d\sigma = l\Sigma\xi_i M_i N_i$$

de sorte que les coëfficients ξ_i pourront servir à définir la forme de S. Il est clair que les ξ_i sont des fonctions de ϵ et de η, développables suivant les puissances de ϵ et de η.

Pour $\epsilon = 0$, l'ellipsoïde E'_1 se réduit à E_0, et tous les ξ_i s'annullent à l'exception de ξ_5, qui se réduit à η.

Pour $\eta = 0$, la surface S se réduit à E_0' ; alors ξ_3 et ξ_4 sont des quantités du premier ordre, ϵ étant regardé comme de premier ordre, tandis que les autres ξ_i seront du deuxième ordre.

Si donc ϵ et η sont regardées comme des quantités du premier ordre, ξ_i (en excluant les valeurs $i = 3,\ 4,\ 5$) sera du deuxième ordre, parce que tous ses termes contiendront en facteur soit ϵ^2, soit $\epsilon\eta$; ξ_3 et ξ_4 se réduiront à $\frac{d\xi_3}{d\epsilon}\epsilon$ et à $\frac{d\xi_4}{d\epsilon}\epsilon$ à des quantités près du deuxième ordre; ξ_5 se réduira à η à des quantités près du deuxième ordre.

Nous devons calculer $W + \epsilon J$.

Dans ce qui va suivre, nous négligerons les quantités du quatrième ordre et en plus ϵ^3 et $\epsilon^2\eta$. Dans ces conditions nous pouvons négliger d'abord tous les monômes du quatrième ordre par rapport aux ξ, et arrêter le développement de W suivant les puissances des ξ au troisième ordre inclusivement. Nous pouvons également négliger les monômes du troisième ordre multipliés par ϵ, et par conséquent arrêter le développement de ϵJ suivant les puissances des ξ au deuxième ordre inclusivement.

Nous négligerons en outre : les ξ_i^2 $(i \neq 3, 4, 5)$ qui sont du quatrième ordre ; les monômes du troisième ordre en ξ_3 et ξ_4 qui sont au quatrième ordre près égaux à un multiple de ϵ^3 ; les termes en $\xi_i\ \xi_k\ \xi_j$ $(i \neq 3,\ 4,\ 5\ ;\ k,\ j = 3,\ 4,\ 5)$, qui sont du quatrième ordre ; les termes en $\epsilon\xi_i$ $(i \neq 3, 4, 5)$, qui pourraient figurer dans ϵJ, parce qu'ils contiennent ϵ^2 en facteur et par conséquent sont, au quatrième ordre près, égaux à un multiple de ϵ^3 plus un multiple de $\epsilon^2\eta$.

Dans ces conditions nous devons conserver les termes suivants :

dans W : $$W = W_0 + G_3\xi_3^2 + G_4\xi_4^2 + Q_3\xi_5^2\xi_3 + Q_4\xi_5^2\xi_4,$$

dans ϵJ : $$\epsilon J = \epsilon J_0 + \epsilon\gamma_0\xi_5^2 + \epsilon\gamma_3\xi_3 + \epsilon\gamma_4\xi_4,$$

d'où :

$$W + \epsilon J = (W_0 + \epsilon J_0) + G_3\xi_3^2 + G_4\xi_4^2 + Q_3\xi_5^2\xi_3 + Q_4\xi_5^2\xi_4 + \epsilon\gamma_0\xi_5^2 + \epsilon\gamma_3\xi_3 + \epsilon\gamma_4\xi_4.$$

Pour $\eta = 0$, cette expression se réduit à

$$W_0' = (W_0 + \epsilon J_0) + G_3\xi_3^2 + G_4\xi_4^2 + \epsilon\gamma_3\xi_3 + \epsilon\gamma_4\xi_4,$$

et ses dérivées doivent s'annuler, puisque le Jacobien est une figure d'équilibre. On aura donc :

$$2G_3\xi_3 + \epsilon\gamma_3 = 2G_4\xi_4 + \epsilon\gamma_4 = 0$$

d'où, à des quantités près de l'ordre de ϵ^2, ou de $\epsilon\eta$,

$$\xi_3 = -\frac{\gamma_3}{2G_3}\epsilon, \qquad \xi_4 = -\frac{\gamma_4}{2G_4}\epsilon.$$

Comme ξ_5 se réduit à η, pour $\epsilon = 0$, on aura

$$W + \epsilon J = (W_0 + \epsilon J_0) + \frac{\epsilon^2}{4}\left(\frac{\gamma_3^3}{G_3} + \frac{\gamma_4^2}{G_4}\right) + \epsilon\eta^2\left(\gamma_0 - \frac{Q_3\gamma_3}{2G_3} - \frac{Q_4\gamma_4}{2G_4}\right)$$

en négligeant ϵ^3, $\epsilon^2\eta$, $\epsilon\eta^3$, η^4. En faisant $\eta = 0$ il vient

$$W_0' = (W_0 + \epsilon J_0) - \frac{\epsilon^2}{4}\left(\frac{\gamma_3^2}{G_3} + \frac{\gamma_4^2}{G_4}\right).$$

D'autre part, comme η joue par rapport à E'_1 le même rôle que ξ_5 par rapport à E_0, on aura avec la même approximation

$$W + \epsilon J = W'_0 + G'_5\eta^2.$$

G'_5 étant le coëfficient de stabilité relatif à l'ellipsoïde E'_0 et au "third zonal harmonic."

On aura donc :

$$G'_5 = \epsilon\left(\gamma_0 - \frac{Q_3\gamma_3}{2G_3} - \frac{Q_4\gamma_4}{2G_4}\right) = \epsilon T.$$

Si nous supposons que E'_0 est plus allongé que E_0, ϵ sera négatif, puisque les vitesses de rotation vont en diminuant dans la série des Jacobiens. D'ailleurs G'_5 sera positif, puisque le coëfficient de stabilité a passé du négatif au positif quand on a franchi l'ellipsoïde critique. Donc

$$T < 0.$$

Revenons aux conditions de stabilité de la poire.

Posons

$$X = 2H + 2H' - \Sigma \frac{Q_i^2}{2G_i}.$$

$$Y = \frac{J_0}{\omega_0^2} - \frac{\gamma_3^2}{2G_3} - \frac{\gamma_4^2}{2G_4}.$$

Nous avons trouvé

$$\xi_5^2 X + \epsilon T = 0,$$

ϵ se rapportant à la poire et non plus à E'_0.

$$\frac{\omega J - \omega_0 J_0}{\omega_0 \xi_5^2} = T - \frac{XY}{T} \qquad Y < 0, \quad T < 0.$$

Pour la stabilité, il suffit que ω soit maximum, c'est-à-dire que ϵ soit négatif; il faut et il suffit que ωJ soit minimum, c'est-à-dire que

$$\omega J - \omega_0 J_0 > 0.$$

Or $\epsilon < 0$, équivaut, puisque T est négatif, à

$$X < 0,$$

C'est donc là une condition *suffisante* de la stabilité.

Supposons maintenant $X > 0$; alors T sera négatif, XY/T positif, et $\omega J - \omega_0 J_0$ négatif ; il y aura donc instabilité.

En résumé la condition nécessaire et suffisante pour la stabilité, c'est que

$$X < 0,$$

ou

$$2H + 2H' - \Sigma \frac{Q_i^2}{2G_i} < 0\,,$$

ou

$$2H - \frac{\pi}{2} \Sigma \frac{\beta_i^2 R'_i S''_i}{2n + 1} \Omega_i - \Sigma \frac{Q_i^2}{2G_i} < 0.$$

Si nous observons que R'_i est positif, S''_i négatif, les G_i negatifs sauf G_3, nous verrons que tous les termes du premier membre sont positifs sauf

$$2H \quad \text{et} \quad -Q_3^2/2G_3.$$

Si donc il y a instabilité, c'est-à-dire si l'inégalité précédente n'a pas lieu, il suffira pour le constater de calculer un nombre fini de termes du premier membre. Si au contraire il y a stabilité, on ne pourra s'en assurer qu'en calculant la somme des termes positifs du premier membre qui sont en nombre infini, ou en évaluant une limite supérieure de cette somme.

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By Sir NORMAN LOCKYER, *K.C.B., F.R.S., and others.*

Received May 21,—Read June 20, 1901.

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PART I.—GENERAL ARRANGEMENTS.

By Sir NORMAN LOCKYER *K.C.B., F.R.S.*

Objects of the Expedition.

THE discussion of the series of photographs taken with the prismatic cameras employed in the last three eclipses indicated that continued work with this form of spectroscope should be undertaken, with the view (1) of obtaining data strictly comparable with the previous photographs, and (2) of extending the inquiry into the comparative lengths of the various arcs.

For the first purpose it seemed desirable to repeat the Indian work with the 6-inch camera having two prisms; while for the second an instrument of longer focus was necessary.

Representations as to the importance of the latter instrument were made to the Royal Society, and ultimately the purchase of a Taylor triple lens, of 6 inches aperture and 20 feet focal length, was authorised.

With these instruments it was hoped to obtain a very complete record of the spectra of the chromosphere and corona, and to obtain data relating to the distribution of different substances.

For comparison with the spectroscopic pictures of the corona given by the prismatic cameras, it was considered desirable to attempt to secure direct photographs of the corona with instruments having lenses of focal lengths nearly the same as those of the prismatic cameras, and arrangements were accordingly made to use coronagraphs of 16 feet and 8 feet focal length for this purpose. Other coronagraphs, of shorter focal length, were also provided, in case sufficient assistance should be available to enable them to be used, more particularly with the view of photographing the coronal extensions.

Some importance was also attached to visual telescopic observations of the inner corona, in order to determine whether the filamentary structure observed in the eclipse of 1871, at a time of maximum sun-spots, was also a feature of the inner corona at a time of sun-spot minimum.

A comprehensive programme of observations of the general phenomena of the eclipse was also arranged.

The Observing Station and Preparations.

The observing station selected for my party was determined upon from information supplied by the Hydrographer, Rear-Admiral Sir W. J. L. WHARTON, R.N., K.C.B., F.R.S. Santa Pola appeared likely to meet the requirements of a man-of-war; and without such assistance as a man-of-war can render, the manipulation of long focus prismatic cameras in eclipse observations in a strange country is impracticable.

Santa Pola lies very near the central line of the eclipse, and good anchorage was available, protected from the North and West winds.

Before leaving England, I communicated with Professor FRANCISCO IÑIGUEZ É IÑIGUEZ, Director of the Madrid Observatory, and Mr. JASPER W. CUMMING, H.M. Vice-Consul at Alicante. These gentlemen, together with DON JOSÉ BONMATI MAS, a large landed proprietor, and father of the Mayor of Santa Pola, very kindly made all the necessary preliminary arrangements with the local authorities, who had also been instructed by the Spanish Government, after representations had been made by the Foreign Office, at the request of the Royal Society.

As a result of the Royal Society's application to the Admiralty, H.M.S. "Theseus," commanded by Captain V. A. TISDALL, R.N., was told off to meet the expedition at Gibraltar, and convey the observers to Santa Pola.

The expedition consisted at first of Dr. W. J. S. LOCKYER, from the Solar Physics Observatory; Mr. A. FOWLER, the Demonstrator in Astronomical Physics, from the Royal College of Science; and Mr. HOWARD PAYN, who joined as a volunteer. I subsequently received orders to accompany and take charge of it.

As the interval between the arrival of the expedition at Santa Pola and the day of the eclipse was somewhat short, owing to the dates of sailing of the Orient Line steamers to Gibraltar being once a fortnight, it was considered desirable that someone should go on in advance to select a site for the camp and arrange matters generally with the local authorities, and also find the necessary accommodation for the party.

Mr. PAYN therefore proceeded to Alicante overland, and on his arrival placed himself in communication with Mr. JASPER W. CUMMING, the British Vice-Consul, who had previously been apprised of his mission. Mr. CUMMING afforded every assistance in his power.

From Santa Pola, a small seaside town of about 5000 inhabitants, the shore stretches away nearly due west for many miles in a flat sandy plain covered with low scrub, the open sea being to the south.

After a very cordial welcome by the Mayor, Mr. PAYN went over the sites which had previously been offered for the use of the expedition through the Vice-Consul, and finally selected a spot on the open shore about half a mile west of the town.

The reasons for the selection of the site were that the south and west horizons were unobstructed; that the ground was slightly higher in elevation, and consequently drier; that it was at a sufficient distance from the town to be well clear of the houses and their surroundings; that it was close to a large bathing establishment built on piles in the sea, which could be used as a landing place for boats from the ship and so avoid the town pier (which was some distance away); and also because there was a coast-guard post on the spot, and the men stationed there could keep an eye on the camp until the arrival of the civil guards promised by the authorities.

With the assistance of the municipal authorities, Mr. PAYN was enabled to make arrangements for the supply of workmen and building materials for the foundations of the stands for the instruments, and also to arrange for the landing and transport of the packages, some eighty in all.

Facilities were also given for the landing of all articles for the camp, duty free, and without examination, on handing in a list to the Director of Customs. Authority was also obtained for keeping the telegraph station open day and night during the stay of the expedition; indeed, nothing could exceed the kindness of the authorities, who were obviously anxious to afford every assistance in their power.

As there were no bricks to be obtained in the town, they were ordered by telephone from Alicante, and were brought over the next day.

After Mr. CUMMING had left for Alicante, the Mayor and the local authorities accompanied Mr. PAYN to the camp, and assisted him to set out the limits and the positions of the various instruments, the Mayor good humouredly driving the first peg. A meridian line was set out roughly, and by dark all the measurements were completed. A good many inhabitants came out from the town to witness the rather unusual sight of the chief authorities engaged in manual labour.

The meridian line was checked the same night by an observation of the Pole Star. When this line was afterwards tested with the ship's instruments by Mr. ANDREWS, the Navigating Lieutenant of the "Theseus," it was found to be correct. The local deviation was 14° west.

The other observers, who had left England on the 11th of May by R.M.S. "Oruba," of the Orient Line, on arriving at Gibraltar, at once went on board H.M.S. "Theseus," and left for Santa Pola, which was reached just before noon the following day, May 17. I was glad to find that great interest had been shown in the expedition on board before our arrival, and that lectures on the work to be undertaken had already been given by the Chaplain, the Rev. G. BROOKE-ROBINSON, M.A.

Assistants were at once forthcoming to take part in working the prismatic cameras, and also for manipulating several cameras which I had brought out to be used by the ship's company in obtaining photographs of the corona. Observing parties in charge of officers of the ship, to make observations along several lines, were at the same time organised.

On our arrival at Santa Pola, the following local officials came on board with Mr. PAYN:—SRS. FRANCISCO BONMATI MAS, Mayor of Santa Pola; ANTOINE BONMATI MAS, Vice-Mayor of Santa Pola; JOSÉ BONMATI MAS, Municipal Councillor; JOSÉ SALINAS PEREZ, Municipal Councillor; ELADIO PONCE DE LEON, Secretary to the Mayor; MICHEL SEMPERE, Justice of the Peace; JOSÉ HERNANDEZ, Captain of the Port; GERONIMO AGNATI, Administrator of Customs; EDUARD FERNANDEZ, 1st Lieutenant of Coast Guards; TOMAS BUENO, Medical Officer.

Work on the piers for the instruments was commenced on the day of arrival. The erection of the instruments, huts, and tents was commenced on the following morning, May 18, and by the evening of May 21 the principal instruments were reported in approximate adjustment. Drills were begun on May 22, and were carried on several times a day up to the day of the eclipse.

By permission of the Captain, three of the officers of the " Theseus," Lieutenants Andrews, R.N., Doughty, R.N., and Pattrick, R.N., occupied quarters on shore to superintend the work of the parties in the camp. On board the Chaplain gave instructions in sketching coronas and recording stars, using for this purpose a lantern which had been placed at the disposal of the expedition by the Orient Steam Navigation Company.

The weather was very favourable for the work of the expedition, but at times the landing and embarking of parties from the ship was rendered difficult by strong sea breezes and the consequent surf.

Both day and night the instruments were carefully guarded by a detachment of " Guardias Civiles," told off for the purpose by the Spanish authorities.

The distribution of the various instruments is shown in the accompanying plan (fig. 1).

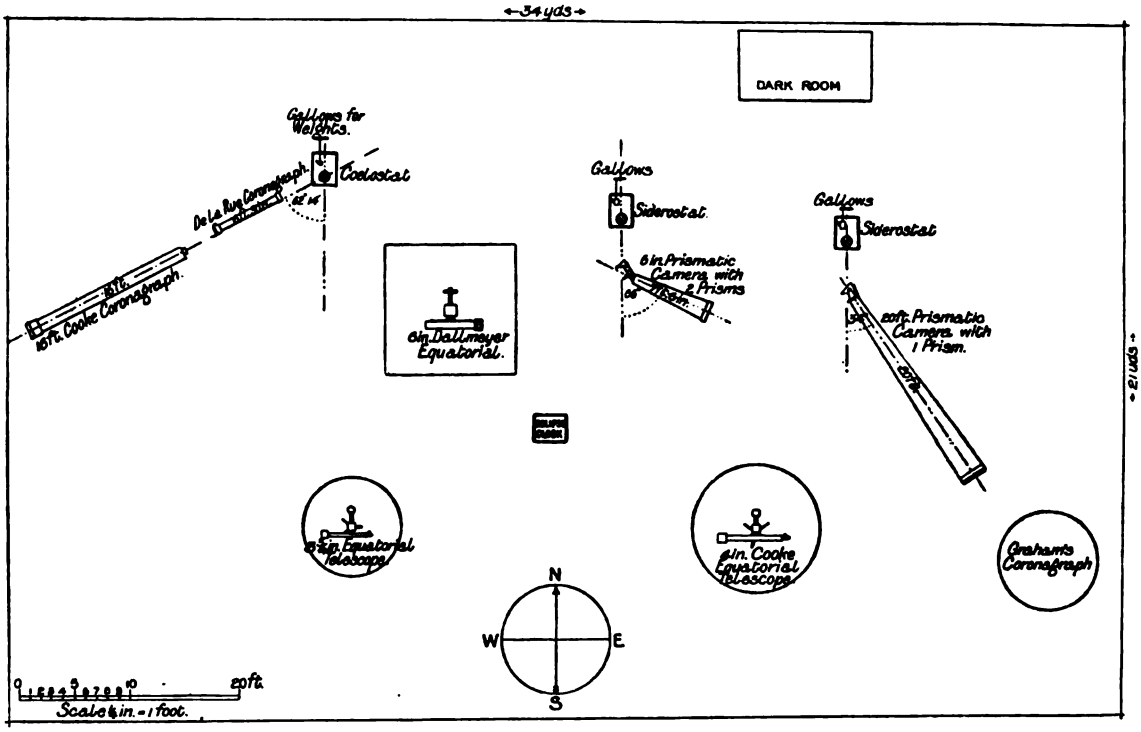

←34 yds→
DARK ROOM
Gallows for Weights.
Coelostat
De La Rue Coronagraph.
62° 14′
16 ft.
16 ft. Cooke Coronagraph.
6 in. Dallmeyer Equatorial.
Gallows
Siderostat.
6 in. Prismatic Camera with 2 Prisms
7½ in.
Gallows
Siderostat
20 ft. Prismatic Camera with 1 Prism.
20 ft.
←21 yds→
ECLIPSE CLOCK
3½ in. Equatorial Telescope
6 in. Cooke Equatorial Telescope
Graham's Coronagraph
N
W
E
S
0 1 2 3 4 5 6 7 8 9 10 20 ft.
Scale ¼ in. = 1 foot.

Fig. 1. Plan of Eclipse Camp at Santa Pola, May, 1900.

Local Conditions of Eclipse.

According to the Admiralty chart, the latitude and longitude of the place of observation are 38° 11′ 20″ N. and 0° 33′·66 W. respectively. For this point, the times and position angles of contact derived from the formulæ given in the ‘Nautical Almanac Circular,’ No. 17, were as follows :—

Beginning of totality, May 28 4h. 12m. 51·7s. G.M.T.
End of totality, May 28 4h. 14m. 10·5s.
Duration of totality, 1m. 18·8s.
Position angle of first contact, 87° 3′·5 from N. towards W.
„ „ last „ 93° 47′·3 „ N. „ E.

The experience of the Indian Eclipse of 1898 suggested that the duration of totality given was too long, and for the practical working during the eclipse the adopted time was 75 seconds, so that there would be no chance of spoiling the coronagraph plates by exposing them after totality. The face of the eclipse clock was graduated accordingly.

The sun’s altitude at mid-totality was calculated to be 33° 23′, and the amplitude 2° 25′ north of west. The apparent semi-diameter of the sun and moon were respectively 15′ 48″·1 and 16′ 5″·9, and the relative motion per second 0″·447. At mid-totality the north point of the sun’s disc, direct view, was 57° 44′ to the right of the vertex, and as the sun’s axis was 17° west of the north point, the sun’s north pole was 74° 44′ to the right of the vertex. The heliographic latitude of the centre of the sun’s disc being −0° 56′, the direct view was as represented in the accompanying diagram; the points of contact with reference to the sun’s axis are also shown, and for 2nd and 3rd contacts they also represent very nearly the disposition with regard to the vertex.

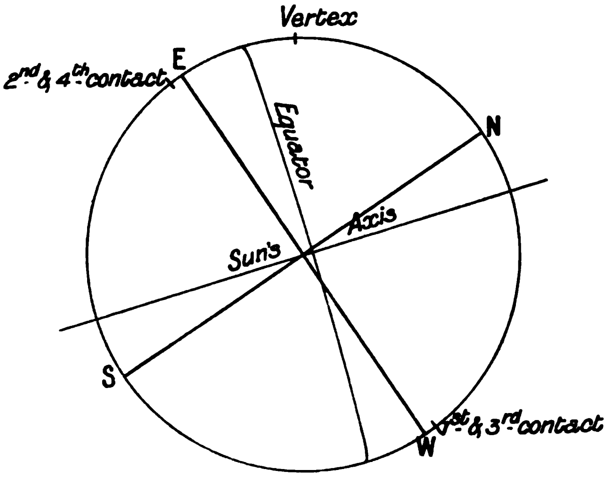


Fig. 2. Illustrating direction of sun’s axis at time of totality, and position angles of contacts.

Time Arrangements.

The arrangements for the time service were similar to those made for the Indian eclipse of 1898. Greenwich time was ascertained by reference to the ship’s chronometers, and for the count of time during totality the eclipse clock, which has been described in my previous reports on eclipse expeditions, was utilised.

The arrangements for securing signals at definite intervals before totality were identical with that employed in Lapland and India. An image of the sun projected by the finder of the 6-inch two-prism prismatic camera was viewed on an adjustable screen, marked in such a way that it was easy to see when the cusps subtended angles of 90° and 55°, which occurred respectively at 16 seconds and 5 seconds before totality. The signals "Go" at the commencement of totality, and "Over" at the end, were given by myself from observations made with the 4-inch Cooke telescope.

The complete system of signals was as follows:—

1. "Rise up," 10 minutes before totality—

    Observers turn backs to sun.
    Clocks to be wound.
    Stops and caps of telescopes, siderostats, and cœlostats to be removed.
    Eclipse clock to be set.

2. "Alert," 5 minutes before totality—

    Disc observers to be blindfolded.
    Observers report all in readiness.

3. "G G," 16 seconds before totality.
4. "G," 5 seconds before totality.
5. "Go," beginning of totality.
6. "Over," end of totality.

At the eclipse clock two men were stationed, one calling the number of seconds remaining up to 30, and the other during the remainder of totality.

In consequence of the perfect drill during the rehearsals, the operations during the eclipse were carried out with great precision.

Acknowledgments of Assistance.

The thanks of the expedition are due especially to those named in the foregoing account, not only for assistance rendered, but also for their great kindness to us. I have already, in a letter, expressed to the Royal Society my deep sense of obligation they have laid us under.

As in the case of the "Volage" and "Melpomene," the officers and men of the "Theseus" not only assisted us with certain instruments, but organised crews for others, and many lines of work which it was impossible for the observers sent out from England to attempt. Their skill, resourcefulness, and steadiness were alike truly admirable.

Thanks are also due to the Managers of the Orient Steam Navigation Company, who conveyed the instruments to and from Gibraltar freight free.

To the Mayor of Santa Pola the whole expedition owes a debt of gratitude for his unwearying kindness. He was accessible at all times, always ready to afford us assistance, and spared no trouble to make things easy and pleasant for all. He was kind enough to open an eclipse account, and all payments to be made locally were made through him, and so useful was this that the stores for the ship were afterwards obtained in the same way; it was found to be not only a check upon the prices, but on the qualities and quantity of the goods. This alone involved considerable labour to him, but it was of the greatest assistance to us. One of our pleasantest recollections of Santa Pola will be the kindness and hospitality of Don FRANCISCO BONMATI. I may add, the Civil Governor of the Province of Alicante, Señor Don HIPOLDO CARAS Y GOMEZ DE ANDINO, visited the camp to assure himself that all the assistance the Spanish authorities could give had been rendered.

PART II.—OBSERVATIONS MADE BY THE OFFICERS AND MEN OF H.M.S. "THESEUS."

(*Forwarded by* Captain V. A. TISDALL, R.N.)

Preliminary Work.

The Chaplain of the "Theseus," the Rev. G. BROOKE-ROBINSON, who undertook to give preparatory courses of instruction to the men, has prepared the following statement:—

"On learning that H.M.S. 'Theseus' was ordered to take a party of astronomers to Santa Pola for the purpose of observing the total eclipse of the sun, May 28, 1900, arrangements were made for the delivery of preliminary lectures on board, previous to the arrival of the eclipse party.

"Lectures were given on the 6th, 7th, and 13th May. The blackboard was used with great advantage on all three occasions. The duration of each lecture was about an hour and a half.

"When Sir NORMAN LOCKYER arrived, he desired that the lectures should continue, giving instructions as to the sort of work to be done preparatory to the eclipse. Three parties of observers were to be trained, a magic lantern, slides, and star charts being provided by the astronomical party. The three sets to be trained were divided as follows:—

Set A. The disc party.
,, B. The corona sketching party.
,, C. The star chart party.

"Set A began their preliminary training on board after erecting the disc poles on shore. They joined the corona sketching party already at work in the submerged torpedo flat where the magic lantern slides were being shown.

"Commencing with 3 minutes in which to sketch a corona, the time was reduced to 2 minutes, then to one minute and a half, and finally to one minute and a quarter, the actual time of totality.

"After a few lectures on board, they returned ashore to complete their training under Lieutenant DOUGHTY and Mr. DANIELS, and to learn to what use the discs were to be put.

"Set B. The corona party continued under instruction on board from Friday, May 18, to Friday, May 25. On Saturday one-half went ashore to complete their drills under Lieutenant DOUGHTY. While on board they, too, had been trained to sketch against time. Three minutes was the time allowed for a sketch at the outset; it was gradually reduced to one and a quarter minutes.

"Set C were selected from amongst those who showed the most aptitude for star charting. Their attention was directed to the importance of accurately noting distance and direction.

"A diagram of constellations to be expected at the time of the eclipse was placed in the lantern, and then repeatedly sketched as it appeared on the sheet.

"From 8.30 P.M. to 9.30 P.M. star charting was tried on deck. The constellation Ursa Major, 'the pointers' and their distance from Polaris forming the preliminary lesson. When proficiency in setting down the seven brighter stars of this constellation was attained, the more intricate work of identifying the constellations, shown in the star maps of the overhead sky, supplied by Sir NORMAN LOCKYER, was proceeded with.

"I wish to lay special stress upon the difficulty I found in training a class to deal with a map upon which 'the line parallel to the horizon' was drawn across one of its angles. I would suggest that in future maps supplied for this class of work have 'the line parallel to the horizon' placed parallel to the lower edge of the map. I think it would be well to omit the cardinal points round the solar disc, since I found they tended to confuse observers who were in an elementary stage of training, and that the reason of their not being shown like a compass card required careful and repeated explanations.

"I recommend that only planets and stars of the first three or four magnitudes be shown on future maps.

"I find that Neptune shown with a large symbol has conveyed the idea that a large body was to be expected in that direction, whereas Neptune was not visible at all at the time of the eclipse.

"Mr. BENNETT, Clerk, was of very great assistance. He gave much valuable aid in the training of the star chart party ashore.

"A. PHILLIPS, Leading Shipwright, did good service in preparing extra tracings of star charts. All his work was noteworthy for its accuracy and extreme neatness. The first tracing he took was submitted to Sir NORMAN LOCKYER, who described it as excellent."

Diary of the Expedition.

A careful diary of the expedition was kept by Midshipman LAMBERT from the time of the arrival of the expedition on board the "Theseus" to their leaving the ship at Gibraltar on the return journey.

The substance of this is as follows :—

Wednesday, May 16.—H.M.S. "Theseus" left Gibraltar at 11 A.M. with the following observers on board : Sir NORMAN LOCKYER, K.C.B., F.R.S., &c., Dr. RALPH COPELAND, Dr. W. J. S. LOCKYER, Mr. A. FOWLER, Mr. T. HEATH, and Mr. FRANKLIN ADAMS. Mr. WYLLIE, A.R.A., also accompanied the expedition.

Thursday, May 17.—Arrived at Santa Pola at 11.15 A.M. Mr. HOWARD PAYN, who had gone in advance by the overland route, came on board with the local authorities, and reported the arrangements made; a party landed with Mr. FOWLER and Dr. LOCKYER in the afternoon, and the site which had been selected by Mr. PAYN was approved. The meridian line laid down by Mr. PAYN was confirmed by Lieutenant ANDREWS, R.N., with a large azimuth compass. The foundation for one of the siderostats was built.

Friday, May 18.—Landed gear. Set up brick piers and some of the instruments.

Saturday, May 19.—Setting up instruments and discs.

Sunday, May 20.—Setting up and adjusting instruments.

Monday, May 21.—Adjusting instruments. Parties were told off for each instrument. At 6 P.M. the chief instruments were reported in approximate adjustment.

Tuesday, May 22.—Commenced drills. In the forenoon drilled coronagraphs and prismatic cameras with eclipse clock. In the afternoon drilled coronagraphs, prismatic cameras, and disc parties. Screens were erected for the observation of shadow bands. In the evening photographs of stellar spectra were taken for focussing the prismatic cameras.

Wednesday, May 23.—In the forenoon the instruments were drilled individually. Mr. FOWLER gave a short lesson to the disc party. In the afternoon the coronagraphs were drilled, and in the case of each instrument one trial plate was exposed to test focus. At eclipse time there was a full rehearsal of all parties.

The meteorological house was erected, and three thermometers and a Watkin aneroid were set up.

No stars visible in the evening.

Thursday, May 24.—Meteorological observations were commenced. At 2.30, drilled disc party. Rehearsals at 3.15, and at eclipse time. At 4.30, drilled disc party. From 8.30 to 11, star trails were photographed with the four coronagraphs, and in each case the focus was found satisfactory. Tried spectrum photograph with Dr. LOCKYER's instrument, but exposure was interrupted by clouds.

Friday, May 25.—At 2.20, drilled disc party. At 3.0 and eclipse time general drills. A photograph of the spectrum of Arcturus was taken with Mr. FOWLER's instrument.

Saturday, May 26.—At 11.45 general drill. Señor DON HIPOLDO CARAS Y GOMEZ DE ANDINO, Civil Governor of the Province, visited the camp. At 3.30, general drill, and full rehearsal at eclipse time. French astronomers from Elche visited the camp. At 5.5 the French astronomers, Spanish Commission, and Governor visited the camp and witnessed a full rehearsal. Two photographs of the spectrum of Arcturus were taken with Mr. FOWLER's instrument.

Sunday, May 27.—General drill at 4.45 P.M. Three photographs of spectrum of Arcturus taken with Mr. FOWLER's instrument.

All plate holders loaded in readiness for eclipse.

Monday, May 28.—Full rehearsal, without plate-holders, at 10.30. The eclipse was observed under perfect conditions, and all operations successfully performed.

A crowd of over 2000 of the inhabitants collected round the camp to watch the eclipse. At 5 P.M. a photograph of the corona, taken with the De la Rue coronagraph, was successfully developed. At 8 P.M. photographs numbers 5 and 10, taken with the two prismatic cameras, were successfully developed.

Tuesday, May 29.—Commenced dismounting and packing instruments. Mr. WYLLIE and Dr. LOCKYER left Santa Pola. Mr. FOWLER made paper prints and glass positives of spectra developed yesterday. In the evening Mr. FOWLER developed photographs numbers 1 and 2 taken by Mr. PAYN with the long focus coronagraph; also spectra, numbers 1 and 2, taken by Dr. LOCKYER, and number 1 taken by Mr. FOWLER.

Wednesday, May 30.—Made glass positives of photograph taken with De la Rue coronagraph, and those taken by Mr. PAYN. Finished packing instruments, negatives, and undeveloped plates. Everything on board except dark room.

Thursday, May 31.—Posted box containing copies of photographs. Settled accounts with local people. Observers returned to the "Theseus," which left for Gibraltar at 8.30 P.M.

Friday, June 1.—At sea.

Saturday, June 2.—Arrived at Gibraltar 5 A.M.

The ship was anchored outside the Mole, and in consequence of the roughness of the sea it was impossible to transfer the instruments to the lighter which was sent for them. The observers also remained on board.

Sunday, June 3.—By permission of the Commander-in-Chief, the "Theseus" proceeded inside the Mole, and the instruments were put into the lighter. At 9.30 A.M. the party left the ship and took up their quarters at the Royal Hotel. Left Gibraltar, per R.M.S. "Cuzco," at 6 P.M. Wednesday, June 6.

The groups of observers were as follows :—

Timekeepers.

Lieutenant F. A. Andrews, R.N.
Mr. Boughey, Midshipman.
Mr. Lambert, Midshipman.
J. Wale, 2nd Yeoman Signals.
W. Webb, Petty Officer, 1st class.
Bugler Sneller, Ordinary Seaman.

6-inch Prismatic Camera.

Dr. Lockyer.
S. Birley, E.R.A.
J. Green, A.B.
C. Fishenden, Ordinary Seaman.
C. Willmott, Ordinary Seaman.
A. Humphries, Ordinary Seaman.
G. Hyatt, Ordinary Seaman.

20-foot Prismatic Camera.

Mr. Fowler.
W. F. Cox, Armourer.
A. Whitbourne, A.B.
F. Burt, A.B.
A. Maskell, A.B.
E. Davies, Ordinary Seaman.
H. Cristopher, Ordinary Seaman.
W. Harrison, Stoker Mechanic.

4-inch Equatorial.

Sir Norman Lockyer, K.C.B.
C. C. Lambert, Midshipman.

3¾-inch Equatorial.

Lieutenant H. M. Doughty, R.N.
A. G. N. Lane, Midshipman.

Long-focus Coronagraph.

Mr. Payn.
T. McGowan, A.B.
E. Woodland, A.B.
H. Eary, A.B.
W. Mann, Ordinary Seaman.
H. Brooks, Ordinary Seaman.

Graham Coronagraph.

Mr. W. J. S. Perkins, Assistant Engineer, R.N.
W. Walker, Leading Stoker.
J. Knowles, Chief Stoker.

De la Rue Coronagraph.

Mr. H. W. Portch, Assistant Engineer, R.N.
W. Waterfield, E.R.A.
H. Frost, Chief Stoker.

Dallmeyer Coronagraph.

Surgeon J. Martin, R.N.
E. Buckingham, E.R.A.
R. Quint, Chief Stoker.

Discs.

Mr. J. B. Bateman, Midshipman, R.N.
W. Fraser, Arm. Crew.
R. S. Bradbrooke, A.B.

Mr. J. A. Daniels, Torpedo Gunner, R.N.
G. Fair, Armourer.
E. Gordon, Ship's Carpenter.

H. W. Richardson, Petty Officer, 2nd class.
E. Voyle, Leading Shipwright.
T. Orange, Boy, 1st class.

W. Tucker, A.B.
W. Brewer, A.B.
B. Salmon, Boy, 1st class.

A. Mason, A.B.
A. Steven, A.B.
C. Paul, Boy, 1st class.

A. May, A.B.
H. Bailey, A.B.
J. Entwistle, Ship Steward's Boy.

Sketches of Corona without Discs (on shore).

W. Butt, M.A.A.
G. Guilliame, A.B.
H. Meacher, Private, R.M.L.I.
H. Schmidtœl, Ordinary Seaman.

Sketches of Corona without Discs (on board).

W. Baxter, A.B.
W. Butts, Private, R.M.L.I.
C. Jacob, Private, R.M.L.I.
J. Wheeler, Private, R.M.L.I.
E. Willis, Sick Berth Attendant.

Observations on Stars (on shore).

Mr. Bennett, Clerk.
W. Riches, Leading Seaman.
A. Pontifex, A.B.
W. Bosworth, A.B.
H. Angus, Ordinary Seaman.
W. Kinvett, Private, R.M.L.I.
W. Oliver, Private, R.M.L.I.

Observations on Stars (on board).

Rev. G. B. Robinson, M.A.
H. Croxon, Ship's Corporal.
A. Phillips, Leading Shipwright.
R. Vigus, Corporal, R.M.L.I.
E. Price, Private, R.M.L.I.
E. Hammond, Stoker.
G. Andrews, Stoker.
G. Nightingale, Stoker.
S. Wilson, Stoker.
E. Savage, Private, R.M.L.I.

Observations of Shadow Bands (on shore).

Commander Hon. R. F. Boyle, R.N.
Mr. T. Slator, Naval Instructor, R.N.
Mr. J. G. Walsh, Midshipman, R.N.
Mr. F. C. Skinner, Midshipman, R.N.

Meteorological Observations (on shore).

Lieutenant Pattrick, R.N.
Mr. G. S. Hallowes, Midshipman, R.N.

Meteorological Observations (on board).

G. Donnelly, Yeoman Signaller.
E. Gant, Leading Signaller.
A. Enstidge, Signaller.
W. Hearne, Signaller.
J. Beach, Signaller.

Meteorological Observations (Wind, &c.).

G. Perrin, Leading Stoker.
H. Clackett, Stoker.
T. W. Empson, Stoker.
G. Guy, Stoker.
J. Wignell, Stoker.

Landscape Colours (on shore).

Captain F. V. Whitmarsh, R.M.L.I.
Ship's Steward D. Green
Lance-Corporal Wade, R.M.L.I.
W. Birkett, Writer.

Landscape Colours (on board).

Fleet Paymaster A. W. Askham, R.N.
Lieutenant W. J. Frazer, R.N.

Shadow Phenomena (on shore).

Mr. C. Prynn, Carpenter, R.N.

Shadow Phenomena (on board).

Lieutenant H. R. Shipster, R.N.

Photographers.

J. KNIGHT, Sick Berth Steward. B. BULBROOK, A.B.

Aide-de-Camp to Sir NORMAN LOCKYER, *K.C.B.*, *F.R.S.*
Mr. C. C. LAMBERT, Midshipman, R.N.

As the expedition was on board the "Theseus" for one day only before the eclipse, and as the ship's parties returned to the ship every evening, it was not possible to give instruction in the observation of spectra, and parties for this branch of eclipse work could not therefore be organised.

Assistance in Time-keeping.

Lieutenant ANDREWS, R.N., who assisted in the important duty of time-keeping, has drawn up the following statement of the procedure adopted :—

"A time signal was made daily, at noon, from the ship by the Commander; the error of the chronometer having been ascertained by telegraph on the 16th May, the day we left Gibraltar.

"The deck watch (which was daily compared with the chronometers) was also landed, so that I could give any comparison or time required.

"On the day of the eclipse I gave the time from the deck watch, 10 minutes before totality, on which the 'Rise up' was sounded on the bugle, and 5 minutes before totality, on which the 'Alert' was sounded."

The observations of the cusps to signal intervals of 16 and 5 seconds before totality were also made by Lieutenant ANDREWS, who remarks that the apparatus provided worked most satisfactorily. During the drills, when the cusps were of course not observable, the corresponding signals were given by reference to the deck watch.

The Coronagraphs.

Three coronagraphs were employed by officers and men of H.M.S. "Theseus," particulars of which are appended :—

(1.) The De la Rue coronagraph. Aperture $4\frac{5}{8}$ inches, focal length 8 feet. Assistant Engineer H. W. PORTCH in charge.

The instrument, which had previously been used in Nova Zemlya and India, was fed by a spare part of the cœlostat mirror used for the long-focus coronagraph. Three exposures were made of approximate durations, 40 seconds, 15 seconds, and 0·5 second respectively, the plates employed being "Sandell" triple coated, 6 inches square.

(2.) The Dallmeyer coronagraph. Aperture 6 inches, focal length 54 inches. Surgeon J. MARTIN in charge.

The instrument was mounted equatorially, and provided with an excellent driving clock.

One photograph was taken with an instantaneous exposure at call of 70 from the timekeeper at the eclipse clock, and another from, as soon as the plate could be changed, to the call of 5, the exposure thus being about 60 seconds. Sandell triple-coated plates, 6 inches square, were employed in each case.

(3.) The Graham coronagraph. Aperture 3 inches, focal length 20 inches. Assistant Engineer W. J. S. PERKINS in charge.

This instrument, together with the small cœlostat with which it was used, was loaned to the expedition by the Marquis of GRAHAM. Only one exposure was made during totality—from the call of 70 to that of 5 from the eclipse clock—the exposure being about 60 seconds. Seven additional plates were exposed at half-minute intervals after totality, with the view of ascertaining how long the corona could be photographed after the sun had reappeared. Sandell plates, 3 inches square, were employed throughout.

The exposures were successfully made in each case.

Discs.

Six discs for cutting out the bright light of the inner corona were erected, with the view of enabling the observers to detect the long extensions if there should be any.

The following are particulars relating to the observations :—

Sun's altitude at mid-totality .. .. .. = 33½°.
„ azimuth „ „ .. .. .. = N. 87½° W.
„ semi-diameter 15′ 48″·12.. .. .. = 948″·12 Radius
Disc to cover 3′ round sun .. .. .. = 1128″·12 „
„ „ 6′ „ „ .. .. .. = 1308″·12 „

Example.

Disc No. 1, 6-inch diameter—
Distance from eye to cover 6′ round sun .. = 39½ feet.
Height above eye .. .. = 22 feet 1 inch.
Height above ground .. .. = 26 feet.

With reference to this branch of work, the following statement has been drawn up by Mr. J. A. DANIELS, Gunner, R.N., who superintended the erection of the discs and eye-pieces, adjustments for azimuth and altitude being made by Lieutenant ANDREWS :—

"The eclipse camp being on perfectly level ground, the six discs were fixed up on poles, rough spars from the ship being found suitable for this purpose.

"Owing to the loose sandy nature of the soil, it was found necessary to secure the heels of the poles in casks sunk in the ground, stones and turf being rammed down tight round them, the heads being further secured by four rope stays, taken to pegs about 4 feet long driven into the ground.

" On the poles being set up, steps were fixed for convenience of mounting the poles to place and adjust discs.

"The discs, which were made of wood, varied from 6 to 2 inches in diameter; they were painted a dead black, and were fixed at the ends of brass rods which projected at right angles to the poles. These brass rods were placed at correct height, allowance being made for height of eye. The rods were further secured and stayed to the pole by twine. They were then turned to an angle of 33½° with the vertical, so as to place them at right angles to the line of observation. The correct angle of each disc was obtained by fixing a plumb line to the edge of a triangle which was ruled with pencil at the angle of 33½°.

"*Eye-pieces.*—These in each case consisted of a small piece of sheet brass, with a hole pierced in it of about $\frac{3}{16}$ inch in diameter, which was fitted on the front face of a framework made sufficiently large for a seat for the observer to be placed inside it. The front face was carefully adjusted so as to be parallel to its corresponding disc, and the eye-piece arranged so as to have a movement on its frame, both in altitude and azimuth, for purposes of final adjustment. The correct position of these frames and eye-pieces required a good deal of very careful observation to arrive at. Compass, spirit level, and a large wooden triangle having an angle of 33½° were used for this purpose."

Arrangement of Observers.—The six discs were each worked by three persons, who were told off as Nos. 1, 2, and 3. Their duties were as follows:—

No. 1 to observe the corona and describe to No. 2.

No. 2 to write down the description given by No. 1.

No. 3 to blindfold No. 1, and to lead him to the eye-piece at the correct time, and to repeat time calls from the eclipse clock.

The routine carried out was as follows:—

10 *minutes before totality* (bugle " Alert ")—

Blindfold No. 1; then Nos. 1 and 2 turn their backs to the sun, No. 3 takes the place to be occupied by No. 1, and keeps eye-piece adjusted.

16 *seconds before totality* (bugle 2 G's)—

No. 1 is led to position at eye-piece by No. 3.

5 *seconds before totality* (bugle 1 G).

Order " Go " at totality, and 75 seconds is called from eclipse clock and repeated by No. 3. At 65 being called from clock the bandage was removed from the eyes of No. 1, who looks through eye-piece and describes to No. 2 what he can see of the corona. No. 3 continues to repeat the time called from the eclipse clock, and makes a rough sketch of the corona to assist No. 1, who makes his sketch from his description given to No. 2 when totality is over.

Training of Observers.—To bring the observers to the necessary stage of efficiency a considerable amount of training was required; those of the disc observers who could be spared after the discs were set up were instructed in sketching coronas of former eclipses, illustrated by magic lantern slides, the time allowed for sketching these being gradually decreased from a period of 3 minutes to a period of 1¼ minutes, the expected duration of totality.

On and after Tuesday, May 22, the disc parties were landed and general rehearsals were commenced; the disc observers were drilled in sketching and describing a typical corona outlined on a large piece of cardboard with chalk, the same time being allowed for exposure of the typical corona as the time totality would last. After the sketches were made by the Nos. 1 from the descriptions given to Nos. 2 they were handed in for inspection and very carefully checked in regard to position and length of streamers, the result being that each day showed an improvement.

A variety of methods were tried before it was finally decided as to the best way of sketching coronas. It was found necessary to use abbreviations as much as possible whilst taking down the descriptions. This at first was found to be very confusing, but it was eventually got over by using ruled forms.

The best means found for sketching coronas was to cut a service pistol target in four parts, and use the back on which to sketch. A small disc was painted in the centre, and the card was marked in concentric circles, each increasing by one diameter of the disc already painted. The position of streamers were described in terms of the clock, the direction of streamers by compass bearing, and length of streamers in diameters of the painted disc.

The forms used for taking down descriptions left very little writing for No. 2 to get through, and no difficulty whatever was experienced by the observers in utilizing these notes for the sketches made immediately afterwards. Of the six discs used, four covered a radius of 6 minutes outside the moon's diameter, and the remaining two covered a radius of 3 minutes. Although the 6-minute discs covered much more of the inner corona according to the sketches and descriptions handed in, practically the same results were obtained from both 6-minute and 3-minute discs, allowance being made for the difference in diameter of discs.

The disc observers were personally instructed by Lieutenant H. M. DOUGHTY, R.N., the rehearsals being held four or five times daily. A different sketch was used on each occasion.

The discs were set up and the eye-pieces fixed under the direction of Lieutenants ANDREWS and DOUGHTY, assisted by Mr. HALLOWES, Midshipman.

It has been thought desirable, instead of forwarding all the original sketches, to select the best of each sort, and to make a separate sketch representing the mean of the results in each case. These are—

(1.) With discs covering 6 minutes of arc outside the moon's perimeter.

(2.) With discs covering 3 minutes.

(3.) Free-hand drawings without discs.

The sketches made with discs agreed fairly well in each case. The mean sketches appended have been made with much care from the originals, and in the opinion of others besides myself very fairly represent the general results.

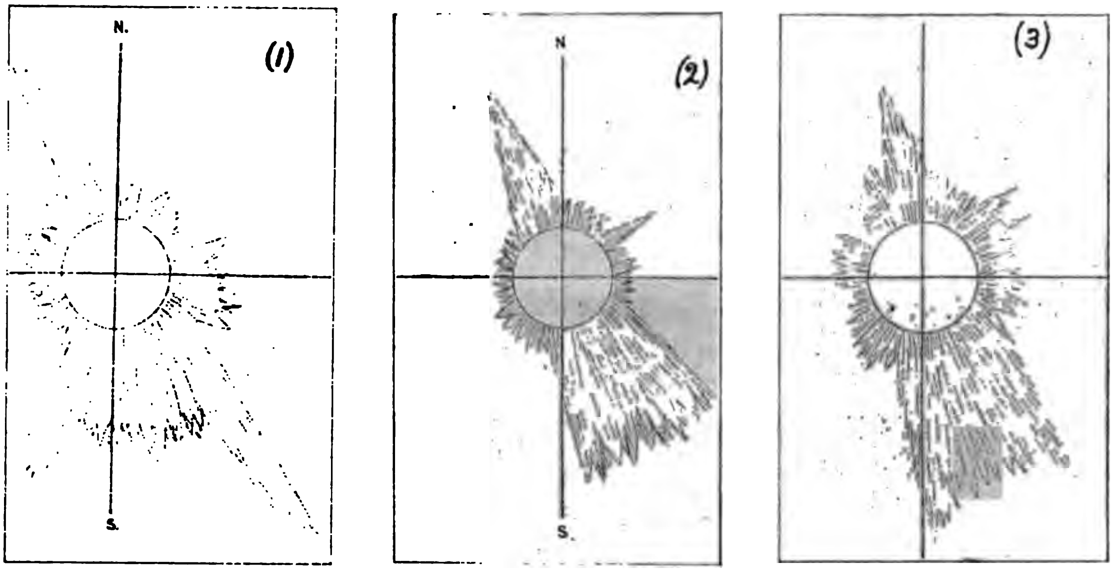


Fig. 3. Sketches of Corona. (1) With discs covering 3 minutes round Sun's limb; (2) with discs covering 6 minutes; (3) without discs.

Sketches of Corona without Discs.

Particulars as to the selection and training of the observers who made drawings of the corona as seen by the unaided eye have already been given.

Observations of Inner Corona with the 3¾-inch Telescope.

A telescope of 3¾ inches aperture, with a magnifying power of thirty-six, mounted on a portable equatorial stand, was employed by Lieutenant DOUGHTY, R.N., in a search for minute coronal structure, such as that observed by Sir NORMAN LOCKYER in the eclipse of 1871. His account of the observations was as follows :—

"For the purposes of description, I propose to make the moon's disc a clock face, 12 o'clock being in the zenith. The corona appeared perfectly white except that the lower portion of the bright ring just after the commencement of totality was of a rosy colour, as also was the upper portion just before the finish of totality. I noticed the following red prominences. Two, straight and radial, very close together, at 12.30 o'clock. Two rather further apart and not so long at about 1.15. One at 4.30. This prominence I lost sight of just before the end of totality. No detailed structure was visible from 1.30 to 4.

"Straight radial lines of bright light were visible outside the bright ring. They appeared to grow out of the ring of bright light and gradually get brighter as they grew, and then as gradually fade. The spaces between these lines started as pale light, and gradually assumed a soft mouse-coloured brown.

"The radial lines visible from 7 to 10.30 were slightly brighter and longer. They were curved in appearance, curling outwards from about 9 o'clock. Round the remaining portions of the moon's disc, outside the bright ring, the light appeared to gradually fade; I could see no dark spaces, and, except those previously mentioned, very few radial lines, these being most noticeable at 10.30 to 11, 7 o'clock and 4.30."

Observations of Stars Visible During Eclipse.

The following account of the preparatory work, and of the observations of stars made during totality, has been prepared by the chaplain, the Rev. G. Brooke-Robinson.

"The party landed for the purpose of star observations consisted of one officer and six men; the men were divided into two sections, each section taking half the heavens; all were provided with maps to assist them in recognising such stars and planets that might appear.

"The arrangements for the party on board were similar to those for the shore party.

"The actual observations of stars during the eclipse were taken during a period extending from 20 minutes before totality to 15 minutes after totality.

"*Venus* showed distinctly throughout the whole period of these observations. A particularly bright body was seen close to the sun at the lower right-hand quadrant; this was taken to be Mercury. The stars α Orionis and α Tauri were both visible during totality. All observations were carried on with the naked eye. No luminous body unmarked in the maps was noted."

Landscape Colours.

Two parties of observers were told off to record the general colour phenomena, one party being stationed on board the "Theseus," and the other on a hill on shore. Fleet Paymaster A. W. Askham has prepared a report on the first set of observations, and Captain F. V. J. S. Whitmarsh, R.M.L.I., has presented a separate report on the shore observations. The following general statement has been combined from these. (See table accompanying.)

Captain Whitmarsh further remarks:—"I did not notice any appreciable difference in the cultivated land in front as regards colour at any time. What I noticed particularly was that the clouds travelled in a northerly direction, and as we neared totality they travelled southwards, and I certainly imagined that a cold breeze came up from a northerly direction as soon as the sun was totally eclipsed. The hills

irection.				Sea.			Observer.
		Befor totalit		Before totality.	During totality.	After totality.	
	On ship.	Light softened to the zon.	n- y e, y	Light slate colour.	Dark blue, with a green tinge.	Light indigo.	Fleet Paymaster, A. W. Askham, R.N. Staff Engineer, F. T. George, R.N. Lieutenant, W. J. Frazer, R.N.
E.	On shore.	Blue . .		Nil.	...	...	Captain Whitmarsh, R.M.L.I.
		3.41 P.M., 4 P.M., ish on zon; 4.12 slate colo		Nil.	Nil.	Nil.	G. Birkett, Writer.
	On ship.	Light softened to the zon, tinge of colour horizon.		Slate colour.	Very dark slate colour, greenish tinge in foreground.	Indigo green tinge in distance.	Fleet Paymaster A. W. Askham, R.N. Staff Engineer F. T. George, R.N. Lieutenant W. J. Frazer, R.N.
E.	On shore.	Yellowish hue on zon, and overhead		Dark blue, and getting still darker blue near totality.	...	...	Captain Whitmarsh, R.M.L.I.
		3.41 P.M., blue; yellow st appeared horizon; changed pink.		3.41, blue. Yellow shadow coming across from S.; 3.57, sea very blue; 4.10, horizon like early dawn.	Dark blue to light blue.	Light blue.	Lance-Corporal Wade, R.M.L.I.
	On ship.	Light blue ened aw the ho slight tin warm c near hori		Slate colour.	Dark blue (indigo), with a green tinge.	Blue, with touches of indigo and cobalt green.	Lieutenant W. J. Frazer, R.N. Fleet Paymaster A. W. Askham, R.N. Staff Engineer F. T. George, R.N.
W		Yellow					

changed their hue from pinkish-brown to dark slate colour. I also noticed that a bird which had been chirupping long before the first contact carried on doing so right through totality; he remained in the same place the whole time. I also noticed a yellow shadow about 6 miles to sea which extended from S.W. to S.E. by S."

Shadow Phenomena.

The moon's shadow was not seen at all, although observers were stationed at the mast-head of the ship at a height of about 80 feet from the water, with careful instructions as to the measurement of its speed, &c., had it been seen. The relative positions of the shore line, the ship, and the island to seaward gave reason to hope for good results; but in this we were disappointed.

Shadow Bands.

Mr. T. SLATOR, B.A., Naval Instructor, who took charge of this section of the observations, presents the following report of the preparatory work and the observations made:—

"Two canvas screens, 18 × 6 feet, were placed vertically, one in the meridian and the other in the prime vertical, on a road crossing the camp, which was as nearly as possible horizontal. The sides of the screens facing the sun, and the 18-foot square included by them on the horizontal plane, were whitewashed.

"Rods 3 feet long were provided to mark the direction in which the bands were travelling on the horizontal plane, and two wooden "T's" were made, with a 3-foot head and a 6-foot handle to secure permanent impressions on the screens.

"The heads were smeared with a mixture of blacklead and tallow, and the long handles enabled the observers to stand at some distance from the screens.

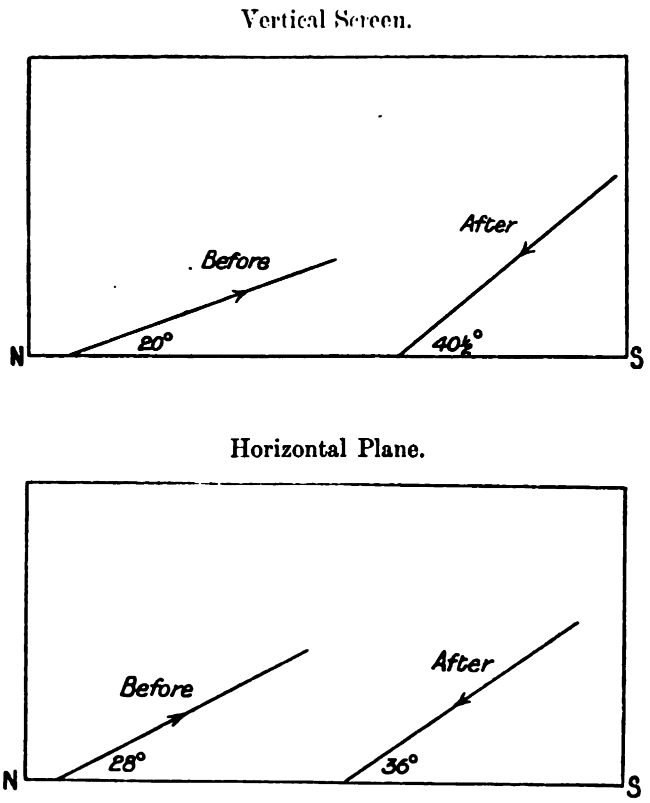
"It was decided to place the rods and mark the screens perpendicular to the directions of the bands, and in the directions in which they were travelling. To make a more accurate estimate of the distance between successive bands the rods and the heads of the 'T's' were painted white at the ends, and the centre foot was painted black.

"Commander Hon. R. F. BOYLE volunteered to mark the screen in the meridian, Mr. J. G. WALSH, Midshipman, was watching the other screen; Mr. F. C. SKINNER, Midshipman, placed the rods on the horizontal plane, and I had a stop watch to note times, and to be used if possible to form an estimate of the speed at which the bands were travelling.

"Unfortunately the bands were very faint and elusive, and on the screen in the prime vertical no shadows were seen at all. Only one mark was made on the other vertical screen before totality began, and this was found to be inclined to the horizontal at an angle of 20°, the bands moving upwards from N. to S. One mark was made after totality, and this was inclined at an angle of 40½° to the horizontal, and the bands were seen to be travelling downwards from S. to N.

"Two rods were placed by Mr. **Skinner** on the horizontal plane, one before and one after totality. Before totality the bands were travelling S. 28 E. and after totality N. 36 W. The bands were first seen at 4h. 12m. 16s. G.M.T., that is, 22 seconds before the beginning of totality, but no reliable estimate was formed as to their width or their speed.

"They were described as being like a mirage, or like the faint rippling on a smooth surface of water when light airs disturb it, and at first it was hard to believe that the slight trembling seen on the screen was not due simply to the shaking of the canvas in the wind."


Vertical Screen.

Horizontal Plane.

Fig. 4. Direction of Shadow Bands.

Contact Observations.

Although observations of the times of the contacts did not form a definite part of the programme of the expedition, it may be useful to state the times recorded. Observations of the first contact were made by Lieutenant **Andrews** and Mr. **Fowler**, while the times of beginning and end of totality were noted from signals given by Sir **Norman Lockyer**.

The times recorded were as follows :—

	G.M.T. h.	m.	s.	Observer.
1st contact . . .	2 2	58 59	30 0	FOWLER (telescopic view). ANDREWS (projected image).
2nd " . . .	4	12	38	LOCKYER.
3rd " . . .	4	13	53	LOCKYER.
4th " . . .	Not observed.			

Meteorological Observations.

The following report on the meteorological observations made at the Eclipse Camp and on board the "Theseus" from the 25th to the 29th May, has been prepared by Lieutenant PATTRICK, R.N.

The results of the observations are given in four separate tables, instead of in their original form, so that comparisons of the data for different days can readily be made. A graphical representation of the temperature observations on the day of eclipse and the preceding and following day is also given.

On Thursday, May 24, a shelter was erected, in the Eclipse Camp, at Santa Pola, for the meteorological instruments. The shelter was about 7 feet square, the sides being made of three thicknesses of bunting, with a foot space between the bottom and the ground, to allow a free current of air. The height was about 7 feet, the roof being of canvas, whitewashed on the outside.

Inside two posts were planted, with a cross-bar between, to which the instruments were hung, viz. :—A "Watkins" aneroid, and three Centigrade thermometers. They were about 3 feet 6 inches from the ground, and suspended from the cross-bar with twine.

The observations were commenced at noon on May 25, and were all taken by Mr. HALLOWES, Midshipman, and myself. A copy of the observations is appended.

It will be noticed that for 3 days before the eclipse the barometer was more or less steady, there being a slight rise till midnight of May 27, after which it fell steadily till the time of totality, when it rose again slightly for 10 minutes, then continued falling till 6 P.M., after which it was unsteady for some hours. The temperature usually rose till noon, and remained the same till about 3 P.M., when it would fall gradually.

On the 28th, from the time of first contact, the thermometers dropped much faster than usual till 10 minutes after totality, falling from 24°·5 C. to 19°·8 C.—a total drop of 4°·7 C. After this it again rose steadily till 5.45 P.M., when it reached its normal height for the time of day.

The wind, which was always from the sea during the afternoon, and usually steady as regards direction and force, became very fitful and uncertain in strength, though constant in direction, for about half-an-hour at totality. This may indicate a current of air from the opposite direction, which was not strong enough to overcome the regular sea breeze prevailing at the time.

Meteorological observations were also kept on board the ship, by the signal staff. These, which are also appended, are nearly identical, as regards the variation of barometer and thermometers, with those taken at the camp.

The more important observations are also illustrated graphically in figs. 5 and 6.

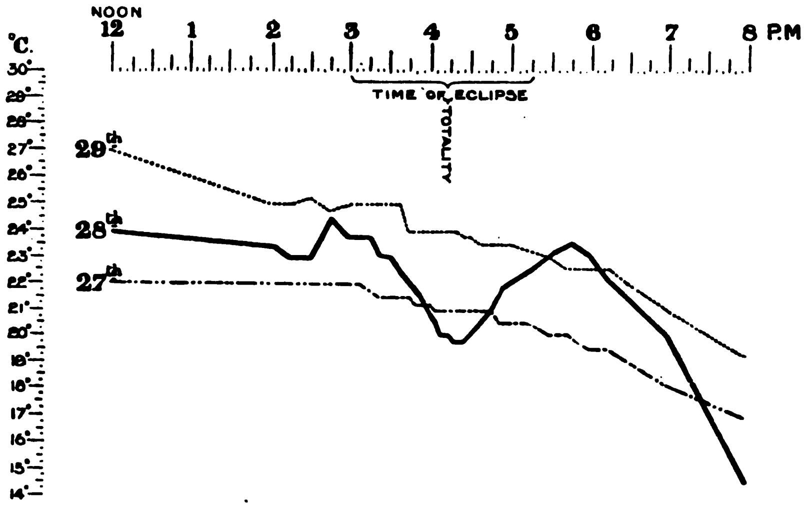


FIG. 5. Curve of temperature observations, May 27, 28, 29.

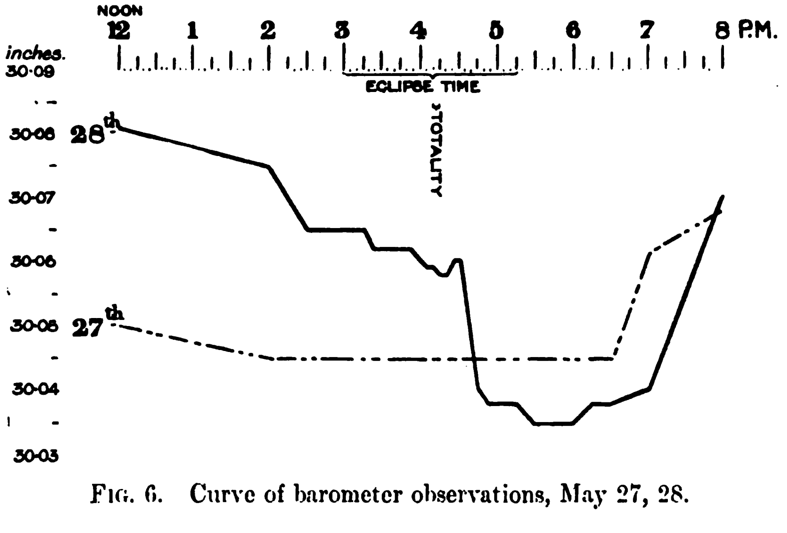


FIG. 6. Curve of barometer observations, May 27, 28.

Table I.—Temperature Variations.

					Time of Eclipse.															
	A.M.			P.M.						Time of Totality.										
Date (1900).	4	8	12	2	3	3.15	3.30	3.45	4.0	4.10	4.15	4.20	4.30	4.45	5	5.15	6	7	8	12
May 25 . .	° —	° —	22°·2	21°·0	21°·0	21°·1	20°·4	20°·4	20°·8	21°·1	20°·8	20°·5	20°·3	20°·4	20°·3	20°·1	19°·4	17°·8	16°·8	14°·6
,, 26 . .	10·5	20·0	21·5	21·4	21·1	20·6	20·6	20·6	20·1	20·1	20·1	20·3	19·9	19·9	20·0	19·6	19·0	17·7	16·3	13·0
,, 27 . .	10·9	22·0	22·0	22·0	22·0	21·8	21·5	21·5	21·2	21·0	21·0	21·0	21·0	21·0	20·5	20·5	19·5	18·2	17·0	11·0
Eclipse day, ,, 28 . .	10·2	20·2	24·0	23·5	24·0	24·0	23·0	22·0	20·8	20·0	20·0	19·8	20·0	21·0	22·0	22·5	23·0	20·0	14·5	12·0
,, 29 . .	16·2	20·3	27·0	25·0	25·0	25·0	25·0	24·0	24·0	24·0	24·0	24·0	23·8	23·5	23·5	23·2	22·5	21·0	19·2	14·2

Table II.—Barometric Readings.

					Time of Eclipse.															
	A.M.			P.M.						Time of Totality.										
Date (1900)	4	8	12	2	3	3.15	3.30	3.45	4.0	4.10	4.15	4.20	4·30	4.45	5	5.15	6	7	8	12
May 25 .	—	—	20·965	29·96	29·95	29·945	29·943	29·943	29·942	29·942	29·940	29·94	29·935	29·93	29·935	29·935	29·935	29·945	29·965	30·00
,, 26 .	29·995	29·99	29·995	29·995	29·975	29·975	29·975	29·975	29·975	29·975	29·975	29·975	29 975	29·975	29·975	29·975	29·975	29·80	29·99	30·00
,, 27 .	30·05	30·025	30·05	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·045	30·061	30·068	30·090
,, 28 .	30·089	30·089	30·081	30·075	30·065	30·065	30·062	30·062	30·06	30·059	30·058	30·06	30·06	30·04	30·038	30·038	30·035	30·04	30·07	30·06
,, 29 .	30·038	30·04	30·022	30·005	29·90	29·90	29·72	29·70	29·70	29·70	29·70	29·70	29·70	29·97	29·967	29·965	29·965	29·97	29·98	29·975

TABLE III.—Wind (Direction and Force).

Date.		A.M.			P.M.	Time of Eclipse.															
											Time of Totality.										
		4	8	12	2	3	3.15	3.30.	3.45	4	4.10	4.15	4.20	4.30	4.45	5	5.15	6	7	8	12
1900. May 25	Direction	—	—	E.S.E.	S.E.b.E.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	E.S.E.	E.S.E.	E.S.E.	E.S.E.	E.S.E.	S.E.b.S.	S.E.b.S.	S.E.	Calm	Calm
	Force ..	—	—	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	1	0	0
26	Direction	N.W. b.W.	S.	E.S.E.	E.S.E.	E.S.E.	S.E.b.E.	S.E.b.E.	E.S.E.	E.S.E.	S.E.b.E.	S.E.b.E.	S.E.b.E.	S.E.b.E.	E. b. S.	E.S.E.	S.E.b.E.	S.E.	S.E.	Calm	Calm
	Force ..	1	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	0	0
27	Direction	Calm	S.E.	E.S.E.	S.E.	S.E.	S.E.b.S.	S.E.	S.E.	S.E.	S.E.	S.E.	S.E.	S.E.	S.E.	S.E.	S.E.	S.E.	S.E.b.E.	Calm	Calm
	Force ..	0	2	2	3	2	2	2-3	3	3	3	3	3	3	3	3	3	2	1	0	0
28	Direction	Calm	W.S.W.	S.E.b.S.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S.S.E.	S. b. E.	S.E.	Calm	Calm
	Force ..	0	1–2	3	3	3	3	3	3	3	2–3	2–3	2–3	3	2	2	2	1	1	0	0
29	Direction	Calm	W.S.W.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.b.S.	S.E.	S.E.b.S.	Calm	Calm
	Force ..	0	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	0	0

TABLE IV.—Clouds (proportion) and Direction of Movement (Upper Wind).

Date.		A.M.			P.M.	Time of Eclipse.															
											Time of Totality.										
		4	8	12	2	3	3.15	3.30	3.45	4	4.10	4.15	4.20	4.30	4.35	5	5.15	6	7	8	12
1900. May 25.	Proportion of cloud .	—	—	3	2	1	1	1	2	3	0	3	2	1	1	1	0	0	0	2	1
	Upper wind . . .	—	—	W.	W.	N.W.	N.W.	N.W.	N.W.	N.W.	N.W.	0	N.N.W.	N.N.W.	N.W.	N.W.	N.W.	N.W.	0	0	0
26.	Proportion of cloud .	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
	Upper wind . . .	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27.	Proportion of cloud .	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Upper wind . . .	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28.	Proportion of cloud .	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Upper wind . . .	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29.	Proportion of cloud .	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
	Upper wind . . .	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

PART III.—PHOTOGRAPHS OF THE CORONA TAKEN WITH A 4-INCH COOKE LENS OF 16 FEET FOCAL LENGTH. *By* HOWARD PAYN.

The Camera.

The instrument I used during the eclipse was a telescope fitted with a camera for taking pictures of the corona. The lens was a Cooke photovisual objective (TAYLOR'S Patent) of 4 inches aperture and 16 feet focal length, the sun's image being therefore 1¾ inches in diameter. The fittings of the telescope were specially drawn for me by Mr. GEORGE SCORER, Architect, of Newman Street, and gave every satisfaction.

The object glass was fitted in a mahogany box, the wood being 1 inch thick; this in its turn slid in and out of a holder of the same thickness; the camera at the other end was similarly fitted.

Between the two ends were two sets of battens, each 8 feet long, supported in the centre by a square wooden frame into which the battens screwed. The screws worked in 2-inch slots, so that by loosening the thumb screws at either end room was afforded for any shrinkage or expansion of any part of the instrument.

The telescope when put together was supported on three solid piers, the two on which the ends rested being of brick. The whole was perfectly steady, even with a strong breeze blowing.

The battens were covered in with brown paper pasted round them, and outside this was a covering of tarred paper. The covering on the whole was fairly light-tight, but the paper was very easily torn, especially when the telescope was covered up at night with sails from the ship. During this operation the wind would flap parts of the sail against the tube, in spite of every care, and the paper was often broken. This could be repaired by pasting on fresh paper, but it was always a source of anxiety, and had any tearing of the paper happened during the exposures, the plates would probably have been spoilt.

I think that it would be better on another occasion to cover the battens with black cloth, with a mackintosh cover to fasten securely over that. The only covering then required at night would be for the object-glass and camera. This would save a good deal of trouble, and would much facilitate working the instruments at night, as, after the bluejackets have returned to the ship, it is often very difficult to open out coverings single-handed, especially in the dark.

The focus was obtained by sliding the object-glass box in and out of the holder. The focus was tested by exposing plates on the sun and by photographing star trails.

MAWSON'S Castle plates were used, size 10 × 10, F No. 56, relative speed 1·5 (WYNNE'S exposure-meter scale.)

The Cœlostat.

The telescope was fed by a cœlostat, with a 16-inch mirror, mounted on a brick and cement pier. This instrument is the property of the Royal Astronomical Society, and was lent for the purpose of the expedition. It had previously been used in India at the last eclipse.

This instrument gave a great deal of trouble at first. Fortunately Mr. HERBERT PORTCH, Assistant Engineer, R.N., of H.M.S. "Theseus," who was working the smaller coronagraph with the spare light from my mirror, was a skilled mechanic, and after taking the clock and the driving gear to pieces several times, was at length able to get it fairly in order. During the 75 seconds of totality it fortunately was at its best.

Before this instrument can be used at another eclipse it will be necessary to repair or alter the present clamping arrangements in R.A.

The Exposures.

The time of totality at Santa Pola, as given by the "local particulars," was 79·4 seconds; but as the American eclipse measurement of the moon was followed, the estimated duration of totality was reduced to 75 seconds.

The times arranged by Sir NORMAN LOCKYER for my instrument were as follows :—

1. At call of 70 expose till 60 = 10 seconds.
2. Snap = 1 second.
3. Expose as soon as possible after this snap until the call of five = 40 to 45 seconds.

I had five bluejackets to assist at the instrument, four being employed to hand and receive from me the plates and carriers and return them to their covers, the other man making the exposures from the object-glass end, by cutting off the light from the mirror by a piece of millboard.

Although we were able to carry out the programme without mistakes during the drills, at the eclipse the 10-seconds exposure of the first plate was accidentally reduced to 5. At the moment of totality the shouts and hand-clappings of 2000 spectators outside the ropes drowned for the moment the time signals, and the first count I heard was 65. I exposed at once, and at 60 the light was cut off by the bluejacket, as previously arranged. It was impossible at that moment to make him hear me, and I was afraid of confusing him in the other exposures, so it stood at that.

The snap and the long exposure were carried out as arranged. Two of the plates were developed at the eclipse camp by Mr. FOWLER. The long exposure was also developed by him after our return to South Kensington.

The Prominences.

The photographs show that at the time of the eclipse there were several prominences visible, the shorter exposure showing about twenty in all, most of them being small, and the whole of them being in the southern hemisphere.

The two most conspicuous in the south-west quadrant are shown in Professor LANGLEY'S photograph taken at Wadesboro, and a copy of both is given here for comparison (Plate 4). The interval between the taking of the two photographs would be about 2 hours, the time occupied by the moon's shadow in crossing the Atlantic. In this interval the prominences have altered somewhat; the dark central rift of the more northerly one has disappeared, and so has the spiky appearance of the rays. In this photograph it has become a broad flame like a prairie fire, the tips of the flame for the most part pointing towards the north.

This prominence appears from the report on the prismatic cameras to be composed chiefly of calcium, but it also shows on the hydrogen, helium, titanium, strontium, 4687 (unknown), asterium and magnesium (*b*) rings.

The other large prominence, somewhat to the south of the last, is shaped like a fleur-de-lys. In Professor LANGLEY'S photograph the three flames are nearly straight; but on this plate the two outside ones have bent over inwards towards the central flame, forming two loops.

This prominence is well defined in the calcium, less well in the hydrogen, and feebly in the helium rings.

The Corona.

The plates show none of the coronal rays, the longest being only about half the sun's diameter in extent. The polar rifts are somewhat sharply separated from the equatorial extensions, and are about 5 minutes of arc in height. The prominences are not situated in any relation to the rays, and appear indiscriminately under rays and rifts.

A comparison which has been made with Professor LANGLEY'S photograph shows that no change in the corona took place in the interval between the taking of the plates.

This is illustrated by the two photographs reproduced in Plate 5, showing the north polar rays in both cases.

PART IV.—THE PRISMATIC CAMERAS.

By W. J. S. LOCKYER, *M.A.*, *Ph.D.*, *and* A. FOWLER.

Description of the Instruments.

The instrument employed by Dr. LOCKYER was that employed in India by Mr. FOWLER, the aperture being 6 inches, and focal length 7 feet 6 inches, and two prisms of 45° being fixed in front of the objective. A different series of exposures, however, was arranged, and unlike those made in India, each was made on a separate plate, so that proper treatment in developing could be given. The spare part of the mirror was utilised for a finder, as in India, with which observations of the cusps for giving signals before totality were also made.

The instrument employed by Mr. FOWLER consisted of a 6-inch Taylor triple objective, of focal length 20 feet 3 inches, with the 9-inch prism of 45° belonging to the Solar Physics Observatory. The camera tube was a skeleton one, similar to that described in Mr. PAYN's report. The lens and prism were provided with a substantial mounting of steel and brass, which rested on a brick pier, and the camera, of solid construction, was placed at the proper distance on another brick pier. Two plate holders were provided, each holding five plates of size 15 inches by 3 inches. The back of the camera was an extended one, so arranged as to protect all the plates except the one which was being exposed through an opening of the same size as the plate. The moon's disc as represented on the photographs is 2¼ inches in diameter, so that a small margin of the plate was available in case there should be a slight error in centering the spectrum. A finder was arranged to view the sun or stars by reflection from the first surface of the prism. It may be mentioned that in all the plate holders employed, the use of springs was discarded, the plates being held in position by india-rubber pads arranged to give even pressure on the edges. This change was considered desirable in consequence of certain slight departures from perfect focus in some parts of the spectra photographed in India, which were attributed to the bending of the plates by the central springs of the usual form.

The 20-foot lens was received so shortly before the expedition left England, that it was only possible to make a rough trial of the instrument before it was set up at Santa Pola.

Both prismatic cameras were worked in conjunction with siderostats, calculations having shown that the position angles of contact were favourably situated after reflection.

In the case of each instrument the necessary assistance was rendered by men from the "Theseus." One man made the exposures at the prism end at pre-arranged signals from Dr. LOCKYER or Mr. FOWLER at the camera end; another recorded the times at which the exposures were made, the beginnings being marked by the signals to expose, and the ends by certain calls from the time-keeper at the eclipse clock. In each case also a bluejacket handed the dark slides as they were required, and another placed them in their bags when the exposures had been made.

In the case of each siderostat a spare portion of the mirror was used to reflect light into a camera with a transmission grating in front of the lens, these instruments being worked by bluejackets.

The instruments were roughly focussed in the first instance by observing the sharpness of the edges of the spectrum when the light of the sun was reflected into them, and afterwards by taking photographs of the spectrum of Arcturus. In the case of the 20-foot, considerable difficulty was experienced with the focussing, for the reason that the focus determined by a star did not hold good for the sun next day; this was especially noticed on the day of the eclipse, when it was further found that between early morning and eclipse time, the focal length, as judged by the sun's edge, was shortened by nearly 2 inches. The stellar focus was accordingly disregarded, and an attempt was made to obtain the focus by observing on the ground glass the Fraunhofer lines given by the disappearing crescent just before totality. It was then found that the range of adjustment was insufficient to allow perfect focussing, and there was no time to make the necessary alteration in the length of the tube. The photographs consequently lack the perfection of definition which had been hoped for, but they nevertheless give a good deal of information, as will appear later. The cause of the variation of focus has not yet been investigated, but it is recalled that in a general way the focus appears to have depended to some extent on the hour angle at which the siderostat mirror was used.

Table of Exposures.

It was intended that the exposures in both instruments should be as nearly as possible alike. The actual times in the case of the 20-foot prismatic camera were as follows:—

Number of plate.	Time of exposure.	Duration.	Remarks.
	hr. min. sec.		
1	4 12 36	Inst.	
2	4 12 38	„	Totality commenced, 4h. 12m. 38s.
3	4 12 40	„	
4	4 12 42	„	
5	4 12 44 to 4 12 50	6 secs.	
6	4 12 58 to 4 13 30	32 „	
7	4 13 34 to 4 13 45	11 „	
8	4 13 47	Inst.	
9	4 13 49	„	
10	4 13 51	„	Totality ended, 4h. 13m. 53s.

Edwards' snap-shot Isochromatic plates were used throughout, and the expedition is greatly indebted to Messrs. **Edwards** for supplying special batches of plates on patent plate glass.

A few of the plates were developed at Santa Pola, and the remainder were developed at the Solar Physics Observatory, pyro-soda developer being used.

Description of Photographs. (*Plate* 6.)

The photographs indicate the same succession of phenomena recorded in the three previous eclipses, but the recent eclipse was specially advantageous, for the reason that the chromospheric arcs at the instant of contact were of greater length.

Taking the 20-foot series as typical of both, we find the following general appearances in the ten photographs :—

(1.) " Instantaneous " exposure 2 seconds before beginning of totality. Chromospheric arcs in great numbers, those corresponding with hydrogen, calcium (H and K), and helium being of great length; the whole crossed longitudinally by streaks of continuous spectrum from the uneclipsed photosphere. A number of small prominences are shown on the H and K arcs, and less strongly on the arcs due to hydrogen and helium. The arcs, like the continuous spectrum, are broken up by the irregularities of the moon's limb.

(2.) " Instantaneous " exposure, at (or just before) beginning of totality (Plate 6, A). Plate almost identical with No. 1, but with fewer streaks of continuous spectrum.

(3.) " Instantaneous " exposure 2 seconds after beginning of totality. Somewhat similar to No. 2, but without marked continuous spectrum from the photosphere. There is a general shortening of chromospheric arcs, more especially noticeable in those which are shortest in Nos. 1 and 2.

(4.) " Instantaneous " exposure 4 seconds after beginning of totality. Nothing now visible but a comparatively small number of arcs due to the upper chromosphere. Chief among these are H and K, $H\beta$, $H\gamma$, &c., of the hydrogen series, D_3 and other arcs due to helium, and others due to strontium, iron, &c. Plate under-exposed.

(5.) Exposed from 6 to 12 seconds after beginning of totality. The increased exposure at this stage has had the effect of increasing the intensity of the chromospheric spectrum as compared with No. 4, the arcs now being nearly as numerous as in No. 3, but relatively less intense. A notable feature is the relative increase in the intensity of the lines of helium and the line 4687, to which reference was made in the report on the Indian eclipse.

Fragmentary rings due to the corona appear on this plate, their position-angles of maximum intensity being quite different from those of the chromosphere and prominences. The brightest ring is that in the green, λ 5303·7, but others are also distinctly seen.

(6.) From 20 to 52 seconds after beginning of totality. In this photograph the decided chromospheric arcs seen in No. 5 are replaced by fragmentary rings representing prominences and crests of the serrated parts of the chromosphere. H and K appear almost as complete rings, a little stronger on the eastern side. Several coronal rings, to which reference will be separately made later, are seen. The general continuous spectrum overlapping the chromospheric and coronal rings, on account of the long exposure, is rather strong in this photograph.

(7.) From 58 to 69 seconds after beginning of totality. The brighter chromospheric arcs now appear on the western side of the sun, together with the images of certain prominences, and parts of coronal rings, the whole being involved in continuous spectrum of moderate intensity.

(8.) "Instantaneous" exposure 6 seconds before end of totality. Only a few of the brighter chromospheric arcs, on the western side of the sun, and a few prominences are seen in this photograph. Plate under-exposed.

(9.) "Instantaneous" exposure, 4 seconds before end of totality. Very similar to No. 8, but more chromospheric arcs are visible.

(10.) "Instantaneous" exposure probably about 2 seconds before end of totality. The photograph is very similar to No. 2, but the arcs are on the opposite (western) side.

The spectra taken with the 20-foot prismatic camera are 10¾ inches long from Hα (which appears on some of the photographs) to the strong titanium line at 3685·34, and nearly 7½ inches from D_3 to K. Those taken with the 6-inch 2-prism instrument have corresponding dimensions of about 8½ inches and 5¾ inches respectively.

Photographs No. 2 (A), taken with the 20-foot prismatic camera, and Nos. 5, 6, and 7 (B, C and D), taken with the 2 prism instrument, are reproduced in Plate 6.

The photographs taken with the small objective gratings are very good, but show no features which are not seen in the larger photographs. They are chiefly of interest as showing what can be done with small and inexpensive instruments.

PART V.—DISCUSSION OF RESULTS.

By Sir NORMAN LOCKYER, *K.C.B.*, *F.R.S.*

The Spectrum of the Chromosphere.

The spectrum of the chromosphere, as shown on the series of photographs taken with the prismatic cameras, greatly resembles that photographed in India in 1898.* It is, therefore, not considered necessary to discuss at present the wave-lengths, intensities, and origins of the chromospheric arcs.

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* 'Phil. Trans.,' A, vol. 197, p. 151

In connection with the intensities it is important to note that the relative intensities are not the same at different stages of the eclipse. Thus in photographs Nos. 1 to 4, the helium line 4026·34 is very much less intense than the adjacent strontium line, 4077·89, while in photograph No. 5, which shows the spectrum of the chromosphere only, the two are of practically equal intensity. As both arcs are of the same length, this change indicates that while the strontium vapour extends down to the sun's limb, the helium exists only in an elevated shell concentric with the photosphere. A similar behaviour is noted in the lines of asterium; in numbers 1 to 4, for instance, the asterium line 4922·10 is only very slightly stronger than the adjacent barium line 4934·24, whereas in No. 5 the latter is scarcely visible, while 4922·10 is a well-marked arc.

The same feature is observed in other arcs due to asterium and helium, and also in the unknown line at wave-length 4687. Arcs due to other substances, however, gradually become shorter and less bright as the moon eclipses more and more of the chromosphere.

The Spectra of the Prominences.

The prominences photographed during the eclipse are few in number, and with two exceptions were of no considerable magnitude. For convenience of reference the accompanying diagram (fig. 7) has been prepared from photographs 5 and 7 of the 20-foot series, by painting out all parts except the images in K light, and the various prominences have been numbered as shown.

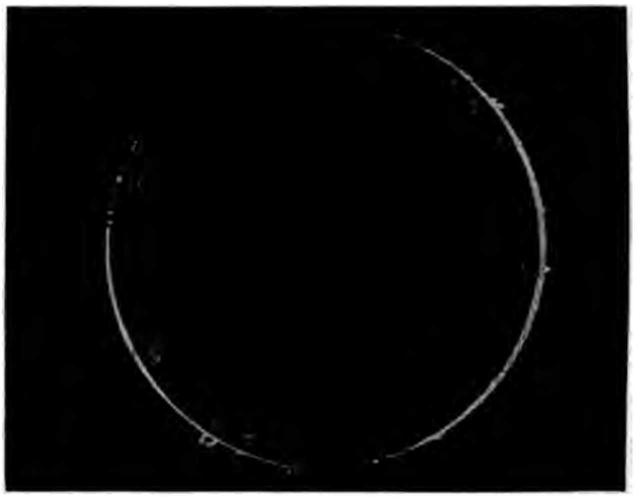

FIG. 7. The chromosphere and prominences in K light.

The prominences numbered 1, 2, 4, 6, 7, and 9 have spectra in which H and K are very bright, the chief lines of hydrogen less bright, and those of helium comparatively faint. The small prominence No. 10 and the long thin one No. 3 appear in H and K and the lower part of the latter is shown feebly in Hβ and Hγ. Those richest in bright lines are numbers 5 and 8, the spectra of which are indicated by the following table. The wave-lengths and probable origins are taken from the table of chromospheric lines which forms part of my report on the eclipse of 1898. In the column of origins the prefix *p* is an abbreviation for *proto*, indicating that an enhanced line in the spectrum of the substance in question agrees in position with the prominence line, enhanced lines being those which are intensified in passing from the arc to the spark spectrum.

SPECTRUM of Prominences Nos. 5 and 8.

Wave-length.	Photographic Intensity.	Probable Origin.	Remarks.
3685·34	1	*p* Ti	
3711·9	<1	H	Hν
3721·8	1	H	Hμ
3734·15	1	H	Hλ
3750·2	2	H	Hκ
3759·45	3	*p* Ti	
3761·46	3	*p* Ti	
3770·7	2	H	Hι
3798·0	3	H	Hθ
3820·59	1	Fe	
3835·6	5	H	Hη
3860·06	1	Fe	
3889·15	8	H	Hζ
3900·68	1	*p* Ti	
3913·61	1	*p* Ti	
3933·83	10	*p* Ca	K
3968·63	10	*p* Ca	H
4026·34	4-5	He	
4045·98	<1	Fe	
4077·89	4	Sr	
4102·00	8	H	Hδ (*h*)
4120·97	<1	He	
4215·70	3	Sr	
4226·90	<1	Ca	
4247·00	<1	Sc	
4340·63	8	H	Hγ (G)
4395·20	<1	*p* Ti	
4471·65	5	He	
4687·0	<1	—	
4713·25	1	He	
4861·53	8	H	Hβ (F)
4922·10	<1	Ast	
5015·73	<1	Ast	
5183·79	1	Mg	b_1
5875·87	6	He	D_3
6563·05	2	H	Hα (C)

A comparison with the chromospheric arcs indicates that the spectrum of the prominences consists of the radiations which are brightest in the spectrum of the chromosphere.

Heights of Chromospheric Vapours.

Measurements of the lengths of the arcs photographed at the commencement of totality have been made in order to determine the heights above the photosphere to which the corresponding vapours are visible. The results obtained from measurements of photograph No. 2 of the 20-foot series are shown in the following table, in which the corresponding results obtained in 1898* are introduced for purposes of comparison.

HEIGHTS of Chromospheric Vapours.

Lines.	Heights in Seconds of Arc.		Heights in Miles.	
	1900.	1898.	1900.	1898.
	″	″		
Proto-calcium (K)	13·0	13·3	5900	6000
Hydrogen	8·9	10·0	4000	4500
Helium (4471·65).	8·7	8·9	3900	4000
Strontium (4077·89)	6·7	6·0	3000	2700
Strontium (4215·70) . . . . } Helium (4026·34) . . . . }	5·8	6·0	2600	2700
Calcium (4226·90)	4·1	4·4	1850	2000
Titanium (3761·46)	3·6	—	1600	—
Scandium (4247·00) . . . . } Mg ultra-violet triplet (3832·45). . . . . }	3·4	4·4	1500	2000
Ti enhanced lines (4572·16, &c.).	3·3	—	1500	—
Fe triplet (4045·98, &c.) . . . . } Al (3944·16, 3961·67) . . . . } $b_{1.3}$ (Mg) (5183·79, &c.). . . . . . }	2·6	3·2	1180	1450
Mn quartet (4030·92, &c.) . . . . } Fe enhanced (4233·25, 4584·02). . . . . }	2·4	2·4	1090	1100
Majority of other arc lines	1·6	—	700	—
Carbon fluting 3883·55 and many other lines	0·6	1·05	270	475

As I pointed out in connection with the eclipse of 1898, these results do not necessarily give the actual heights reached by the various vapours, as only the brightest lower portions of the chromospheric arcs may be registered on the photographs. That the heights given in the table do not represent the upper limits of the respective vapours is indicated by measures which have been made of photograph No. 5, which had an exposure of 6 seconds; some of the arcs in this photograph are as long or longer than the corresponding ones in Nos. 1–4. Notwithstanding the less favourable position of the moon, correcting for the moon's motion, the measured arcs in photograph No. 5, reduced to miles, are as follows :—

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* 'Roy. Soc. Proc.,' vol. 64, p. 37.

Proto-calcium (K)	7800 miles.
Hydrogen	7000 „
Helium, 4471·65	5800 „
„ 4026·34	5000 „
Strontium, 4077·89	4800 „
„ 4215·70	3800 „

The Spectrum of the Corona.

Coronal rings are shown on photographs 5, 6, and 7.

The green ring at 5303·7 is the brightest, but that at 3987·0 is also very distinct and more continuous than the green ring. The rings generally appear to be less bright than in 1898, and many of the fainter rings recorded in 1898* have not been detected on the 1900 series of photographs. The following table shows the wave-lengths of the rings which have been noted, with general remarks indicating the distribution of intensity in them.

TABLE of Coronal Radiations.

Wave-length.	Average Intensity. Max. = 10.	Remarks.
3800	2	Brightest from W. to N.W.
3987·0	5	A nearly continuous ring a little brighter in west than elsewhere.
4231·3	3	Brightest in west and south-west, very similar to chief ring 5303.
4359·5	3	Brightest from S. to N.E.
4568·5	3	Very similar to 3987·0 in distribution of intensity.
5303·7	10	Brightest between west and south-west.

The above correspond with the principal rings recorded in 1898. Others are suspected, but they are too indistinct for satisfactory measurement; one of them, not previously recorded, is near λ 5537.

The dissimilarity of form of the rings indicates that they have probably not all the same chemical origin, as I have previously pointed out,† three different substances probably being in question.

Besides the bright rings, the spectrum of the corona shows a considerable amount of continuous spectrum. This is brightest in the parts corresponding to the inner corona generally, as indicated by its being broken up into bands by the elevations and depressions of the moon's limb, but it also has places of maximum brightness agreeing in position with the brighter parts of the green coronal ring. In the eclipse of 1898 the feature last mentioned was also noted, and the opinion was expressed that the action which produced a brightening of the green ring also produces a brightening of the continuous spectrum, not only in the region where the gaseous mass is rendered more luminous, but in the region immediately overlying it. It is only in the large-scale photographs of 1900 that the effects of lunar irregularities have been directly traced, and it is possible that some of the brighter parts of the continuous spectrum in the 1898 series attributed to concealed elevations of the green ring should be ascribed to these irregularities.

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* 'Roy. Soc. Proc.,' vol. 66, p. 191.

† 'Roy. Soc. Proc.,' vol. 66, p. 191.

A detailed examination of the photographs has further shown that some of the brighter streaks of continuous spectrum correspond with polar rays of the corona, more particularly with those extending outwards in a direction nearly coincident with the plane of dispersion.

Comparison of the Green Coronal Ring with the Inner and Outer Corona.

The general results as to the distribution of intensity in the coronal rings are the same as those arrived at from the photographs of 1898, namely :—

(1.) The positions of greatest brightness of the coronal rings have apparently no connection with the positions of the prominences.

(2.) The brightest parts of the green ring correspond very closely with the brightest parts of the inner corona, but are apparently independent of the outer corona. This distribution, however, is less marked than in 1898, when distinct prominence-like masses in the inner corona, corresponding to the brightest parts of the green ring, were photographed. Some of the bright parts of the inner corona in the north-east quadrant appear to be unrepresented on the green ring only because of their unfavourable situation with reference to the plane of dispersion.

These results are illustrated in the accompanying diagram (fig. 8).

The Differences between the Coronas observed at the Periods of Sun-spot Maxima and Minima.

My attention was called especially to these differences, because I saw the minimum eclipse of 1878, while the phenomena of that of 1871 (maximum) were still quite fresh in my mind. My then published statements have been amply confirmed during the eclipses which have happened since 1878, but certainly the strongest confirmation has been obtained during the present one, which took place two more spot periods after 1878.

(1.) *Form.*—With regard to form, at the instant of totality I saw the 1878 corona over again, the wind-vane appearance being as then most striking.

This is also the appearance presented by the drawings made by some of the "Theseus" observers and reproduced in Part II. of this report; and also by the photographs obtained with the various coronagraphs.

Great equatorial extensions corresponding with those observed by **Newcomb**, in 1878, were not seen by any of the observers, either with or without the aid of discs.

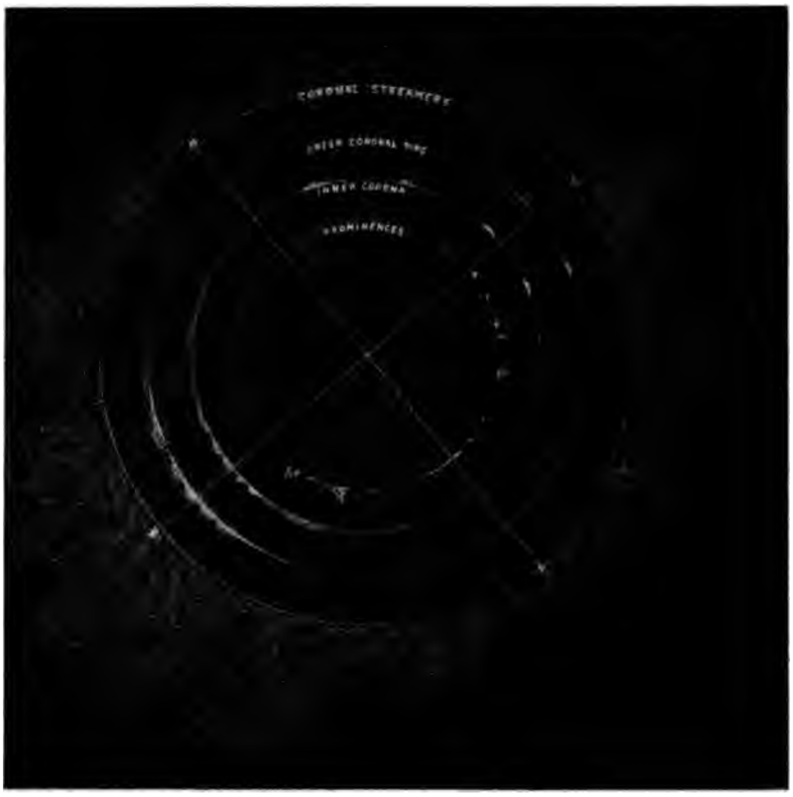

Fig. 8. The green coronal ring compared with prominences and inner and outer corona.

The atmospheric conditions at the time of eclipse were excellent, and it may be therefore that the feature observed by **Newcomb**, in 1878, was exceptional.

(2.) *The Spectrum.*—In connection with the eclipse of 1878 (minimum), I pointed out that, whereas in 1871 (maximum) the spectrum of the corona viewed by small dispersion was remarkable for the brightness of the lines, in 1878 they were practically absent, and the continuous spectrum was remarkably brilliant.

I determined therefore to make a similar observation in this year of minimum, and, as in 1878, used a grating-first-order spectrum placed near the eye. The result was identical with that recorded in 1878. I saw no obvious rings or arcs, but chiefly a bright continuous spectrum.

The photographs taken with the prismatic cameras, as already pointed out, confirm this view that the bright rings are feebler near a time of minimum. The green ring was observed visually by Mr. **Payn** with a powerful direct-vision spectroscope deprived of its collimator, and by Mr. **Fowler** with a prismatic opera-glass.

The latter remarked that the green ring was decidedly dimmer than in 1898, when it was observed by him with the same instrument.

(3.) *The Minute Structure of the Inner Corona.*—Lieutenant DOUGHTY, R.N., and myself made observations on the minute structure of the corona, in order to see if any small details could be observed, and whether they were the same I saw so well and recorded during the eclipse of 1871, at a period of sun-spot maximum. This question was specially taken up this year, as exactly two sun-spot periods have elapsed since 1878.

In 1871 I used a 6-inch object-glass, and distinctly observed marked delicate thread-like filaments, reminding one of the structure of the prominences, with mottling and nebulous indications here and there; some of these distinct markings were obvious enough to be seen till some minutes after totality.*

This year, with a perfect 4-inch Taylor lens and a higher power, not the slightest appearance of this structure could be traced; the corona some 2′ or 3′ above the chromosphere was absolutely without any detailed markings whatever.

Lieutenant DOUGHTY duplicated and confirmed these observations with a 3¾-inch Cooke. Here, then, is established another well-marked difference between maximum and minimum coronas.

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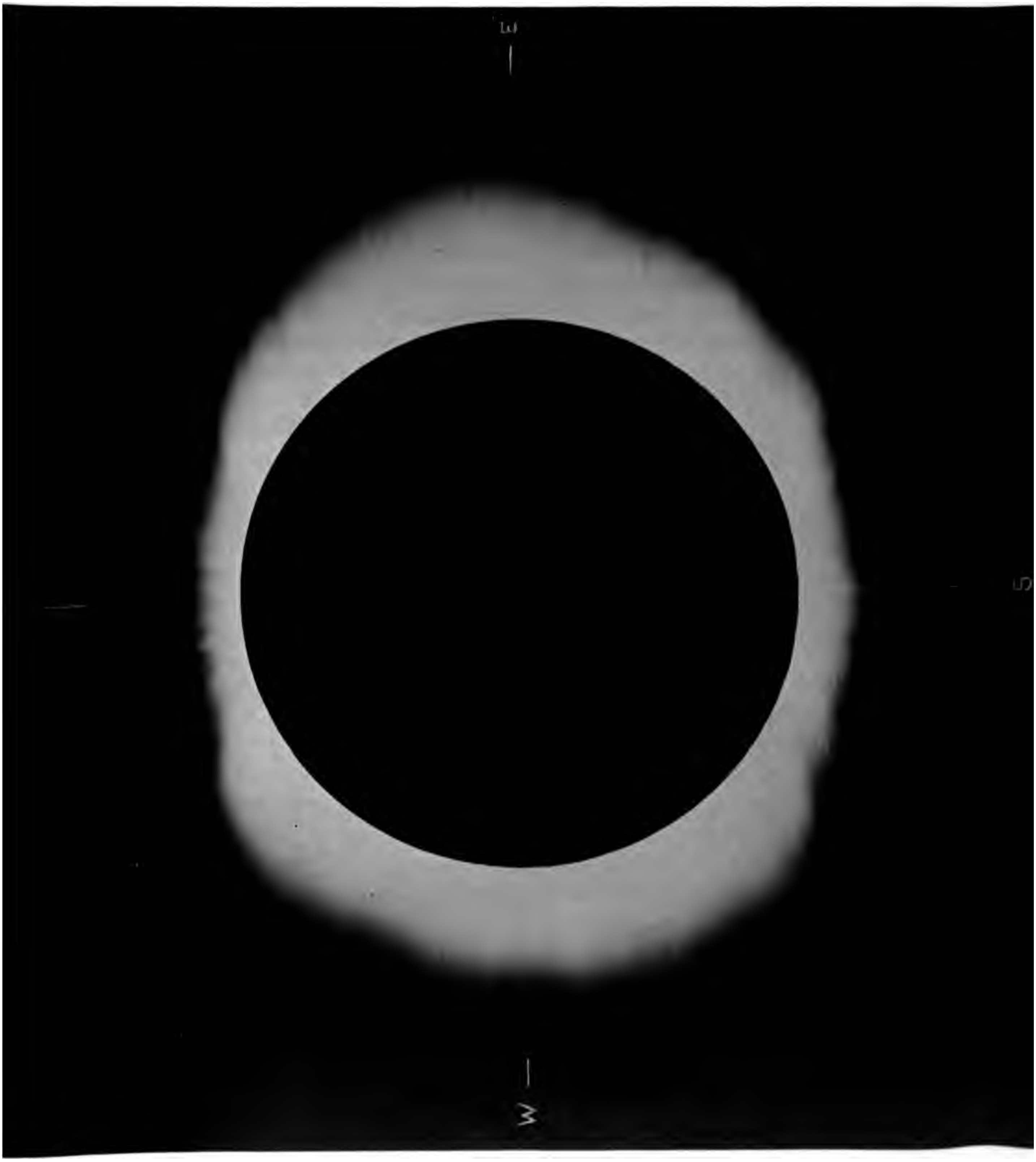
* 'Solar Physics,' p. 372.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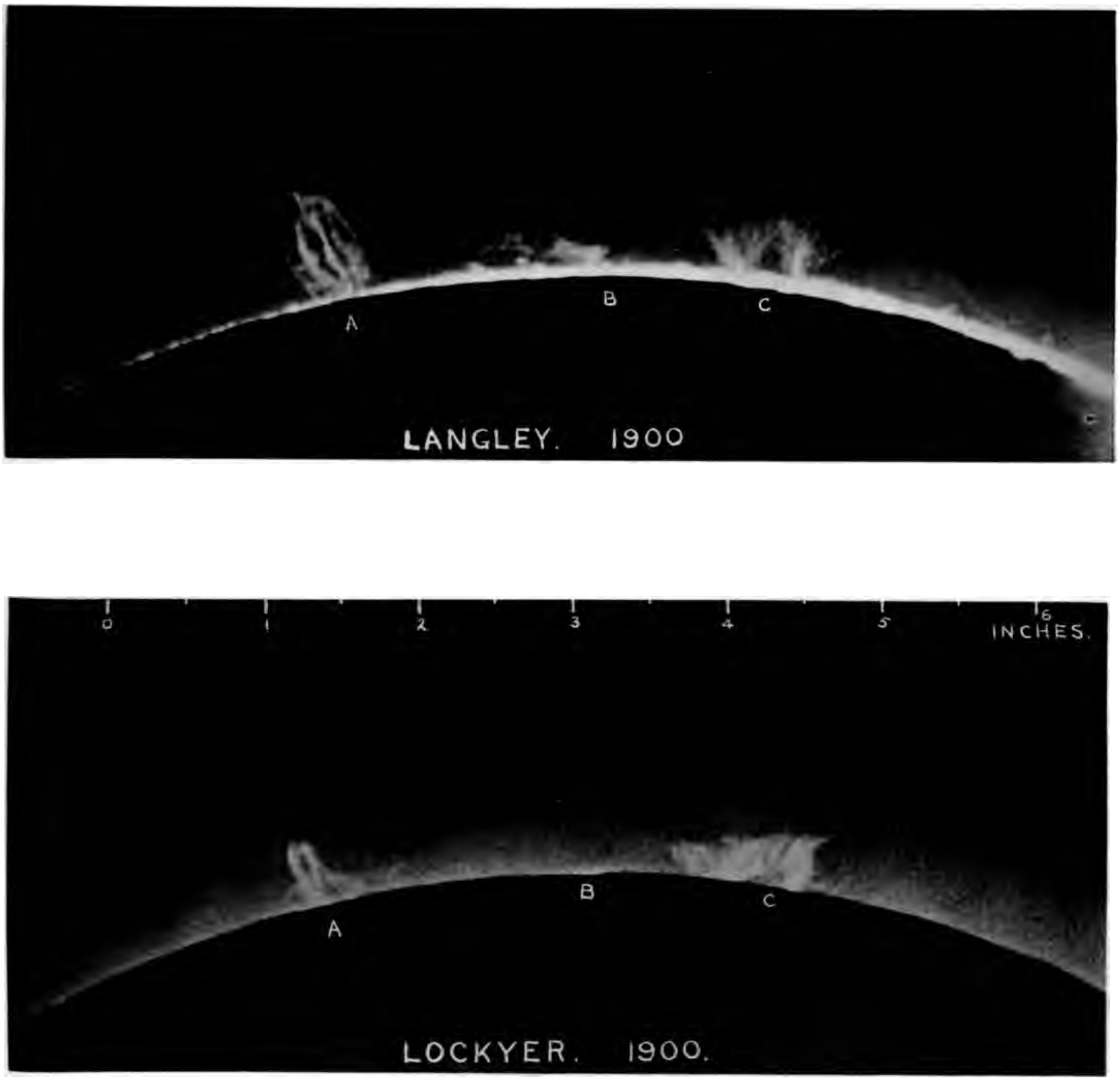
TOTAL SOLAR ECLIPSE.

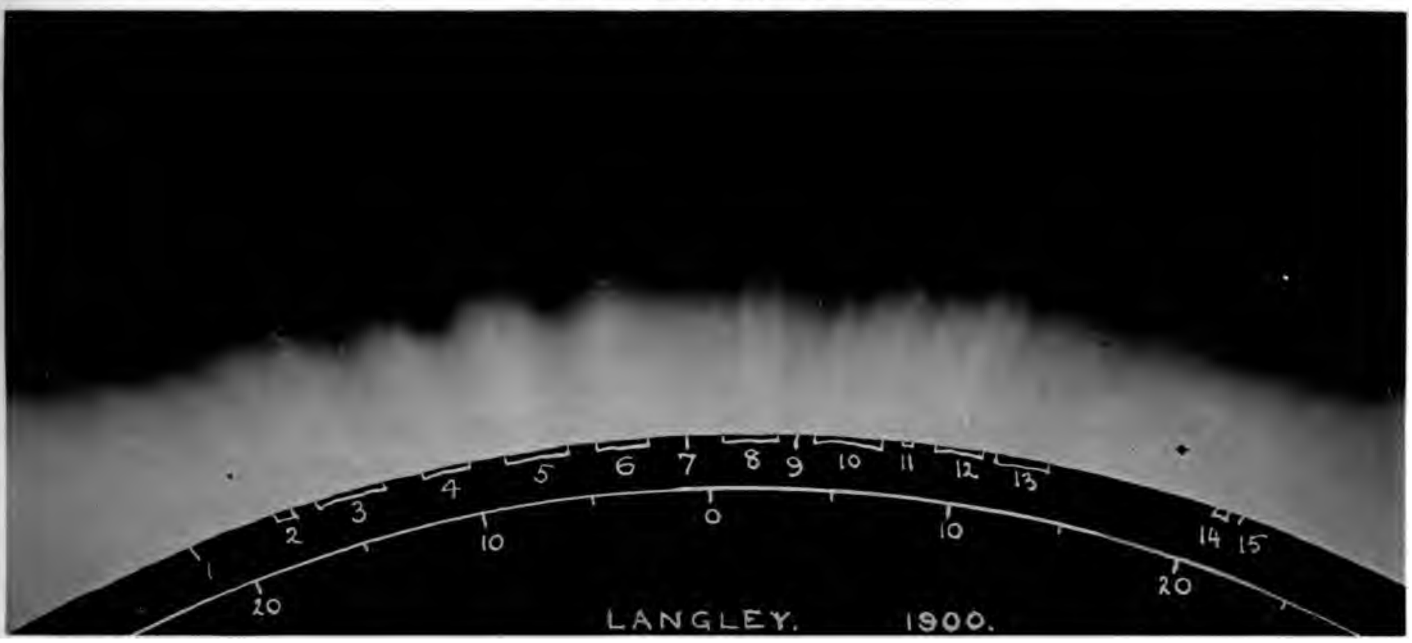
MAY 28, 1900.

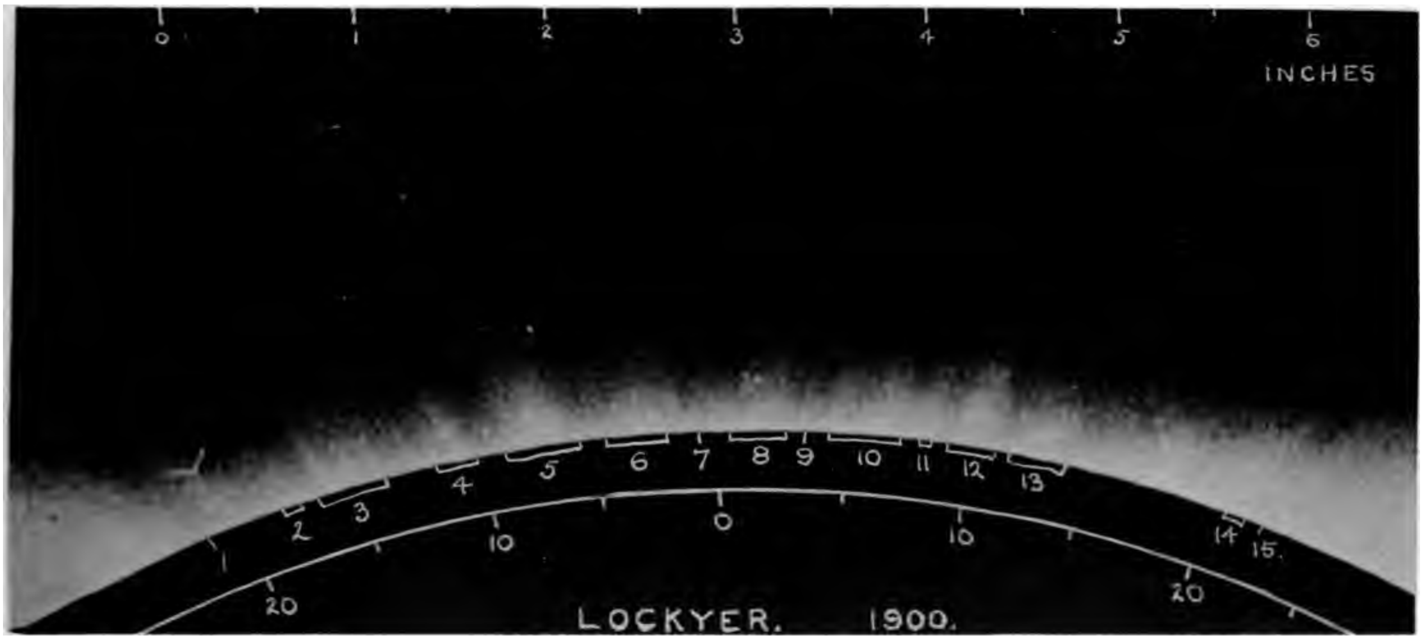


PROMINENCES AT S.W. LIMB.

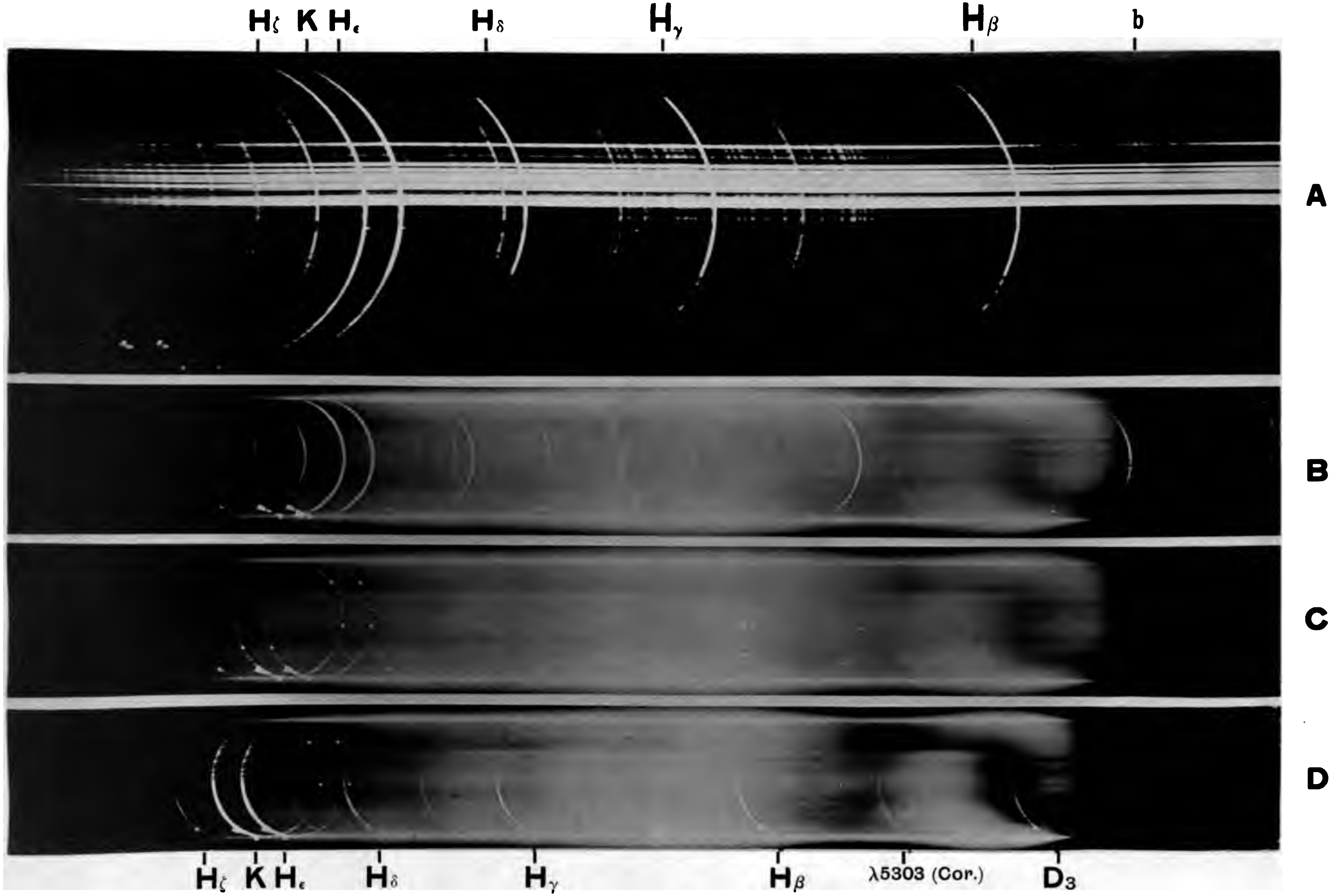
TOTAL SOLAR ECLIPSE.

MAY 28, 1900.





NORTH POLAR REGION.



A.—20 foot prismatic camera, near beginning of totality. B. C. D.—7 foot 6 in. prismatic camera, near beginning, middle and end of totality respectively.

INDEX SLIP.

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RAYLEIGH (Lord).
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IX. Bakerian Lecture.—*On the Law of the Pressure of Gases between 75 and 150 Millimetres of Mercury.*

By Lord Rayleigh, *F.R.S.*

Received January 15,—Read February 27, 1902.

In a recently published paper* I have examined, with the aid of a new manometer, the behaviour of gases at very low pressures, rising to 1·5 millims. of mercury, with the result that Boyle's law was verified to a high degree of precision. There is, however, a great gap between the highest pressure there dealt with and that of the atmosphere—a gap which it appeared desirable in some way to bridge over. The sloping manometer, described in the paper referred to, does not lend itself well to the use of much greater pressures, at least if we desire to secure the higher proportional accuracy that should accompany the rise of pressure. The present communication gives the results of observations, by another method, of the law of pressure in gases between 75 millims. and 150 millims. of mercury. It will be seen that for air and hydrogen Boyle's law is verified to the utmost. In the case of oxygen, the agreement is rather less satisfactory, and the accordance of separate observations is less close. But even here the departure from Boyle's law amounts only to one part in 4000, and may perhaps be referred to some reaction between the gas and the mercury. In the case of argon too the deviation, though very small, seems to lie beyond the limits of experimental errors. Whether it is due to a real minute departure from Boyle's law, or to some complication arising out of the conditions of experiment, must remain an open question.

In the case of pressures not greatly below atmosphere, the determination with the usual column of mercury read by a cathetometer (after Regnault) is sufficiently accurate. But when the pressure falls to say one-tenth of an atmosphere, the difficulties of this method begin to increase. The guiding idea in the present investigation has been the avoidance of such difficulties by the use of manometric gauges combined in a special manner. The object is to test whether when the volume of a gas is halved its pressure is doubled, and its attainment requires two gauges indicating pressures which are in the ratio of 2 : 1. To this end we may employ a pair of independent gauges as nearly as possible similar to one another, the similarity being tested by combination in parallel, to borrow an electrical term. When connected below with one reservoir of air and above with another reservoir, or with a vacuum, the two gauges should reach their settings simultaneously, or at least so nearly that a suitable correction may be readily applied. For brevity we may for the present assume precise similarity. If now the two gauges be combined *in series*, so that the low-pressure chamber of the first communicates with the high-pressure chamber of the second, the combination constitutes a gauge suitable for measuring a doubled pressure.

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* 'Phil. Trans.,' A, vol. 196, p. 205, Feb., 1901.

The Manometers.

The construction of the gauges is modelled upon that used extensively in my researches upon the density of gases, so far as the principle is concerned, although of course the details are very different. In fig. 1 A and B represent 3/4 size the lower and upper chambers. As regards the glass-work, these communicate by a short neck at D as well as by the curved tube ACB. Through the neck is carried the glass measuring-rod FDE, terminating downwards at both ends in carefully prepared points E, F. The rod is held, at D only, with cement, which also completely blocks up the passage, so that when mercury stands in the curved tube the upper and lower chambers are isolated from one another. The use of the gauge is fairly obvious. Suppose for example that it is desired to adjust the pressure of gas in a vessel communicating with G to the standard of the gauge. Mercury standing in C, H is connected to the pump until a vacuum is established in the upper chamber. From a hose and reservoir attached below, mercury is supplied through I until the point F and its image in the mercury surface nearly coincide. If E coincides with its image, the pressure is that defined; otherwise adjustment must be made until the points E, F both coincide with their images, or as we shall say until both mercury surfaces are *set*. The pressure then corresponds to the column of mercury whose height is the length of the measuring-rod between the points E, F. The verticality of E F is tested with a plumb-line.

The measuring-rods appear somewhat slender; but it is to be remembered that the instruments are used under conditions that are almost constant. So far as the comparison of one gas with another is concerned, the qualification "almost" may indeed be omitted. The coincidence of the points and their images is observed with the aid of four magnifiers of 20 millims. focus, fixed in the necessary positions.

General Arrangement of Apparatus.

In fig. 2 is represented the connection of the manometers with one another and with the gas reservoirs. The left-hand manometer can be connected above through F with the pump or with the gas supply. The lower chamber A communicates with the upper chamber D of the right-hand manometer and with an intermediate open through AB to the pump. When the mercury is raised to the middle mark H, the volume is halved, and the pressure to be dealt with is doubled. Gas sufficient to exert the single pressure (75 millims.) must be supplied to the intermediate chamber, which is now isolated from the pump by the mercury standing up in AB. Both manometers can now be set, and the doubling of the pressure verified.

The communication through F with the pump is free from obstruction, but on a side tube a three-way tap is provided communicating on the one hand with the gas supply and on the other with a vertical tube, more than a barometer-height long and terminating below under mercury, by means of which a wash-out of the generating vessels can be effected when it is not desired to evacuate them. The five tubes leading downwards from A, E, C, I, K are all over a barometer-height in length and are terminated by suitable hoses and reservoirs for the supply of mercury. When settings are actually in progress, the mercury in the hoses is isolated from the reservoirs by pinch-cocks and the adjustment of the supply is effected by squeezing the hoses. As explained in my former paper, the final adjustment must be made by squeezers which operate upon parts of the hoses which lie flat upon the large wooden mercury tray underlying the whole. The adjustment being somewhat complicated, a convenient arrangement is almost a necessity.

The Side Apparatus.

By the aid of these manometers the determination of pressure is far more accurate than with the ordinary mercury column and cathetometer, but since the pressures are defined beforehand, the adjustment is thrown upon the volume. The variable volume is introduced in the side-tube JK. This was graduated to ½ cub. centim., in the first instance by mercury from a burette. Subsequently the narrow parts above and below the bulb (which as will presently be seen are alone of importance) were calibrated with a weighed column of mercury of volume equal to ½ cub. centim. and occupying about 80 millims. of the length of the tube. The whole capacity of the tube between the lowest and highest marks was 20½ cub. centims. The object of this addition is to meet a difficulty which inevitably presents itself in apparatus of this sort. The volume occupied by the gas cannot be limited to the capacities susceptible of being accurately gauged. Between the upper mark G and the mercury surface in C when set, a volume is necessarily included which cannot be gauged with the same accuracy as the volumes between G and H and between H and I. The simplest view of the side apparatus is that it is designed to measure this volume. In the notation subsequently used V_3 is the volume included when the mercury stands at C, at G, and at the top mark J. Let us suppose that with a certain quantity of gas imprisoned it is necessary in order to set the manometer CD, the upper chamber being vacuous, to add to V_3 a further volume V_5, amounting to the greater part of the capacity of the side-tube, so that the whole volume is $V_3 + V_5$.

From this equation V_3 has been eliminated and B is expressed by means of P_1/P_2, and the actually gauged volumes V_1, V_2, $V_5 - V_5'$, $V_4 - V_4'$. It is important to remark that only the *differences* $(V_5 - V_5')$, $(V_4 - V_4')$ are involved. The first is measured on the lower part of the side-apparatus and the second on the upper part, while the capacity of the intervening bulb does *not* appear.

If the principal volumes V_1 and V_2 are nearly in the right proportion, there is nothing to prevent both $V_5 - V_5'$ and $V_4 - V_4'$ from being very small. When the temperature changes are taken into account, V_3, V_4, V_5 are not fully eliminated, but they appear with coefficients which are very small if the temperature conditions are good.

Thermometers.

As so often happens, much of the practical difficulty of the experiment turned upon temperature. The principal bulbs were drowned in a water-bath which could be effectively stirred, and so far there was no particular impediment to accuracy. But the other volumes could not so well be drowned, and it needed considerable precaution to ensure that the associated thermometers would give the temperatures concerned with sufficient accuracy. As regards the side-tube, a thermometer associated with its bulb and wrapped well round with cotton-wool was adequate. A third thermometer was devoted to the space occupied by the manometers and the tube leading from C to J. It was here that the difficulty was greatest on account of the proximity of the observer. Three large panes of glass with enclosed air spaces were introduced as screens, and although the temperature necessarily rose during the observations, it is believed that the rise was adequately represented in the thermometer readings. A single small gas flame, not allowed to shine directly upon the apparatus, supplied the necessary illumination, being suitably reflected from four small pieces of looking-glass fixed to a wall behind the glass points of the manometers.

As regards the success of the arrangement for its purpose, it is to be remembered that by far the larger part of any error that might arise is *eliminated in the final result*, since it is only a question of a *comparison* of observations with and without the large bulbs. Any systematic error made in the first case as regards the temperature of the undrowned capacities will be repeated in the second, and so lose its importance. A similar remark applies to any deficiency in the comparison of the three thermometers with one another.

Comparison of Large Bulbs.

This comparison needs to be carried out with something like the full precision aimed at in the final result, although it is to be noted that an error enters to only equality, to maintain the same sloped position during the subsequent use when the gauges must be combined in series. But in this case it would hardly be advisable to trust to wood-work in the mounting. At any rate in my experiments the gauges were erected with measuring-rods vertical, an arrangement which has at least the advantage that a displacement is of less importance as well as more easily detected At the close of the observations upon the various gases it became necessary to compare the gauges with full precision.

For this purpose, they were connected (without india-rubber) in parallel, the upper chambers of both being in communication with the pump, and the lower chambers of both in communication with the gas reservoirs GI. Had the lengths of the measuring-rods been absolutely equal, this equality would be very simply proved by the possibility of so adjusting the pressure of the gas and the supply of mercury to the two manometers that all four mercury surfaces could be *set* simultaneonsly. It was very evident that no such simultaneous setting was possible, and the problem remained to evaluate the small outstanding difference. To pass from one manometer to the other, either the volume or the temperature had to be varied.

In principle it would perhaps be simplest to keep the volume constant and determine what difference of temperature (about half a degree) would be required to make the transition. But the temperature of the undrowned parts (now increased in volume) could not be ascertained with great precision, so that I preferred to vary the volume and to trust to alternations backwards and forwards for securing that the mean temperature in the two cases to be compared should not be different. Thus in one set, including seven observations following continuously, four alternate observations were settings with one manometer and three were settings with the other. According to the thermometers, the mean temperature in the first case was for the drowned volume 11°·38 and for the (much smaller) undrowned volume 12°·76. In the second case the corresponding temperatures were 11°·39 and 12°·80, so that the differences could be neglected. The volume changes were effected in the side-tube J K, and the mean difference in the two cases was ·411 cub. centim. It will be understood that in order to define the volume *both* manometers were always set *below*. The whole volume was reckoned at 294 cub. centims., of which about 258 cub. centims. represents the capacity of the bulbs G I drowned in the water-bath. According to these data the proportional difference in the lengths of the measuring-rods, equal to the proportional difference of the above determined volumes, is ·00140. Two other similar sets of observations gave ·00136, ·00137; so that the mean adopted value is ·00138. The measuring-rod of the manometer on the *right*, fig. 2, is the longer.

As in the case of the volumes, any error in the above comparison is halved in the actual application. If H_2 be the length of the rod in the right-hand manometer, H_1 the length on the left, we are concerned only with the ratio $H_1 + H_2 : H_2$. And from the value above determined we get

$$\frac{H_1 + H_2}{H_2} = 1\cdot99862 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

I.	V_1	θ_1	V_3	τ_1	V_5	t_1
II.	V_2	θ_2	V_3	τ_2	V_4	τ_2
III.	—	—	V_3	τ_3	V_5'	t_3
IV.	—	—	V_3	τ_4	V_4'	τ_4

In the first observation V_1 is the volume of the two large bulbs and θ_1 the temperature of the water-bath, reckoned from some convenient neighbouring temperature as a standard. V_3 is the ungauged volume already discussed whose temperature τ_1 is given by the upper thermometer. V_5 is the (larger) volume in the side apparatus whose temperature t_1 is that of the lower thermometer. In the second observation V_2 is the volume of the upper bulb and θ_2 its temperature. V_4 is the volume in the side apparatus whose temperature, as well as that of V_3, is taken to be τ_2, the mean reading of the upper thermometer. III. and IV. represent the corresponding observations in which the large bulbs are not filled. The reading of the water-bath thermometer is in every case denoted by θ, that of the upper thermometer by τ, and that of the lower thermometer by t. The temperature of the columns of mercury in the manometers is also represented by τ.

As an example of the actual quantities, the observations on air between October 28 and November 5 may be cited. The values of V_1 and V_3 are approximate. As appears from the formulæ, V_3 occurs with a small coefficient, as does also V_1, except in the ratio $V_1 : V_2$ otherwise provided for. We have

$$V_1 = 258{\cdot}4, \qquad V_3 = 19{\cdot}05\,;$$

$$V_4 = {\cdot}810, \qquad V_5 = 20{\cdot}493\,;$$

$$V_4 - V_4' = {\cdot}0841, \qquad V_5 - V_5' = {\cdot}0266\,;$$

$$\theta_1 = -\,{\cdot}077, \qquad \theta_2 = -\,{\cdot}059\,; \qquad t_1 = {\cdot}257, \qquad t_3 = {\cdot}141\,;$$

$$\tau_1 = {\cdot}092, \qquad \tau_2 = {\cdot}186, \qquad \tau_3 = -\,{\cdot}033, \qquad \tau_4 = {\cdot}100.$$

The volumes are in cubic centimetres and the temperatures are in Centigrade degrees reckoned from 14°.

An effort was made, and usually with success, to keep all the temperature differences small, and especially the difference between θ_1 and θ_2. It is desirable also so to adjust the quantities of gas in the two cases that $V_4 - V_4'$, $V_5 - V_5'$ shall be small.

The Reductions.

The simple theory has already been stated, but the actual reductions are rather troublesome on account of the numerous temperature corrections. These, however, are but small.

We have first to deal with the expansion of mercury in the manometers. If, as in

(5), the actual heights of the mercury (at the same temperature) be H_1, H_2, we have for the corresponding *pressures* $H/(1 + m\tau)$, where $m = \cdot 00017$. Thus in the notation already employed

$$P_1 = \frac{H_2}{1 + m\tau_1}, \text{ or } \frac{H_2}{1 + m\tau_3},$$

and

$$P_2 = \frac{H_1 + H_2}{1 + m\tau_2}, \text{ or } \frac{H_1 + H_2}{1 + m\tau_4}.$$

The quantity of gas at a given pressure occupying a known volume is to be found by dividing the volume by the absolute temperature. Hence each volume is to be divided by $1 + \beta\theta$, $1 + \beta\tau$, $1 + \beta t$, as the case may be, where β is the reciprocal of the absolute temperature taken as a standard. Thus in the above example for air (p. 426), $\beta = \frac{1}{273 + 14} = \frac{1}{287}$. Our equations, expressing that the quantities of gas are the same at the single and at the double pressure, accordingly take the form

$$\frac{H_2}{1 + m\tau_1}\left\{\frac{V_1}{1 + \beta\theta_1} + \frac{V_3}{1 + \beta\tau_1} + \frac{V_5}{1 + \beta t_1}\right\} = \frac{B(H_1 + H_2)}{1 + m\tau_2}\left\{\frac{V_2}{1 + \beta\theta_2} + \frac{V_3 + V_4}{1 + \beta\tau_2}\right\},$$

$$\frac{H_2}{1 + m\tau_3}\left\{\frac{V_3}{1 + \beta\tau_3} + \frac{V_5'}{1 + \beta t_3}\right\} = \frac{B(H_1 + H_2)}{1 + m\tau_4}\,\frac{V_3 + V_4'}{1 + \beta\tau_4},$$

where B is the numerical quantity to be determined—according to BOYLE's law identical with unity.

By subtraction we deduce

$$\begin{aligned}
&\frac{1}{(1 + m\tau_1)(1 + \beta\theta_1)} - \frac{BV_2(H_1 + H_2)}{V_1H_2(1 + m\tau_2)(1 + \beta\theta_2)} \\
&\quad = \frac{V_3}{V_1}\left\{\frac{B(H_1 + H_2)}{H_2(1 + m\tau_2)(1 + \beta\tau_2)} - \frac{B(H_1 + H_2)}{H_2(1 + m\tau_4)(1 + \beta\tau_4)}\right. \\
&\qquad\qquad\qquad\qquad \left. - \frac{1}{(1 + m\tau_1)(1 + \beta\tau_1)} + \frac{1}{(1 + m\tau_3)(1 + \beta\tau_3)}\right\} \\
&\quad + \frac{B(H_1 + H_2)V_4}{H_2V_1}\left\{\frac{1}{(1 + m\tau_2)(1 + \beta\tau_2)} - \frac{1}{(1 + m\tau_4)(1 + \beta\tau_4)}\right\} \\
&\quad - \frac{V_5}{V_1}\left\{\frac{1}{(1 + m\tau_1)(1 + \beta t_1)} - \frac{1}{(1 + m\tau_3)(1 + \beta t_3)}\right\} \\
&\quad + \frac{B(H_1 + H_2)(V_4 - V_4')}{H_2V_1}\,\frac{1}{(1 + m\tau_4)(1 + \beta\tau_4)} \\
&\quad - \frac{V_5 - V_5'}{V_1}\,\frac{1}{(1 + m\tau_3)(1 + \beta t_3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).
\end{aligned}$$

The first three terms on the right, viz., those in V_3, V_4, V_5, vanish if $\tau_1 = \tau_3$,

$\tau_2 = \tau_4$, $t_1 = t_3$. In the small terms we expand in powers of the small temperatures (τ, t), and further identify $B(H_1 + H_2)/H_2$ with 2. The five terms on the right then assume the form

$$\begin{aligned}
&\frac{V_3}{V_1}\{(m+\beta)(\tau_1 - \tau_3 - 2\tau_2 + 2\tau_4) + \beta^2(2\tau_2^2 - 2\tau_4^2 - \tau_1^2 + \tau_3^2)\} \\
&-\frac{2V_4}{V_1}\{(m+\beta)(\tau_2 - \tau_4) - \beta^2(\tau_2^2 - \tau_4^2)\} \\
&-\frac{V_5}{V_1}\{m(\tau_3 - \tau_1) + \beta(t_3 - t_1) + \beta^2(t_1^2 - t_3^2)\} \\
&+\frac{2(V_4 - V_4')}{V_1}\{1 - (m+\beta)\tau_4 + \beta^2\tau_4^2\} \\
&-\frac{V_5 - V_5'}{V_1}\{1 - m\tau_3 - \beta t_3 + \beta^2 t_3^2\},
\end{aligned}$$

in which $m\beta$ and m^2 are neglected, while β^2 is detained. In point of fact, the terms of the second degree were seldom sensible.

Taking the data above given for the observations on air October 28—November 5, we find

$$\begin{array}{lll}
\text{Term in } V_3 \ldots\ldots\ldots & = & -{\cdot}000012 \\
\quad ,, \quad V_4 \ldots\ldots\ldots & = & -{\cdot}000002 \\
\quad ,, \quad V_5 \ldots\ldots\ldots & = & +{\cdot}000034 \\
\quad ,, \quad (V_4 - V_4') \ldots\ldots & = & +{\cdot}000652 \\
\quad ,, \quad (V_5 - V_5') \ldots\ldots & = & -{\cdot}000103 \\
\hline
 & & +{\cdot}000569
\end{array}$$

For the first term on the left of (6), we find

$$\frac{1}{(1 - m\tau_1)(1 + \beta\theta_1)} = 1{\cdot}000256\,;$$

so that

$$B = \frac{V_1 H_2 (1 + m\tau_2)(1 + \beta\theta_2)}{V_2(H_1 + H_2)} \times {\cdot}999687,$$

or when the numerical values are introduced from (4), (5),

$$B = 1{\cdot}00002.$$

The deviation from BOYLE'S law is quite imperceptible.

It may be noted that a value of B exceeding unity indicates an excessive compressibility, such as is manifested by carbonic acid under a pressure of a few atmospheres.

The Results.

Little now remains but to record the actual results. All the gases were, it is needless to say, thoroughly dried.

Air.

Date.	B.
April 15–29, 1901	·99986
May 22–28, 1901	1·00003
October 28–November 5, 1901	1·00002
Mean	·99997

Hydrogen.

Date.	B.
July 6–13, 1901	·99999
July 16–23, 1901	·99996
Mean	·99997

The hydrogen was first absorbed in palladium, from which it was driven off by heat as required.

Oxygen.

Date.	B.
June 7–17, 1901	1·00022
July 21–July 1, 1901	1·00044
September 18–30, 1901	1·00005
October 10–18, 1901	1·00027
Mean	1·00024

The two first fillings of oxygen were with gas prepared by heating permanganate of potash contained in a glass tube and sealed to the remainder of the apparatus. The desiccation was, as usual, by phosphoric anhydride. In the third and fourth fillings the gas was from chlorate of potash and had been stored over water.

Nitrous Oxide.

Date.	B.
July 31–August 5, 1901	1·00059
August 8–24, 1901	1·00074
Mean	1·00066

Argon.

Date.	B.
December 28–January 1, 1902	1·00024
January 2–9, 1902	1·00019
Mean	1·00021

The argon was from a stock which had been carefully purified some years ago and has since stood over mercury. In this case the two sets of observations recorded related to the same sample of gas imprisoned in the apparatus. In all other cases the gas was renewed for a new set of observations.

With regard to the accuracy of the results it was considered that systematic errors should not exceed $\frac{1}{10000}$. In the comparison of one gas with another most of the systematic errors are eliminated, and the mean of two or three sets should be accurate according to the standard above stated. That nitrous oxide should show itself more compressible than according to BOYLE'S law is not surprising, but there appear to be deviations also in the cases of oxygen and argon. Whether these deviations are to be regarded as real departures from BOYLE'S law, or are to be attributed to some complication relating to the glass or the mercury cannot be decided. At any rate they are very minute. It will be noted that the oxygen numbers are not so concordant as they ought to be. I am not in a position to suggest an explanation, and the discrepancies were hardly large enough to afford a handle for further investigation.

If we are content with a standard of $\frac{1}{5000}$, we may say that air, hydrogen, oxygen, and argon obey BOYLE'S law at the pressures concerned and at the ordinary temperatures (10° to 15°).

Throughout the investigation I have been efficiently assisted by Mr. GORDON, to whom I desire to record my obligations.

X. *A Determination of the Value of the Earth's Magnetic Field in International Units, and a Comparison of the Results with the Values given by the Kew Observatory Standard Instruments.*

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THE discrepancies found by Professor RÜCKER and the author* to exist between the values for the horizontal component of the earth's magnetic field, as measured by the absolute instruments in use in the various British observatories, were so great that it seemed of interest to measure the field at one of the observatories by an entirely different method, in order, if possible, to obtain some indication as to the reliability of the various instruments.

Further, in the ordinary method of measuring H, a correction has to be applied on account of the distribution of the magnetism in the magnets employed, about the value of which there is some uncertainty.†

It was therefore decided to make a measurement of the horizontal component of the earth's field, by comparing it with the field produced at a certain point by a known current flowing through a coil of known dimensions. The comparison was to be made by suspending a small magnetic needle at the centre of the coils, and noting its deflection when acted upon by the earth's field and the field due to the coils. In this way a direct comparison would be obtained between the value of the unit magnetic field, as deduced from absolute magnetic measurements, and the value of the unit field produced by a known current, the value of the current being deduced from an entirely different set of measurements from those used in determining the constants of the magnetometers. The following paper contains an account of such a comparison.

For constructional reasons, as well as to secure a uniform magnetic field at the centre of the coil where the magnetic needle is suspended, and so to reduce any error which any want of accuracy in the centring, or any departure from the solenoidal condition of this magnetic needle would produce, the HELMHOLTZ arrangement was adopted, in which two equal coils are placed with their planes parallel, and at a distance from each other equal to the radius of either. For the comparison of the field produced within the coils, when they are traversed by a known current, the sine method was chosen, the coils being mounted on a horizontal circle so that they could be turned about a vertical axis till the magnetic axis of the needle at the centre of the coils was at right angles to the axis of the coils when the current was passing. Thus, if F is the field produced at the centre of the coils when they are traversed by unit current, and the angle through which the coils have to be turned when they are traversed by a current C is θ, the value of the earth's horizontal component, measured in terms of a unit derived from the unit of current, is given by $H = CF/\sin\theta$.

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* 'Brit. Assoc. Rep.,' p. 87, 1896.

† CHREE, 'Roy. Soc. Proc.,' vol. 65, p. 375, 1899.

In order to measure the current, use was made, in the first place, of a silver voltameter. As it is not convenient to make the deflection experiments, involving as they do a continual reversal of the current, and hence interruptions of the current of uncertain duration, at the same time as a silver deposition is being made, an intermediate standard of current was adopted. This was obtained by balancing the potential difference between the terminals of a standard resistance coil against the E.M.F. of a standard cadmium cell. This addition has the further advantage that, if we know the value of the resistance, then we can use the results for the E.M.F. of the cadmium cell obtained by other observers to obtain the value of the current, and thus check the values obtained by the voltameter.

Degree of Accuracy aimed at in the Measurements.—In designing the apparatus and arranging the method to be adopted, a determination of the value of H by means of the coil accurate to 1 or 2 parts in 10,000 was aimed at.

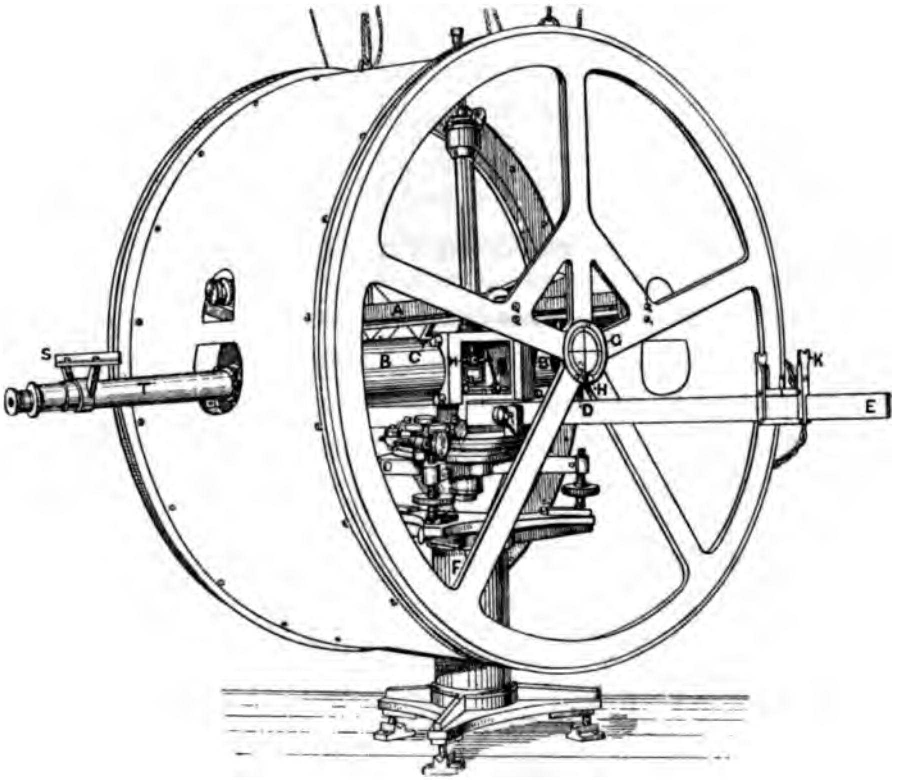
That the current may be known in terms of the electrochemical equivalent of silver to the required degree of accuracy, we must be able to measure the weight of silver deposited, and the time during which the current passes, as well as to maintain the current constant, each to about one part in 10,000. Since the weight of silver deposited during each experiment was about 1·6 grammes, this involves the weighings being correct to within 0·16 milligramme. The interval during which a deposition lasted being two hours, the time has to be measured to within 0·7 second. Further, owing to the use of the subsidiary standard, namely, the resistance coil and standard cell, we have to know the change of the resistance of the coil and that of the E.M.F. of the cell to within 1 in 10,000 of the value of the resistance of the coil, or the E.M.F. of the cell, as the case may be.

With reference to the accuracy with which the dimensions of the coils have to be known, it is shown in most text-books that if a is the mean radius of either coil and $2x$ is the distance between the mean planes of the coils, b the axial breadth of each coil, and d the radial depth, that is, the section of either of the coils is a rectangle of length b and depth d, N the number of turns in both coils together, F the coil

The Helmholtz Galvanometer.—Since the radii of the coils and the distance between their mean planes has to be measured, it is important that the radius should be as large as possible. It was decided that the largest manageable coil was one having a diameter of 60 centims. The construction of a pair of coils of this diameter having the grooves in which the wire is to be wound true to within about ·02 centim. is a problem of considerable difficulty, especially when, as in the present case, the coils have to be capable of rotation about a vertical axis, and this rotation has to be measured with accuracy.

The coils were made in the Physical Laboratory of the Royal College of Science under my immediate supervision, the construction being entirely performed by Mr. J. W. Colebrook, the instrument maker attached to the laboratory, and very great credit is due to him for the way in which he acquitted himself of the task. In this connection it must be remembered that the tools available were not such as would be considered indispensable for such a job in any engineering or instrument-making workshop.

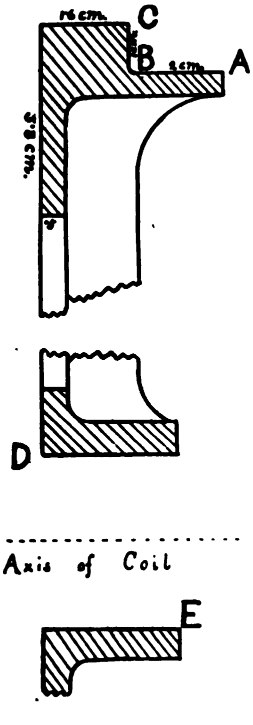
Fig. 1.



In order to measure the angle through which the coils are rotated, it was decided to use the azimuth circle of a Kew-pattern magnetometer, and so the coils were made to fit the magnetometer and allow of the use of the telescope and scale to determine the position of the magnet. The magnetometer employed is one by **Elliott**, and numbered 70. It was used in the magnetic survey of Great Britain and in the series of comparisons between the different magnetic observatories carried out by Professor **Rücker** and the author. The accuracy of the circle was tested, during a long series of comparisons made between this instrument and the Kew standards, by measuring the angle subtended by the two fixed marks used for the declination observation at Kew with different parts of the circle. In this way no error amounting to 10 seconds of arc, the smallest angle which can be read off on the circle, was detected in any part of the circle. Hence, as the average value of the angle used in the following measurements was 46°, the accuracy attainable was amply sufficient.

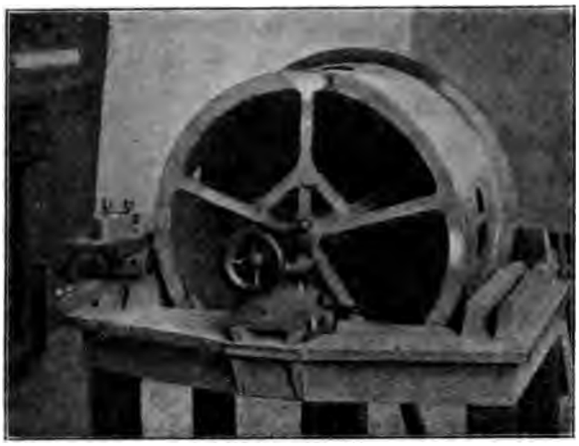
A general view of the coils, as they appeared when in use, is shown in fig. 1. The coils were carried by two cross bars, A, the points of four screws, C, attached to these cross bars resting on the cylindrical parts, B B, of the magnetometer. These screws, instead of being fixed directly to the cross bars, were attached to two metal plates which could slide along the bars. Thus, by moving these sliding pieces, as well as by the screws C, the relative positions of the coils and the magnetometer could be adjusted. The coils were prevented from turning about a horizontal axis by two screws, D, resting on the top of the deflection bar E. The position of the magnet, M, was observed by means of a plane mirror attached to the magnet and the telescope, T, and scale, S. The magnetometer rested on a stand, F, which passed through a hole in the cylindrical side of the coils. The space between the spokes of the flanges was such that when the telescope and the glass suspension tube were removed the magnetometer could be inserted. In order to relieve the magnetometer from the full weight of the coils, part of the weight was supported by strings attached to the top of the flanges, which, after passing over a pulley, had a counter weight attached.

Fig. 2.



As the construction of such large coils with the degree of accuracy aimed at in these experiments is a problem of considerable difficulty, and one which has frequently arisen in experimental researches, a short description of the method employed may be of some interest. The flanges of the coils in which the grooves to receive the wire are turned out are formed of two gun-metal castings. The section of a portion of one of them is shown in fig. 2. The spokes were strengthened by a rib 1·5 centim. deep. In order to turn up these castings a large cast-iron bed was made with planed ways on the top to take the fixed and movable headstocks of one of the lathes in the laboratory. This bed is shown in fig. 3, and was of such a size that the finished coil could be swung between centres. A gun-metal face-plate, slightly larger in diameter than the coils, was also cast and turned up on the bed itself. Teeth were cut round the circumference of this face-plate, and it was driven by a spur-wheel gearing into these teeth. Each of the flange castings was bolted to this face-plate and the surfaces AB and BC, fig. 2, were turned up. The central hole DE was also turned to fit accurately a steel mandrel on which a square threaded screw had been cut. The castings were then mounted on the mandrel, and by means of lock nuts were held at the right distance apart. A sheet of hard rolled brass, 0·1 centim. thick, the edges of which had been cut parallel, was then bent round so as to lie along the surfaces AB (fig. 2), and was held in place by screws passing into the castings. The brass was of sufficient length to overlap for about 30 centims. at the bottom where the hole for the passage of the stand had to be cut. The attachment of the cylinder of brass to the flanges was rendered additionally secure by tinning the surfaces which came in contact before putting the parts together, and then, after the insertion of the screws, by heating the coils, solder was run into the joint. The necessary holes in the cylindrical side of the coils were cut out by means of a milling cutter. By this method of construction the two metal flanges, in which the grooves to hold the wire were to be turned, were held very rigidly in the same relative positions.

Fig. 3.



In order to turn out the grooves the coil was mounted on the mandrel between centres and driven by two pins attached to the face-plate. Light cuts were taken, particularly at the finish, when the tool being left untouched the coil was reversed between the centres so that the depths of the two grooves should be the same. The flanges were marked with twelve equi-distant lines, parallel to the axis of the coil, to act as reference lines when measuring the dimensions of the coil. Two diametral lines at right angles were also ruled on each of the flanges near the central hole. Plates of mica were then clamped over the central holes by means of brass rings, G, fig. 1. The intersection of fine lines ruled on these mica plates, to form a continuation of the diametral lines ruled on the flanges, marked the axis of the coils.

That the rigidity of the coils is quite satisfactory is shown by the measurements given in the subsequent pages, which were made eighteen months after the grooves were turned. That the edges of the grooves lay accurately in one plane was shown by clamping a microscope to the lathe and rotating the coils while the cross wire of the microscope was adjusted on the edge of the groove. In this way it was found that the sides of the grooves nowhere departed from a plane by 0·002 centim.

Small ebonite bushes were inserted in the flange where the wires leading to the coils had to pass. The wire employed was number 34 S.W.G., the uncovered wire having a diameter of 0·023 centim. This wire was covered with three coatings of white silk. In order to allow of a test of the insulation being made after the coils were wound, the wire was wound double so that each coil consisted of two independent circuits, the wires of which lay alongside each other throughout their whole length. Hence by testing the insulation between these two circuits, and between each circuit and the flanges, a satisfactory test of the insulation of the wire could be made. These tests, however, would not allow of the detection of the short-circuiting of one or more turns of either of the circuits. This point could be tested by sending the same current through the two circuits in opposite directions. The absence of any magnetic field at the centre of the coils showed that the magnetic effect of each of the circuits was the same, and hence, as it is very unlikely that exactly the same number of turns of each circuit could be short-circuited without a connection being formed between one circuit and the other, these tests may be taken as sufficient. The width of the grooves was made such that twelve turns of wire, that is six turns of each circuit, just fitted in. There were eight layers in each coil, and since the two circuits were always used in parallel this gave 48 turns in each coil. After winding each layer it was given a coating of weak shellac varnish, made by dissolving shellac in absolute alcohol. This varnish was soaked up by the silk covering of the wire and served the purpose of preventing the silk taking up moisture from the air and thus losing in part its insulating properties. The wires joining the two coils were led along the cylindrical surface of the coils in the same horizontal plain as the centre of the coils. In this position these wires produce at the centre of the coils no magnetic field which has a horizontal component, and hence the fact that they are distant from one another by about 1 millim. from centre to centre can produce no error.

The insulation of the circuits was tested before and after the experiments and was found to exceed 200 megohms both from one circuit to the other and from each of the circuits to the flanges. The insulation of the rubber-covered leads and of the Pohl commutator used to reverse the current was also tested and found practically infinite.

The metal of which the flanges were cast was specially mixed by the founders so as to avoid the presence of any iron, and these castings, as well as all metal used in the construction of the coils, were carefully tested for magnetism by means of a delicate magnetometer. The fact that the instrument is free from magnetism was also tested by using the magnetometer which carries the coils to measure H at Kew while the coils were in place. The difference between the value obtained and that given by the Kew Observatory magnetograph curves was the same as the difference when the coils were removed.

When measuring the dimensions of the coils two micrometer microscopes were employed. These microscopes are attached to two carriages which move along a V-groove in a heavy iron bed-plate and can be clamped at any distance apart. By means of two brackets attached to the same bed-plate, a standard metre could be supported in front of the microscopes. Readings on the standard metre were taken before and after each set of measurements made on the coils, and so the screws of the microscopes were only depended upon to measure a fraction of a millimetre at each end. The runs of the micrometer screws were compared with the graduations of the metre. The metre actually employed was a brass one, with the scale divided on silver. It is of the standard H section and was made by the Geneva Instrument Company. The errors of this metre for the divisions actually employed were determined by comparison with a nickel-steel standard metre by the same makers, which has been compared with the International Standards at the Bureau at Sèvres. The temperature coefficient and the correction to each reading having been determined at the Bureau, the error of the brass metre could be calculated. The temperature coefficient of the brass metre does not come into the measurements, as the measurements on the coil and the comparison with the nickel-steel metre were made at the same temperature.

Since when the measurements were actually made, the nickel-steel metre had not been returned by the Bureau, the following measurements are given in terms of the brass metre, and a correction will have to be applied at the end for the difference between the two metres. This difference was found to be proportional to the length measured, so that it is probably due to the fact that the temperature coefficient of the brass metre which was assumed in reducing the results is not quite correct. The final result, however, is free from any error on this account.

In order to determine the distance between the mean planes of the coils, the distance between the outside edges and between the inside edges was measured for twelve positions equi-distant round the circumference. To allow of the setting of the cross wires of the microscopes being made with precision, the edges of the grooves were turned quite sharp.

Three independent sets of measurements were made in this way, one set before the magnetic measurements and the other two after. The results obtained, reduced to a temperature of 16° C., are given in the following table :—

turned so that it had a concave curvature equal to that of the coil as deduced from the tape measurements. These small curved gauge-pieces were held against the outside layer of the wire by clamps, and their internal curved edges formed a sharp line to which the cross wires of the microscopes could be adjusted, these microscopes looking through the holes in the flanges. Two independent sets of measurements were made for each coil, and the results are given in the following table:—

DIAMETERS outside Layer 8.

Station.	Coil A.		Coil B.	
	I.	II.	I.	II.
1- 7	60·670	60·670	60·677	60·679
2- 8	60·670	60·671	60·678	60·680
3- 9	60·662	60·660	60·689	60·688
4-10	60·657	60·658	60·684	60·680
5-11	60·648	60·647	60·666	60·667
6-12	60·664	60·664	60·666	60·669
Means	60·662	60·662	60·677	60·677
Radius	30·331	30·331	30·339	30·339
Correction for temperature of scale	+ ·009	+ ·009	+ ·009	+ ·009
Radius at 16°	30·340	30·340	30·348	30·348

It thus appears that the maximum departure from a perfect circle amounts to 0·014 centim. in coil A and to 0·011 centim. in coil B. That is, to about two parts in ten thousand.

The results for the external radius obtained by the two methods are collected in the following table:—

RADIUS of Coils outside Layer 8 at 16°.

Method.	Coil A.	Coil B.
From circumference (1) . . .	30·346	30·348
„ „ (2) . . .	30·342	30·349
„ diameters (1)	30·340	30·348
„ „ (2)	30·340	30·348
Means	30·342	30·348

Of the two values obtained by the tape method, the first measurement was made before the magnetic measurements, and for fear of damaging the insulation of the wire no attempt was made to smooth down any slight roughness produced by the shellac varnish having stiffened projecting filaments of the silk covering of the wire. Before the second set, however, such roughness was removed by lightly rubbing the surface of the coil with a smooth piece of brass. Hence the fact that the first measurements gave a larger value for the radius is not to be wondered at.

The radius below the first layer was in the same way measured by the two methods. The individual measurements of the diameters are given in the following table:—

DIAMETERS at Bottom of Groove.

Station.	Coil A.		Means.	Coil B.		Means.
	I.	II.		I.	II.	
1–7	59·960	59·962	59·961	59·962	59·961	59·961
2–8	59·962	59·961	59·962	59·960	59·962	59·961
3–9	59·962	59·962	59·962	59·962	59·962	59·962
4–10	59·963	59·962	59·962	59·960	59·962	59·961
5–11	59·961	59·960	59·961	59·960	59·960	59·960
6–12	59·960	59·961	59·960	59·961	59·962	59·962
Means			59·961	—	—	59·961
Radius.			29·981	—	—	29·981
Correction for temperature of scale . . .			+·008	—	—	+·008
Radius at 16°			29·989	—	—	29·989

RADIUS of Coils at Bottom of Groove at 16°.

Method.	Coil A.	Coil B.
From circumference (1)	29·991	29·990
" " (2)	29·991	29·991
" diameters (1)	29·989	29·989
" " (2)	29·989	29·989
Means	29·990	29·990

The mean radius of a coil is only equal to the mean of the external and internal radii if the distribution of the wire throughout the cross-section of the coil is uniform. To test this point, measurements made during the winding of the coil are useless, as the winding on of the upper layers is likely to compress the lower layers. If, however, as in the present case, the wire has been soaked in shellac, so that the turns are bound together, the measurements taken when unwinding the coil are not subject to this error. The circumferences over each layer were measured as the coils were unwound, and the readings of the scale on the steel tape are given in the following table :—

CIRCUMFERENCES over the different Layers.

Over layer.	Coil A.	Difference.	Coil B.	Difference.
0	3·0		3·0	
		5·9		6·1
1	8·9		9·1	
		5·5		5·7
2	14·4		14·8	
		5·7		5·5
3	20·1		20·3	
		5·3		5·5
4	25·4		25·8	
		5·4		5·4
5	30·8		31·2	
		5·2		5·8
6	36·0		37·0	
		5·3		5·4
7	41·3		42·4	
		5·1		4·9
8	46·4		47·3	

This table shows that on the whole the lower layers of wire are further apart than the upper layers; that is, the mean radius of the coil is really greater than the mean of the external and internal radii. The differences between the circumference below and above the first layer give the change in circumference due to the diameter of the wire. This quantity is not given by the difference of the circumferences of any other layers, since the circumference over any layer cannot be taken as the true circumference below the next layer, for each layer sinks down a little into the hollows between the wires of the layer beneath. The mean of the values obtained from the two coils is 6·0 divisions. That this value is probably very near the truth, is shown by the fact that, if we divide the mean value of the width of the groove by the number of turns of wire which fill it, namely 12, and express the quotient as a change in circumference and in terms of divisions of the steel tape, we get 6·0 as the value.

Hence, taking 3·0 as the amount which has to be deducted from the circumference over any layer to give the circumference of the circle coinciding with the axis of the wire, we get the following values :—

CIRCUMFERENCE corresponding to the Axis of the Wire.

Layer.	Coil A.	Coil B.
1	5·9	6·1
2	11·4	11.8
3	17·1	17·3
4	22·4	22·8
5	27·8	28·2
6	33·0	34·0
7	38·3	39·4
8	43·4	44·3
Means	24·9	25·5
Mean of external and internal circumference of coils . .	24·7	25·2
Difference.	0·2	0·3

This table shows that, to obtain the true mean radius of the coils, we have to increase the mean of the external and internal circumferences in the case of coil A by 0·2, and in that of coil B by 0·3. Since one division of the tape corresponds to 0·015 centim., this corresponds to increasing the mean radius of coil A by 0·0016 centim., and that of B by 0·0024 centim. Hence:—

	Coil A.	Coil B.
Mean of external and internal radii	30·166	30·169
Correction for distribution of wire.	+·002	+·002
Mean radius of coil.	30·168	30·171

The mean radii of the two coils being so nearly alike, we can take the mean of the two numbers for the radius of the pair of coils. Thus the mean radius of the coils is $30{\cdot}169_5$ centims. at 16°. This value of the radius, as well as the value of the distance between the mean planes of the coils given on p. 439, is expressed in terms of the brass metre. To reduce these numbers to true centims., we have to deduct 0·0008 centim. from the mean radius and 0·0004 centim. from the half distance between the mean planes. Thus the dimensions of the coils are as follows:—

Mean radius	30·1687 centims.
Half the distance between mean planes . . .	15·1536 „
Number of turns in the two coils	96.

The coil constant F, which is the field produced at the centre of the coils when one ampère is passing, is thus

$$F = 1{\cdot}42671$$

at a temperature of 16° C.

The value of F changes slightly with temperature, and assuming the coefficient of expansion of the coils to be 0·0000157, a table of the values of F for each degree of the range of temperature met with during the observations was drawn up.

A consideration of the figures given in the tables of measurements will show that, as might be expected, the greatest variations occur in the measurements of the external radii of the coils. Taking into account the fact that the internal radii can be determined with more accuracy, the mean radius ought not to differ from the truth by as much as 0·005 centim. The distance between the mean planes is probably known to within about the same amount. Thus the uncertainty in F appears to be less than 5 parts in 30,000. This corresponds to an uncertainty in H of about 3 in the fifth place.

The adjustments which have to be made before the coils are used are as follows :—

1. The axis of the coil must be horizontal, and the axis about which it turns vertical.
2. The axis of the coil must be perpendicular to the magnetic axis of the magnet.
3. The centre of the coil must lie on the vertical axis of the magnetometer, and the magnet must be at the centre of the coil.

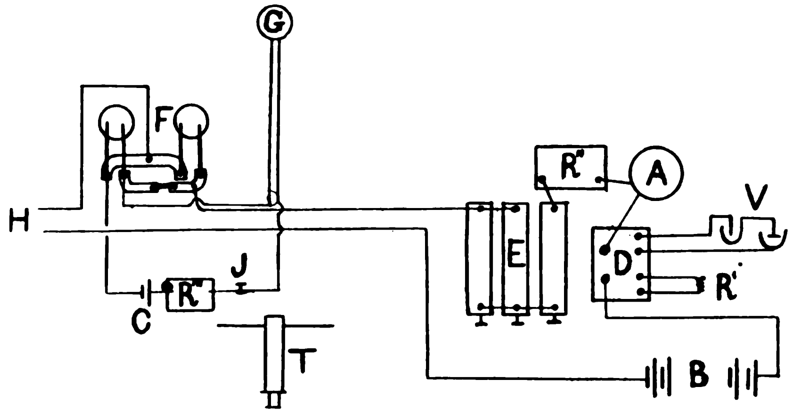
The first of these adjustments was made by means of a striding level which rested on the flanges of the coils. The magnetometer was first levelled so that the reading of the striding level remained the same when the coils were rotated. When this was complete the axis about which the coils turned was vertical. To make the axis of the coils horizontal, the screws which bear on the deflection bar (D, fig. 1) were adjusted till the level reading remained the same when the striding level was reversed.

In making adjustment 2 it was at first assumed that the magnetic axis of the needle was perpendicular to the mirror attached to the needle. If this be so, the axis of the coil will be perpendicular to the axis of the magnet when it is so adjusted that it is perpendicular to the line joining the axis of suspension of the magnet to the zero of the scale, S fig. 1. It was ascertained by experiment that this line was at right angles to the axis of the small tube which is used in the magnetometer for adjusting the height of the deflected magnet when this tube was laid in the V's used for supporting the magnet in the deflection experiment. Thus it was sufficient to make the axis of the coil coincide with the axis of this sighting tube. The position of the coil was adjusted by means of the screws which bear upon the magnetometer till the two crosses on the mica discs which indicate the centres of the two coils were on the axis of the sighting tube. Separate experiments were made in the case of the two magnets used in the experiments to measure the angle included between the magnetic axis of the magnet and the normal to the mirror. In the case of the longer magnet this angle was [illegible], while in that of the shorter magnet it was [illegible]. As the correction which would have to be applied owing to the axis of the magnet not being perpendicular to the axis of the coil, is proportional to the cosine of this angle, there will be no correction in the case of the longer magnet. The readings obtained with the shorter magnet will, however, require to be corrected on this account, that is, the values of H calculated from the deflections made with this magnet will require to be multiplied by cos 38′. This correction amounts to a little more than one unit in the fifth place in H.

Adjustment 3 was made by supporting a scribing block so that the pointer was on a level with the centre of the coils and then adjusting the position of the coils, with reference to the magnetometer, till the edges of the flanges just cleared the pointer when the coils were rotated. This adjustment was further checked by means of a right-angled slider, which rested on the deflection bar. This slider was brought to bear against a flat-headed screw, H fig. 1. The heads of these screws were adjusted by means of distance pieces and a straight edge to be at the same distance from the mean planes of the two coils. When the adjustment was complete, it was found that the reading of the slider on the deflection bar was the same of the two sides. It was found, by rotating the magnetometer when a plummet was suspended in place of the magnet and viewing the fibre just above the plummet with a telescope, that the axis of the fibre did not differ from the axis about which the coils turned by more than half a millimetre. The height of the magnet was adjusted by bringing its centre to lie on the axis of the sighting tube which was used in adjusting the axis of the coils.

Since making one of the above adjustments generally affected the others, a method of successive approximations had to be adopted. This was continued until the centre of the magnet did not differ from the centre of the coils by a millimetre, while the axis of the coil was accurately horizontal, and the axis about which it turned vertical.

Fig. 4.



General Arrangement of Circuits.—The general arrangement of the circuits employed in the experiments is shown diagrammatically in fig. 4. In this figure B represents a battery of 14 accumulators, and D is a switch, by means of which either the silver voltameters, V, or a balancing resistance, R′, can be inserted in the circuit. The resistance of the circuit could be roughly adjusted by means of the resistance, R″, which consisted of a box of manganine coils, adjustable by tenths of an ohm. The value of the current was roughly indicated by a Weston ammeter, A. The final adjustment of the resistance of the circuit was obtained by means of the three adjustable carbon resistances, E. Two of these resistances were in parallel, and the other in series with these. Each of them consisted of about 50 carbon plates in a narrow wooden box, the size of the plates being 9 centims. by 6 centims. and 0·6 centim. thick. By means of a screw passing through one end of the box the carbon plates could be compressed, and thus the resistance altered.

The standard resistance coils, the potential between the terminals of which was kept equal to the E.M.F. of the standard cell, were placed in an oil bath at F. The leads to the coil in the magnetic experiments could be inserted in the circuit at H. A Pohl commutator inserted in these leads allowed of the current being reversed in the coil without its reversal in the resistance coils F.

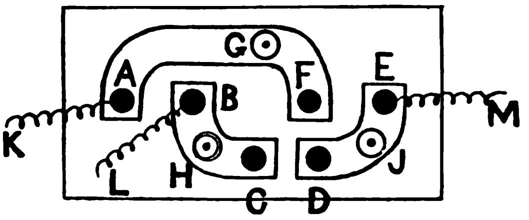
The potential circuit included the two standard resistance coils F, the cadmium cell C, a resistance of 10,000 ohms R‴, a key J on the depression of which the circuit was broken and hence the galvanometer zero could be determined, and a galvanometer G. The galvanometer was one designed by the author and constructed in the Laboratory. It has a resistance of about 570 ohms, and has four coils each of half-an-inch in diameter. The needle system consists of two sets of magnets, the individual magnets being about 3 millims. long, suspended by a long quartz fibre. The magnet system was rendered astatic, so that the distance between the two sets of magnets only amounting to ·5 inch, the field produced by the Helmholtz galvanometer when at a distance of 3 metres produced no deflection. A small magnetized sewing needle was used to bring the needle system into any desired azimuth. The position of the magnet system was observed by means of a telescope and scale, and the sensitiveness of the arrangement was such that a change in the resistance of the main circuit of 1 in 12,000 produced a deflection of 6 scale divisions (millimetres). It was a matter of comparative ease, by manipulating the screws of the carbon resistances, to keep the current so constant that the galvanometer deflection never exceeded a millimetre, that is, to keep the current constant to within 1 part in 120,000.

By means of two resistance boxes and a single accumulator a potentiometer arrangement was provided by means of which the E.M.F.'s of the standard cells could be compared amongst themselves to within 1 part in 30,000. Such a comparison was always made before and after each silver deposition. As no difference, amounting to 1 part in 10,000, was ever detected, the E.M.F. of the cell actually used, which was changed from one experiment to the next, was taken as being the same as the mean E.M.F. of all the cells.

In performing the magnetic experiments the two standard resistance coils F were a standard thermometer graduated in tenths of a degree which has been tested at the Reichsanstalt.

Since the maximum current which could be used in the magnetic experiment was such that, if this current had been used when making the silver depositions, then in order to obtain a sufficiently heavy deposit of silver the deposition would have taken eight hours, an arrangement was employed by which the two resistance coils F could be placed in series when making the magnetic measurements, and in parallel when making the silver depositions. Thus the resistance was four times as great during the magnetic experiments as during the silver depositions. The resistances of the two coils were so very nearly equal that no error was caused by taking their resistance in parallel as equal to a quarter of the sum of their resistances. In order to make the change from the series arrangement to the parallel arrangement without disturbing the coils the connector shown in fig. 5 was used. Stout copper bars AF, BC, DE (section 3 sq. centims.) were screwed to a marble base. The terminals of one of the resistance coils fitted into two cups A, B and those of the other into E, F. The terminals were soldered into these cups by amalgamating them and also the cups and then fusing some Wood's metal in the cup. Two mercury cups C and D could be connected by a short copper rod. Wires M, L and K were soldered into the cups E, B, A, and were used to connect the potential circuit. When the coils were used in series, the connector between C and D was removed, and the main circuit connected to the binding screws H and J, the wires L and M being used to connect the potential circuit. When the coils were to be used in parallel the connector was inserted and the leads of the main circuit moved to the binding screws G and J, the wires K and M being used for the potential circuit. Experiment showed that the increase in the resistance of one arm, when the coils are in parallel, due to the connector between C and D, is inappreciable.

Fig. 5.



The Cadmium Cells.—It had been originally intended to use Clark cells, and two dozen cells were prepared in 1897 according to the specification given by Kahle.[*] An equal number of cadmium cells was prepared at the same time, according to instructions given by Jäger and Wachsmuth.[†] The materials used in preparing both kinds of cells were obtained from Kahlbaum of Berlin. When, in the autumn of 1900, the Clark cells were compared amongst themselves, it was found that although the cells were kept in an ice safe for two weeks, they did not agree, the differences amounting to about 4 parts in 10,000. The cadmium cells, on the other hand, in no case exhibited a differences of 1 part in 15,000, and this the kathode, **Richards**, **Collins**, and **Heimrod** have devised a new form of voltameter in which the anode is surrounded by an earthenware porous pot, the level of the solution being kept lower inside the pot than outside. They find that this porous pot form of voltameter consistently gives values for the electro-chemical equivalent, which are 0·082 per cent. lower than the values given by the ordinary Rayleigh form of voltameter.

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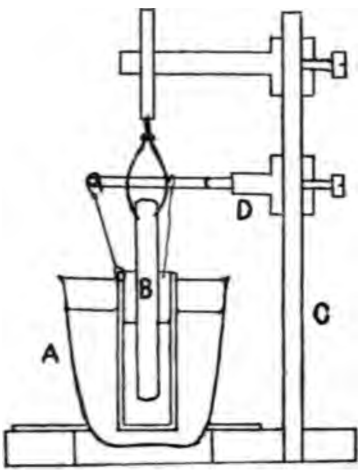
[*] 'Zeits. für Instrumentenkunde,' 1893, p. 191.

[†] 'Wied. Ann.,' vol. 59, 575, 1896.

As this new form of voltameter appeared as if it might offer some advantages over the Rayleigh pattern, two voltameters were always placed in the circuit when making the measurement, one being a porous pot voltameter, and the other a Rayleigh voltameter.

The porous pot voltameter was a close copy of that originally used by the inventors, and is shown in fig. 6. The platinum basin A rested in a circular hole cut in a piece of sheet copper attached to an ebonite base. The silver anode B was held in a spring clip attached to an arm carried by the metal upright C. A second arm D, carried a glass ring from which the porous pot was suspended by platinum wires. When in use, the platinum basin contained about 75 cub. centims. of solution, and afforded a kathode surface of about 80 sq. centims. The anode was a rod of pure silver 1 centim. in diameter, and the anode surface was about 10 sq. centims. The silver from which the anodes were made in both forms of voltameter, was obtained from the Royal Mint through the kindness of Sir **William Roberts-Austen**, and Dr. T. K. **Rose**.

Fig. 6.



The Rayleigh voltameter consisted of a platinum bowl 7·5 centims. in diameter, containing about 55 cub. centims. of solution, the kathode surface being about 80 sq. centims. The anode consisted of a small silver plate with upturned corners, the anode surface being about 10 sq. centims. The anode was wrapped in a small Swedish filter paper.

A solution containing 20 grammes of crystallized silver nitrate to 80 grammes of water was used. The solution was always tested, and found to be neutral to test paper. The same solution was never used in the voltameter twice, so that even at the end of a deposition the amount of silver deposited from the solution in the porous pot voltameter was only 0·02 gramme per cub. centim., and in the Rayleigh voltameter, 0·03 gramme per cub. centim.

The weight of silver deposited was obtained by means of a Bunge balance fitted with a microscope to read the movements of the pointer. The sensitiveness of this balance was such, that when loaded with one of the platinum basins, a tenth of a milligramme produced a deflection of three scale divisions. The weights employed advantage in this respect. The advantage of the porous pot would probably be much greater in those cases where a larger weight of silver per cub. centim. of the solution was deposited during the electrolysis. The porous pot voltameter has one disadvantage compared with the Rayleigh pattern in that its resistance is considerable, and that this resistance, unless suitable precautions are taken, will vary very considerably. Of course this change of resistance will not be of any consequence, when we simply wish to compare the weights of silver deposited in two voltameters placed in series. More frequently, however, the problem is to use the voltameter to measure a current, so that changes in the resistance of the circuit, accompanied as they must be by changes in current, are objectionable. If the porous pot used in the voltameter is dry when inserted in the voltameter or has been soaking in water, then for the first hour the resistance of the voltameter will decrease very considerably and irregularly. Thus, it was found that the increase in the resistance which had to be made to keep the current in the circuit constant, even when the total resistance of the circuit amounted to over 120 ohms, was so great as to make it almost impossible to maintain a proper balance in the potential circuit. Further it was impossible to adjust a balancing resistance, to be switched out of the circuit when the voltameter was switched in, so that no harmful change of current took place at the start of a deposition.

The above objections were to a great extent got over by keeping the porous pots, when not in use, in a jar of the same solution as that used in the voltameters. Thus the pores of the porous pot were filled with the solution even at the commencement of a deposition. The balancing resistance was generally so well adjusted that on switching in the voltameters the current did not differ from its proper value by more than 1 part in 10,000, and even this small want of balance could be set right by altering the carbon resistances in a few seconds. The current was kept constant throughout the deposition by means of these same carbon resistances, which as long as the plates are not too slack, are to all intents and purposes perfect in the way they behave.

In the following table are given the weights of silver obtained in the various experiments, some of which were made before and some after the magnetic observations. In order to be able to compare the different measurements the E.M.F. of the cadmium cell reduced to 20° as calculated from the weight of silver deposited, assuming the electro-chemical equivalent of silver as 0·001118, is given in each case.

The following table contains a summary of the results obtained. The value of H given in the twelfth column is derived from the reading of the recording magnetograph, the value of the base line of the trace being deduced from the sets of absolute measurements made with the Observatory standard instruments, by the Observatory staff, during January and February, 1901. The values of H obtained with the coil when using the shorter of the two magnetic needles, which are indicated by an S placed in the ninth column, have been corrected for the inclination of the magnetic axis of this needle. That is, the values of H have been multiplied by cos 38′. This reduces the value of H by 0·000012.

In obtaining the value of H, given in this table, the same values for the resistance of the coils have been used as were employed in deducing the E.M.F. of the cadmium cells in the table on p. 455. Also the mean value of the E.M.F., as deduced from the Rayleigh voltameter there given, was used as corresponding to the E.M.F. of the cell at 20° C. Hence the value of H given does not depend in any way on the absolute value of the resistances, or of the E.M.F. of the cadmium cell, but only on the value assumed for the electro-chemical equivalent of silver, which was taken as 0·001118 in the case of the Rayleigh form of voltameter.

Column 1 contains the number of the experiment, col. 2 the date, 3 the time, 4 the temperature of the cadmium cell, 5 the temperature of the resistance coils, 6 the temperature of the Helmholtz galvanometer, 7 the number of cadmium cells which were included in the potential circuit, 8 the numbers of the resistance coils between the terminals of which the difference of potential was adjusted to balance the E.M.F. of the number of cells given in column 7, 9 the deflection of the Helmholtz galvanometer, 10 the value of H deduced from the galvanometer deflection, 11 the value of H as obtained from the magnetograph trace, and 12 the difference between the values of H given in the two preceding columns.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
	1901.	h. m.	° C.	° C.	° C.			° ′ ″			·0000
1	Feb. 4	15 25	17·3	11·0	11	1	2+3	23 12 45	·18443	·18441	+2
2	„	15 35	17·2	11·5	11	1	2+3	23 12 54	·18440	·18439	+1
3	„	15 46	17·1	11·6	12	1	2+3	23 12 47	·18441	·18440	+1
4	„	15 55	17·0	11·6	12	1	2+3	23 12 36	·18443	·18440	+3
5	„	16 1	17·0	11·8	12	1	2+3	23 12 39	·18442	·18441	+1
6	Feb. 5	12 29	19·5	11·3	14	1	2+3	S 23 12 29	·18441	·18440	+1
7	„	12 38	19·5	11·8	14	1	2+3	S 23 12 21	·18443	·18441	+2
8	„	13 44	19·9	14·6	18	1	2+3	S 23 11 28	·18447	·18446	+1
9	„	13 52	19·8	14·8	18	1	2+3	S 23 11 31	·18446	·18446	±0
10	„	14 0	19·8	14·8	18	1	2+3	S 23 11 28	·18447	·18445	+2
11	„	14 6	20·3	15·0	18	1	2+3	S 23 11 32	·18445	·18444	+1
12	„	14 22	20·4	15·1	18	1	2+3	23 11 45	·18443	·18444	−1
13	„	14 32	20·5	15·2	18	1	2+3	23 11 40	·18443	·18443	±0
14	„	14 41	20·5	15·4	18	1	2+3	23 11 39	·18443	·18444	−1
15	„	14 48	20·4	15·4	19	1	2+3	23 11 34	·18443	·18444	−1
16	„	14 55	20·3	15·5	19	1	2+3	23 11 32	·18443	·18444	−1

The coils are so constructed that they can be turned through 180° with reference to the magnetometer on which they are carried. Observations 1 to 17 and 46 to 63 were made with the coils in one position, and observations 18 to 45 with the coils reversed. The mean values for the difference in the two positions are 0·000002 and 0·000013, which agree to nearly 1 part in 20,000.

Since the values of H, given in column 10 of the above Table, are obtained by taking the electro-chemical equivalent of silver as 0·001118, they correspond to international units. Thus the measurements lead to the result, that the absolute measurements of the earth's field made at Kew Observatory with the standard magnetic instruments, give a value 0·000007 C.G.S. unit lower than the value of this field, measured in international units.*

The mean value for the difference when the 6 centims. long magnet was used in the Helmholtz galvanometer is 0·000008, while the mean value obtained with the 3 centims. needle is 0·000006. It is thus evident that the two needles give the same value, and hence the neglecting of the length of the needle in the expression used to reduce the observations is justified.

It will be seen that the value of H obtained with the Helmholtz galvanometer agrees, within the limits of errors of experiment, with the value given by the Kew Observatory standard instruments. These measurements therefore afford evidence of the accuracy of the Kew instruments, as against the values given by many instruments of a similar type which have been compared with those of the Observatory. Since, however, the accuracy of the values given by the galvanometer depends on the accuracy with which we know the value of the electro-chemical equivalent of silver, it is necessary to discuss the measurements which have been made of this quantity.†

The various values which have been obtained for the electro-chemical equivalent of silver by different observers are given in the following table :—

1. MASCART (1882)	0·0011156.	'J. de Phys.,' ii, 1, 109, 1882, and ii, 3, 283, 1884.
2. KOHLRAUSCH, F. and W. (1884)	0·0011183.	'Wied. Ann.,' 27, 1, 1886.
3. RAYLEIGH (1884)	0·0011179.‡	'Phil. Trans.,' 411, Part ii, 1884.
4. POTIER and PELLAT (1890) . .	0·0011192.	'J. de Phys.,' ii, 9, 381, 1890.
5. PATTERSON and GUTHE (1898) .	0·0011192.	'Phys. Rev.,' 7, 257, 1898.
6. KAHLE (1899)	0·0011183.	'Wied. Ann.,' 59, 532, 1896.

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* Owing to a mistake in the corrections originally applied to the weights used in the silver depositions, the numbers given in the abstract, which appeared in the Proceedings of the Royal Society, vol. 69, p. 1, require correction. January, 1902.

† By an international ampère is meant the current which, when passed through a solution of silver nitrate in water prepared in accordance with a certain specification, deposits silver at the rate of 0·001118 gramme per second. The international ohm is the resistance of a column of mercury at the temperature of melting ice, the mass of which is 14·4521 grammes, the cross-section constant and the length 106·3 centims. The international volt is the E.M.F., which, applied at the ends of a conductor of resistance one international ohm, produces a current of one international ampère.

‡ The weights of silver used in obtaining this result are those after the silver had been heated to incipient redness. If the weights after heating to 160° C. are taken, the value 0·0011181 is obtained.

voltameter. In the silver depositions fresh solution was always used, and the silver was always deposited in basins which had already a coating of silver. The value for the electro-chemical equivalent obtained was 0·0011185. This number has to be reduced by 1 part in 10,000 if the deposit takes place on a platinum surface, as has been shown by KAHLE. Thus for a deposit made on a platinum surface the value of the electro-chemical equivalent is 0·0011184. KAHLE also measured the E.M.F. of a cadmium cell by comparison with the Clark, and then used this cell to measure the electro-chemical equivalent of silver. In this way he obtained the value 0·0011183 for a deposit on silver, or 0·0011182 for a deposit on platinum. Thus the mean value for a deposit on platinum is 0·0011183.

Taking into consideration that these measurements of KAHLE'S appear to have been made with great care and with the resources of the Reichsanstalt, and further, that the conditions under which the silver was deposited are well defined and are the same as those used by the author, it would seem that the best available value for the electro-chemical equivalent at the present date is 0·0011183.

A consideration of what has been said above will show what grave objections there are to the silver voltameter as a means of measuring a current where an accuracy greater than 1 part in 1000 is required. For this reason KAHLE has recommended that a standard cell (Clark or cadmium) together with a known resistance is a much more trustworthy means of measuring a current. Now in the experiments described in this paper the resistance coils employed were compared together immediately after the magnetic experiments, and then coil No. 1941 was sent to the Reichsanstalt to be compared with the German standards. Hence, knowing the values of the resistances used in the potential circuit, we can calculate the current employed in the magnetic experiments if we know the E.M.F. of the cells.

In order to employ this method a second set of cadmium cells* and of Clark cells were prepared according to the German directions, from a fresh batch of chemicals obtained from KAHLBAUM by Mr. F. E. SMITH, one of the Demonstrators in the Physical Laboratory at the Royal College of Science. This new batch of cadmium cells was compared with the old cells, and they were found to have an E.M.F. 0·00017 volt higher than the old. The Clark cells were compared with the cadmium cells by means of a potentiometer arrangement which contained one standard 1000-ohm coil, three standard 100-ohm coils, one standard 1-ohm coil, and a box containing 1 ohm sub-divided into tenths. Two accumulators sent a current through these coils placed in series and through a box of resistance coils and a carbon adjustable resistance. Two potential circuits were arranged which could by means of a switch be connected to the galvanometer in turn. One of these circuits included

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* These new cadmium cells had as negative poles an amalgam containing 12·7 per cent. of cadmium, while in the old cells the amalgam contained 14 per cent. of cadmium. The reason for the change is that the amalgam containing the smaller percentage of cadmium is more stable (see 'Drude Ann.,' vol. 3, p. 366, 1900).

	Clark at 15°.		Clark at 0°.		Cadmium at 20°.	
	C.G.S.	Int. v.	C.G.S.	Int. v.	C.G.S.	Int. v.
RAYLEIGH*	1·4344	1·4344	—	—	—	—
GLAZEBROOK and SKINNER†	—	1·4342	—	—	—	—
KAHLE‡	1·4324	1·4328	1·4488	1·4492	1·0183	1·0186
CARHART and GUTHE§ . .	1·4333	—	—	—	—	—

From the above table it will be seen that the value obtained by KAHLE for the E.M.F. of the Clark is lower than the values obtained by other observers. This cannot be entirely due to a true difference in the E.M.F. of the cells employed, for direct comparisons have been made between the German cells and those of GLAZEBROOK and SKINNER and of CARHART and GUTHE. In this way it was found that the E.M.F. of the cells used by GLAZEBROOK and SKINNER, which were of the Board of Trade pattern, were 0·0005 volt higher than the German (H-form), while the cells used by CARHART and GUTHE (H-form) had the same E.M.F. to within 1 part in 10,000 as the German. The difference seems to be due to the determination of the E.M.F. In the case of the values expressed in international volts, this difference is most probably due to the silver voltameter used in measuring the current. Thus, if the silver nitrate solution is not fresh, so that the electro-chemical equivalent is really greater than 0·001118, the value obtained for the E.M.F. on the supposition that the electro-chemical equivalent is 0·001118 will be too great.

CARHART and GUTHE used the same instrument for measuring the current as was employed by PATTERSON and GUTHE in measuring the electro-chemical equivalent. It consists of a torsion electro-dynamometer in which the movable coil is wound on ebonite. In addition to the difficulty in measuring with the required accuracy the mean radius of a coil of the size employed (radius 5 centims.)|| there is the further objection that the coil was wound on an ebonite reel which was by no means of constant size. Thus CARHART and GUTHE found that the shrinkage was sufficiently great to warrant them in giving a set of measurements made only four days before the measurements of radii only half weight. For the above reasons the author does not consider that these observations are of as much weight as those of RAYLEIGH or KAHLE. RAYLEIGH'S cells were prepared under very much the same conditions as those of GLAZEBROOK and SKINNER, who, as a matter of fact, took one of RAYLEIGH'S

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[*] 'Phil. Trans.,' p. 411, Pt. II., 1884, and p. 781, Pt. II., 1885.

[†] 'Phil. Trans.,' A 183, p. 567, 1892.

[‡] 'Wied. Ann.,' 59, 532, 1896.

[§] 'Phys. Rev.,' 9, 288, 1899.

[||] To obtain an accuracy of 1 part in 10,000 in the E.M.F., the mean radius must be known to within 0·0005 centim.

cells as a standard in their determinations. Hence we may apply the difference found between **Kahle's** cells and those of **Glazebrook** and **Skinner** to **Rayleigh's** cells. Thus, reduced to the German type of cell, the values obtained by these two observers for the E.M.F. of a Clark cell at 15° are:—

	C.G.S.	International volts.
Rayleigh . . .	1·4339	1·4339
Kahle	1·4324	1·4328
Difference	0·0015	0·0011

The divergence between the two results is still greater than one would expect, taking into consideration the precautions taken in the experiments.

The only experimenter who has made accurate measurements of the E.M.F. of the cadmium cell is **Kahle**.

Taking the mean of **Rayleigh's** and **Kahle's** values for the E.M.F. of the Clark cell as being the E.M.F. of the South Kensington **Clark's**, and **Kahle's** value for the E.M.F. of the cadmium cell as corresponding to the mean of all the cadmium cells set up at South Kensington, we can calculate what is the E.M.F. of the old cadmiums as deduced from these various data. The results, together with that deduced from the silver depositions, are given in the following table.

E.M.F. of the Old Cadmium Cells at 20°.

	International volts.
From silver depositions	1·01909
From **Kahle's** value for Cd. cells	1·01855
From the Clark cells	1·01898
Mean	1·01888

Hence, giving equal weight to each of these methods of obtaining the E.M.F. of the cadmium cells used in the magnetic experiments, we get a value 0·00021 volt lower than that used in the reductions already considered. Making this change we find that the difference between the values of H, as given by the coil and by the observatory instruments, is — 0·00002 C.G.S. unit.

Using all the available data, we may therefore say that the number given by the Kew Observatory standard magnetic instruments for the horizontal component of the earth's magnetism is higher than the value measured in international units by 0·00002 C.G.S. unit.

If the true value of the electro-chemical equivalent of a fresh solution of silver nitrate is 0·0011183, as found by KAHLE, then the difference between the galvanometer value for H and the Observatory value becomes − 0·00004 C.G.S. unit; in this case the field, as measured by the coil, is expressed in C.G.S. units, and the number is derived from the silver depositions *only*.

Since the observations with the galvanometer were made in the new magnet-house, while the absolute measurements with the Observatory instruments are made in the old magnet-house, which is at a distance of about 35 yards from the new one, any difference in the value of H inside the two houses will appear as a difference between the two methods. With regard to this the Director of the National Physical Laboratory writes:—"Such observations as we have do not show any evidence of a systematic difference in the value of H in the two houses. They are not, however, accurate to less than 2γ or 3γ (2 or 3 in the fifth place); and the point is one which we intend to investigate more fully. With regard to the difference in the values of H as given by the galvanometer method and magnetometer method respectively, I should like to point out that Dr. CHREE in some recent papers has called attention to one or two sources of small error or uncertainty in the magnetometer method which may possibly go some way to account for the difference. He hopes to investigate this point shortly; when this has been done it would be desirable to have some further comparisons between the two methods, more especially as the temperature conditions in the early part of February were not well suited for the work."

In conclusion, the author wishes to offer his thanks to Dr. GLAZEBROOK and Dr. CHREE for their kind assistance while he was making the observations at Kew. He is also very much indebted to his colleague, Mr. F. E. SMITH, who gave him invaluable assistance in making the observations at Kew, as well as in the silver depositions and the preparation of the standard cells. His thanks are also due to the Government Grant Committee, who defrayed the cost of constructing the instruments used in the investigation.

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VINCENT, J. H Phil. Trans., A, vol. 198, 1902, pp. 463–481.

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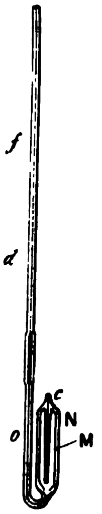
a hundred years previously. (Musschenbroek, 'Essai de Physique,' Leyden, 1739.) Mairan also supported it by experiments published ten years later. (Mairan, 'Dissertations sur la Glace,' Paris, 1749.) But Heinrich in 1807 had obtained a positive coefficient. (Heinrich, 'Gilbert's Annalen,' vol. 26, p. 228, 1807.) His result, obtained by the direct determination of the change in length of a bar of ice, yields the value ·000024 as the linear coefficient for a degree centigrade. This observer also found the density of ice to be ·905. Thus the subject stood when Brunner commenced his experiments.

Brunner started experimenting in the direction of preparing air-free ice from boiled distilled water, but failed to obtain it free from air bubbles. Even when he covered the surface of the water with turpentine immediately after boiling, the product had still to be rejected owing to its being full of small cracks; so that he was led to use selected pieces of river ice.

The method consisted in weighing the ice in air, and in either turpentine or petroleum, which latter liquid had the advantages: 1. Of its smaller density; 2. Its freedom from solvent action. He determined the density of the liquid by weighing a piece of glass in it immediately before and after weighing the ice, and he subsequently weighed the same piece of glass in water at different temperatures. He satisfied himself by direct experiment that by determining the temperature of the oil he also obtained the temperature of the ice suspended in it. The whole of the operations were conducted in a laboratory, the temperature of which never rose above freezing point. After making due allowance for the buoyancy of air, the result for the specific gravity of ice at 0° C., referred to water at 0° C., was ·9180, or ·9179 as the density in grammes per cub. centim. The linear coefficient of expansion was ·0000375, which Brunner remarks was greater than that previously found for any other solid.

Fig. 1.



The paper of Petzholdt also set Struve to work about the same time (Struve, 'Pogg. Ann.,' vol. 66, p. 298). He obtained the value ·0000531 for the linear coefficient per 0° C., using long bars of artificial ice in his experiments.

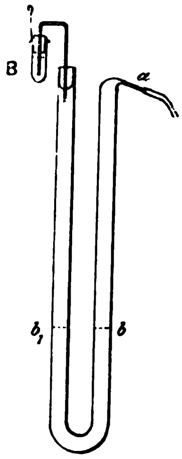
Marchand ('Journ. f. prakt. Chemie,' vol. 35, p. 254), using a dilatometer of glass containing mercury and the ice to be experimented on, obtained ·0000350 for the linear coefficient, but did not state the kind of ice used.

The dilatometric method was also employed in 1852 by Plücker and Geissler ('Pogg. Ann.,' vol. 86, p. 265, 1852), who determined the density and dilatation of artificial ice. They used a dilatometer of a remarkably elegant design (see fig. 1). The cylinder, M, of thin glass is open at the bottom, and has a capillary tube, *c*, inside it. This tube is sealed into the cylinder, M, and also into the outer cylinder, N, at one end. At the other end of the outer cylinder was another capillary tube, and in the preliminary part of the experiment, this tube reached as far as *o* only. The coefficient of expansion of the glass was first determined by preliminary experiments with mercury. To partially fill the inner cylinder with water, a bulb provided with a small opening at the top was sealed on at *c*. This bulb was filled with distilled water, and after this had been boiled for some time the upper orifice ot the bulb was sealed. During this operation the opening at *o* had been closed, but, after cooling, both the upper sealed point in the bulb and *o* were simultaneously opened. Water flowed into M, expressing an equal volume of mercury. The tube at *o* was again closed, and the water in the fine capillary at *c* was displaced by slightly warming the apparatus. This caused the mercury in which the central capillary dipped to rise, and the bulb was then removed by sealing off at *c*. Finally the capillary tube *f d o* was sealed on at *o*, and the position of the end of the column of mercury marked on the tube after the whole apparatus had been reduced to 0° C. On freezing, the water in the inner glass vessel expanded, and breaking the inner cylinder, relieved itself from constraint.

The mean result for the increment in volume of unit of volume of water at 0° C., on changing to ice at 0°, was ·09195, which is equivalent to ·91567 for the density at 0°, while ·0001585 was obtained for the coefficient of cubical expansion.

The next observations with which it is necessary to deal are those of DUFOUR ('Comptes Rendus,' vol. 54, p. 1080). Having previously experimented by finding the density of a mixture of alcohol and water in which ice floated in neutral equilibrium, he published in 1862 an account of experiments in which a mixture of chloroform and petroleum was used in preference to the former liquid, which dissolves ice. By taking the mean of 16 experiments, he obtained ·9178, with a probable error of ·0005, as the specific gravity referred to water at 0° C. He employed the value ·000158 for the coefficient of expansion for reducing his results. The ice used was prepared from water boiled *in vacuo*, and although free from air bubbles was "opaline" in appearance.

Fig. 2.



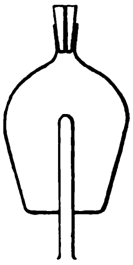
BUNSEN'S celebrated paper on Calorimetry appeared in 1870 ('Pogg. Ann.,' vol. 141, p. 1, 1870). Amongst other researches included in this memoir is a determination of the density of ice at 0° C., by a dilatometric method which, according to the illustrious author, completely eliminated the errors which had rendered previous estimations uncertain.

BUNSEN'S dilatometer is shown in fig. 2. It consisted of a thick-walled U-tube of hard glass drawn out at *a*, and this was filled with mercury up to the level, b_1 *b*, which was boiled for some time. Boiled water was sucked into the apparatus, and rested on the mercury at *b*. This water was then boiled in the tube for half an hour, the end *a* being, by means of a rubber tube *c*, led under the surface of water, which was also kept boiling. The dilatometer was allowed to cool, while the vessel into which c dipped was kept boiling; the side of the tube ab then became completely filled with air-free water, and the point a was then sealed off. By weighing before and after filling with water, the mass of water taken was obtained. The other limb of the U-tube was now filled with mercury, and the water was frozen by subjecting it to cold in such a way that the water froze from above downwards. The ice thus formed was absolutely clear. The cork and capillary tube shown in the figure were then inserted, and the whole apparatus surrounded with dry snow. The mass of the vessel B and its contained mercury was noted. On melting and again reducing to 0°, re-weighing the vessel B provides the other datum requisite to compute the density of ice at 0° C. BUNSEN'S mean value was ·91674.

No experiments on these subjects seem to have been published again until quite recent times, when NICHOLS brought out his paper on the density of ice in 1899 ('Physic. Review,' vol. 8, January, 1899). Leaving the theoretical portion of this memoir out of consideration for the present, we find that NICHOLS determined the density of artificial and natural ice by several methods.

Method 1. *Specific Gravity Bottle.*—The apparatus consisted of a specific gravity bottle fitted with a tube (see fig. 3), round which a cylinder of ice was formed. The unfrozen water was shaken out, and the whole again subjected to cold; by weighing in a laboratory, whose temperature was below freezing point, the mass of ice taken was found. The bottle was now filled up with cold mercury, the stopper inserted, and the whole left in an ice bath overnight with the stopper dipping in mercury. Finally, the stopper was dipped into a weighed quantity of mercury, the ice permitted to melt, and the whole apparatus again reduced to 0° C. The loss of weight of the mercury into which the stopper dipped, gave the means of computing the density of the ice mantle at 0° C. free from any error due to deformation of the flask on filling with mercury. The result for the density from a single experiment was ·91619.

Fig. 3.



Method 2. BRUNNER'S *Method.*—NICHOLS employed refined petroleum, and weighed several varieties in it, again working in a laboratory below freezing point. The results were reduced to 0° C., by employing the value ·00015 for the coefficient of cubical expansion. The results obtained were :—

Kind of ice.	Density at 0° C.	Kind of ice.	Density at 0° C.
Artificial	·91603	Natural . . . . . (new pond ice)	·91792
Natural . . . . . (icicles)	·91795	Natural . . . . . (pond ice, 1 year old)	·91632

Method 3. *Determination of the Volume of the Ice by Displacement.*—NICHOLS next attacked the question by the employment of an absolutely original method. An iron box of special construction, having a capacity of about 2 litres, and rectangular in shape, was nearly filled by means of a regular block of new pond ice, and the rest of its interior was filled up with mercury. From the weighings of this mercury and the ice, and a knowledge of the volume of the box, the density of the ice at 0° C. was computed. The ice contained a small quantity of air, the amount of which was separately determined and allowed for. The final value for the density came out at ·91760.

Nichols next turned his attention to the determination of the linear coefficient of expansion of ice. No work had been done on the dilatation of ice since 1852. The method employed (' Physic. Review,' vol. 8, p. 184, 1899) was similar to that used by Struve in 1845. A bar of commercial artificial ice, which had been manufactured some months previously, was used, and Nichols again had the felicity of working in a laboratory which was never warmer than − 3° C. during the work. The readings were obtained by measuring, by means of a dividing engine, the distance between the centres of two tiny drops of mercury resting in depressions in the ice about 40 centims. apart. The range of temperature was from −8° to −12° C. Four sets of readings were taken, with the mean result ·0000540 for the linear coefficient.

Nichols's *Theory*.

In order to explain the remarkable discrepancies between the values obtained by previous observers, Nichols put forward the theory that in reality there are two kinds of ice which have been under experiment; the density of artificial ice being about ·916, and that of natural ice more than one part in a thousand greater. This immediately throws the subject into a more tangible form, but the serious consequences of such a dual character for ice demand most careful consideration. It seems to me that if there are really two kinds of ice, differing in density so largely, these varieties would also have different latent heats, and, what is perhaps more important still, different melting points.

Nichols's theory is, however, supported entirely by his own work, and also by most of the results of previous observers. In this connection it must be pointed out that, according to one of Nichols's experiments, natural ice assumes a density approaching to that of artificial ice if the natural ice has been kept some time.

The value obtained for the density by Dufour for artificial ice is larger than that of other observers using the same variety. But the method of neutral equilibrium is far inferior in exactitude to the methods employed by Plücker and Geissler, Bunsen, and Nichols. Neither must the results of Bunsen be accepted as being of extraordinary reliability in spite of his assurance that he had eliminated all error. The U-tube dilatometer suffers from the disability that any small difference in the method of holding it may cause considerable change in its voluminal contents. It must also be remembered that the ice in Bunsen's experiment was probably under considerable pressure, since each new layer as it was formed became the vehicle of transference of heat upwards from the underlying water. Ice is one of the most contractible of solids by fall of temperature, and thus when the whole of the water was frozen, it must have been considerably denser than it would be at 0° C. The sides of the somewhat narrow tube would tend to prevent the ice assuming its proper density as the temperature rose to 0° C. It should be noted that if the mean temperature of BUNSEN's ice column was 4° or 5° below zero, this would suffice for the somewhat high value which he obtained. There is another and more serious objection to BUNSEN's method. Any attempt to get ice exactly at 0° C. by surrounding it with an ice jacket may result either in the resulting temperature being lower than 0° C., through the observer not allowing a sufficiently long time for the equalisation of temperature or, on the other hand, may result in some of the ice melting. In either case the value obtained for the density will be too high. None of these objections apply to PLÜCKER and GEISSLER's work.

BARNES ('Physic. Review,' July, 1901) has recently determined the density of natural ice by weighing selected specimens in water. His results give the same density for old and new ice. The mean value obtained (expressed in grammes per cub. centim.) was ·91649.

Synopsis of Previous Work.

In order to facilitate reference, the results of previous workers have been set out in Table I., which gives the methods adopted, the variety of ice used, and the results obtained by different observers.

In Table II. the results for the two kinds of ice are separately set forth. The work of MARCHAND is omitted altogether from this table, as he did not state what kind of ice he used. The value of the density obtained for old pond ice by NICHOLS is also not included. The mean result for the density of *natural* ice at freezing point is ·9176 gramme per cub. centim., while that of *artificial* ice is ·9165 gramme per cub. centim.

If, however, we neglect DUFOUR's value, we obtain the result ·9162 gramme per cub. centim. for artificial ice.

Only one estimation of the dilatation of natural ice is available. It is ·0001125 for the cubical coefficient of dilatation for 1° C., while three results are available for artificial ice. The mean value is ·000160 for the cubical coefficient of dilatation for 1° C.

Only one direct determination of the cubical expansion of artificial ice is to hand. This was obtained by PLÜCKER and GEISSLER, and is ·0001585 for the cubical coefficient of dilatation for 1° C.

In both Tables I. and II. the cubical coefficient only has been tabulated. In those cases in which the linear coefficient was actually determined, the cubical coefficient

Table I.

Date.	Observer.	Method.	Kind of ice.	Density.	Coefficient of cubical expansion.
				(grammes per cub. centim.)	
1845	**Brunner**	Weighing in liquid	Natural	·9179	·0001125
1845	**Struve**	Direct measurement of linear coefficient	Artificial	—	·0001593
1845	**Marchand**	Dilatometric	?	—	·0001050
1852	**Plücker** and **Geissler**	Dilatometric	Artificial	·91567	·0001585
1862	**Dufour**	Neutral equilibrium in liquid	Artificial	·9177	—
1870	**Bunsen**	Dilatometric	Artificial	·91674	—
		Dilatometric	Artificial	·91619	—
			Artificial	·91603	—
			Natural (icicles)	·91795	—
1899	**Nichols**	Weighing in liquid	Natural (new pond ice)	·91792	—
			Natural (old pond ice)	·91632	—
		Volume by displacement	Natural (new pond ice)	·91760	—
1899	**Nichols**	Direct measurement of linear coefficient	Artificial	—	·0001620
1901	**Barnes**	Weighing in water	Natural	·91649	—

Table II.

Observer.	Natural.		Artificial.	
	Density.	Co. of Cub. Exp.	Density.	Co. of Cub. Exp.
Brunner	·9179	·0001125	—	—
Struve	—	—	—	·0001593
Plücker and **Geissler**	—	—	·91567	·0001585
Dufour	—	—	·9177	—
Bunsen	—	—	·91674	—
Nichols	—	—	·91619	—
	—	—	·91603	—
	·91795	—	—	—
	·91792	—	—	—
	·91760	—	—	—
	—	—	—	·0001620
Barnes	·91649	—	—	—
Mean	·9176	·000112[illegible]	·9165 or ·9162 neglecting **Dufour**	·000160

has been tabulated as three times the linear, but the legitimacy of this procedure is open to grave doubt in the case of a body like ice which is endowed with hexagonal symmetry of structure.

Principle of the Method employed.

Since the question of the density of ice was still, in spite of all the labour that had been spent upon it, in a far from satisfactory state, and since a direct determination of the Cubical Coefficient of Expansion had not been attempted since 1852, I was desirous of employing a method which should yield both results. In order that the work should have any value, it was necessary to employ some device other than any which had been used previously.

The method of weighing ice in mercury was one which naturally suggested itself. For the purpose of a sinker two metals are available, tungsten and platinum. Tungsten is difficult to work, but is readily procurable in any amount; if mercury attacks tungsten this could be avoided by protecting it by an iron shell. Although this direct method was not employed, I believe that the use of a platinum or tungsten sinker for weighing ice in mercury would be well worth the attention of future workers.

The necessity of a sinker can be done away with if the buoyancy of the ice is obtained by determining the tension of a wire which moors it to the bottom of the vessel. The tension of the wire may be found by passing it through a small hole in the bottom of the mercury-containing vessel, which latter must be closed at the top so that the mercury will not pour out of the hole through which the wire comes. This method obviates the use of a balance, for a scale pan may be hung on the wire, and equilibrium obtained by suitably adjusting the suspended weights. JOLY has used this principle in the construction of a balance (JOLY, 'Phil. Mag.,' September, 1888), and my apparatus differs from his, in that I introduce the material whose density is to be determined and use it as the float.

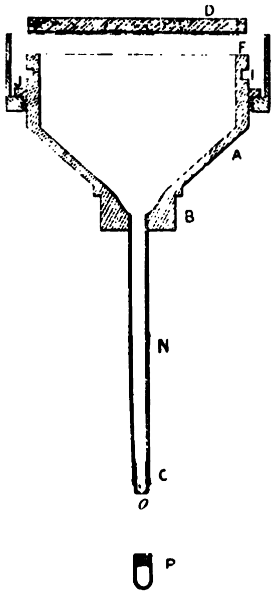
The advantages to be derived from the use of mercury are several. It can easily be obtained quite pure, is without solvent action upon either water or ice, and its coefficient of expansion is of the same order of magnitude as that of ice. Further, this coefficient is known with greater accuracy. The use of a liquid whose coefficient of expansion is near that of ice is helpful in determining the density at 0° C., and also in the determination of the coefficient of cubical expansion.

In order to determine the density at 0° C. it would be sufficient under ideal conditions of temperature to find the buoyancy of an inverted vessel in mercury, to introduce air-free water into this vessel, to determine the buoyancy of the vessel and also that of the water in its liquid and solid state. The experiments were performed by equilibrating the ice at temperatures below zero, and to find the density at these temperatures we must allow for the contraction of the mercury and of the vessel containing the ice. The equilibration of the water was always performed at zero.

Description of Apparatus.

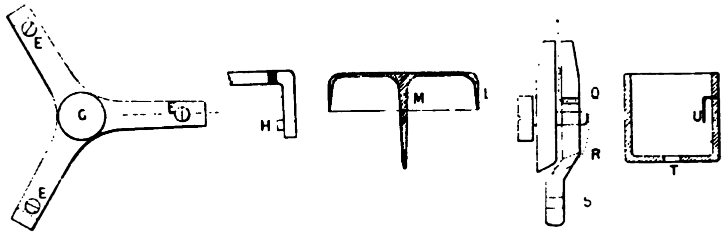
The vessel which constituted the reservoir for the liquid used in the hydrostatic balance is shown in section in fig. 4. It consisted of a funnel-shaped vessel, A, of cast iron. This was turned up in a lathe, and care was taken to have the inside surface quite free from "blow-holes." Into this vessel at B, a tube of steel, N, was driven. The lower end of this tube carried a tightly-fitting screwed piece C, the bottom of which was closed, except for a central hole about ·4 millim. in diameter. This hole was for the wire to pass through downwards to support the scale pan, and upwards to tether the vessel which held the ice or water. The vessel shown in fig. 4 may be called (to save circumlocution) the funnel.

Fig. 4.
(¼ size.)



The top of the funnel could be closed by an accurately fitting steel circular plate, shown in section as D. In order to fasten this plate down securely on to the top of the funnel, a three-armed piece of iron, shown in fig. 5*a*, was used. This was provided with three screws, E, the lower ends of which bore directly on the circular plate D, immediately over the annular plane-bearing surface, F. A central screw, G, with a milled head, served to hold the plate D in position in some manipulations which did not require that the surfaces at F should fit very closely. The three-armed piece was provided with three stout pegs, one of which is shown as H in fig. 5*b*, which is a section of one of the portions of the tri-radiate clamp of which fig. 5*a* is a plan. These pegs could traverse round the channel, I, fig. 4, and their pressure on the upper roof of this channel, provided reactions for the pressure of the four screws E, E, E, G. In order to place the tri-radiate clamp in position, three small gaps were cut at angular intervals of 120° in the outside of the top rim of the funnel, and three similar gaps were provided in the plate D.

Figs. 5*a*—8.



5*a* (¼ size.) 5*b* (¼ size.) 6 (¼ size.) 7 (5 times size.) 8 (¼ size.)

The funnel was provided with a screwed collar, J, which was permanently shrunk on to the cylindrical portion of its outer surface. This collar served to support a removable ring of iron, K. The walls of this ring (which we may call the mercury collar) were higher than the top surface of D, when this latter was in position. The mercury collar served two purposes; it provided a means of sealing the whole of the top of the funnel by flooding with mercury when the closing plate D was in position, the mercury it contained surrounded the bulb of the thermometer which was used to find the temperature of the contents of the funnel. When the funnel was not in position for the actual determinations of buoyancy, it could be held in a vice by the flat surfaces cut in the thick metal at B.

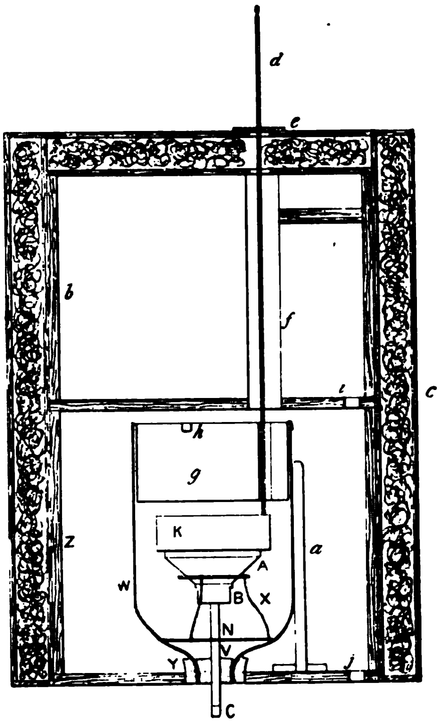
The vessel which served to contain the water or ice, while it and its contents floated in the mercury in the funnel, is shown in fig. 6. It was somewhat of an umbrella shape, and was cut out of a solid block of mild steel in the lathe. It was perfectly smooth, and provided no lurking places for air. The sides, L, were made of decreasing thickness downwards as also was the central stem, M, which was pierced at its lower end with a hole which served to attach the wire by which the scale pan was supported.

Steel wires of two diameters were used in the experiments—one, about ·17 mm. in diameter, was used in the preliminary investigation of the dilatation of the umbrella, the other, about ·2 mm. in diameter, was used in the actual experiments when the umbrella held ice or water. The wire passed through the mercury in the funnel, down the centre of the tube, N, through the small hole in C, and had a specially constructed clamp attached to the end outside C. This clamp was made so that when held up close to the hole *o* by the buoyancy of the umbrella and its contents, it could be enveloped by the screwed closed tube, P (fig. 4). The clamp is shown in fig. 7. The peculiarity in the construction of this clamp was that it was pierced by the holes Q and R, through which the wire passed as well as being held by the jaws. This arrangement made the chance of the wire slipping very small. The course of the wire is indicated by a broken line in fig. 7. The lower portion of the clamp was also drilled with a screwed hole, S. This hole served to receive a hook by which the scale pan used to hold the weights in the equilibrations was supported, and was screwed inside so as to enable the clamp to be attached to the piece C (fig. 4) always at a definite distance by means of a second screwed piece which could be attached to C. The object of this arrangement was to ensure always that the wire was the same length.

In fig. 8 an iron cylindrical vessel is shown in section. This reservoir could be put on the tube N (fig. 4), which was made slightly conical at its lower end in order to fit into the hole T. A bent wire, U, was fixed into the side of the reservoir, and served to keep the stopper, P, submerged when the reservoir was full of mercury. The use of this portion of the apparatus will be referred to in describing the process of filling the funnel with pure, dry, air-free mercury.

The method of holding the funnel during the determinations, and the arrangements for surrounding it with ice or freezing mixture, are shown in fig. 9.

Fig. 9.
($\frac{1}{9}$ size.)



The tube, C N, of the funnel, A, passed through an india-rubber bung, V, which closed the lower orifice of a large glass jar, W. The funnel was supported by a framework of iron, X, which consisted of two rings joined by three bent wires. The lower ring rested in the concavity of the glass jar and bore the weight of the funnel. The jar in turn was supported by a larger rubber bung, Y, which fitted into a hole in the bottom of the lower wooden box, Z. Three stout brass pieces, *a*, only one of which is shown in the drawing, prevented the jar from tipping sideways.

The upper wooden box, *b*, rested on the lower box, Z, while the whole was surrounded on five sides by the non-conducting cases, *c*. These cases were removable boxes of wood loosely filled with cotton wool.

The thermometer, *d*, passed through a piece of ebonite, *e*, which rested on the top of the non-conducting case, through a hole in the case, down a wide brass tube, *f*, in the upper wooden box, and rested with its bulb in the mercury in the collar, K. The removable thin metal vessel, *g*, was supported by three lugs, *h*, which were bent over so as to engage the rim of the glass jar. The upper and lower boxes were fitted with the holes *i*, *j*, which could be closed by bungs.

The whole of the apparatus shown in fig. 9 rested on three levelling screws, which in turn were supported by a strong table having a hole in its centre through which the wire from the funnel passed.

The lower surface of the lower box was protected from the access of heat by filling in the space between it and the table with loosely packed cloth. The description of any other parts of the apparatus which may be necessary will be more conveniently given when dealing with the conduct of the experiments.

Determination of the Buoyancy of the Umbrella.—In order to find the buoyancy of the ice and water, it was necessary to determine that of the umbrella at different temperatures. If we know the load on the scale pan necessary to obtain equilibrium when the umbrella only is tending to float in the mercury, then the weights added when the umbrella and its contents are equilibrated, gives us the buoyancy of the contents. The funnel was taken and a thin steel wire (·17 millim. in diameter) passed upwards through the hole *o* (fig. 4) until it could be threaded through the hole in the stem of the umbrella, when it was fixed by twisting the end round itself. This operation was done in such a manner as to ensure that the length of wire thus used in the fastening was the same in each experiment. The wire was then pulled tight and the little clamp attached with its jaws about 1 centim. distant from *o* by the method previously mentioned. The whole length of wire used was thus always the same. The stopper, P (fig. 4), was then screwed on C and mercury poured into the funnel.

The mercury used throughout these experiments was first cleaned by the ordinary chemical methods, then boiled in air, and distilled twice in a vacuum. After being used in one experiment it was filtered, boiled and distilled twice again before being used in a fresh determination.

The plate D was placed on the funnel and the whole was turned on one side to enable any air imprisoned under the umbrella to escape. This was repeated until no more mercury could be poured into the funnel, when the latter was heaped up with mercury and the plate D was slid on to the top of the funnel and firmly screwed into position by means of the tri-radiate clamp. It was found necessary to carefully grind the bearing surfaces of the closing plate and of the funnel together before each experiment in order to obtain a perfect fit.

After thus filling with mercury, the funnel was removed from the vice and placed in an inverted position on a tripod. The stopper was then unscrewed from the tube, when the little clamp could be seen supported on the top of a straight piece of wire projecting a few millimetres through the hole at the end of the tube. The mercury reservoir (fig. 8) was slipped on to the conical tube and the mercury in the funnel was then boiled by applying the flame of a large Bunsen burner to the closing plate. The mercury which came out of the small hole in this process, partially filled the reservoir, which on removing the flame was filled with mercury boiled in another vessel. The whole was left to assume the ordinary temperature, and then the stopper was inverted and plunged beneath the mercury in the reservoir, where it was prevented from rising to the surface by the wire U.

The boiling was again performed, and then the mercury in the tube and reservoir was also boiled. Operations of alternate heating and cooling were continued until I felt satisfied that no air or water remained in the funnel. The stopper was then taken from the wire in the reservoir, and manipulating it so as never to permit the fingers to come beneath its orifice, it was screwed on to the piece C, and the funnel was thus closed.

The reservoir was then removed, and the funnel was again put in the vice; this time with the mercury collar round it ready to be screwed on. The tri-radiate clamp and the closing plate were removed and the bearing surfaces were covered with a thin film of vaseline before pouring an excess of recently boiled mercury into the funnel and sliding the closing plate again into position. The plate was then fastened down securely and the mercury collar screwed on.

The funnel was then installed in the apparatus shown in fig. 9, when the collar was filled with mercury and the top of the funnel thus completely sealed. The stopper was then unscrewed and a small vessel of hot recently boiled mercury was placed so that the hole *o* dipped beneath its surface. This caused the mercury in the tube to expand and to displace any air from about the orifice of the tube as the exuded mercury flowed through it.

The funnel was now levelled by adjusting the screws on which the bottom box rested, the level being placed with its legs on the top of the closing plate. The wire mooring the umbrella down then coincided with the axis of the tube. In order to find the buoyancy of the umbrella at 0° C., the vessel *g* was not used, but the jar and the boxes were filled with table ice, and the cases *c* placed round the apparatus. Cloths were packed under the lower box, and the whole was left overnight.

When the equilibration was to be performed, the small mercury vessel into which C dipped was removed, the scale pan attached, and the weights adjusted so that equilibrium was attained when the wire projected 4 millims. from the hole. The reading having been taken, the pan was removed and the hole was again closed with hot mercury.

In order to get readings below 0° C., the ice was all removed, *g* was put in position filled with ice and salt, and the boxes were filled with the same mixture. Then the thermometer was inserted so as to have its bulb in the mercury collar.

The space round the funnel was clear of the freezing mixture, and the funnel thus changed in temperature so slowly that the thermometer readings could be relied on as giving the temperature of the funnel and its contents. All readings were taken with the thermometer slowly rising.

The thermometer used in these experiments was made by Hicks, and had the portion which projected above the cotton wool case graduated from 1° C. to 10° C. in tenths of a degree. I tested its accuracy at 0° C., and could find no error. The temperature rose (after the apparatus had been left a day or so) about a degree in three hours, and readings of the buoyancy could be obtained at intervals.

The results for the weights necessary to be added to the pan, which itself weighed about 60 grammes, are set out in Table III., and shown also in fig. 10.

It will be seen that the last weighing taken after five others agrees closely with the first, showing that no air gained access to the umbrella in the process of equilibration.

A new set of platinized weights by Oertling was used in these experiments. They were tested after the experiments, and were found to be consistent with themselves. Their absolute mass is not involved in the determination.

Fig. 10.

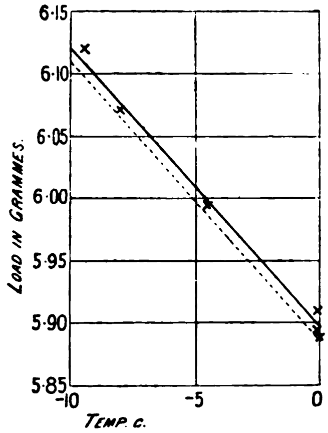


TABLE III.—Buoyancy of Umbrella.

Order of Weighing.	Circumstances.	Temperature.	Weights in grammes.
1st	After 15 hours in ice	0°	5·890
2nd	After 21 hours in 1st freezing mixture	− 8°	6·071
3rd	After 33 hours in 1st freezing mixture	− 4°·48	5·995
4th	After 45 hours in 1st freezing mixture	− 0°·1	5·895
5th	After 45 hours in 2nd freezing mixture	− 9°·42	6·120
6th	After 72 hours in 2nd freezing mixture	− 0°·1	5·910

Determination of the Density of Ice at different Temperatures.

The wire and clamp used in equilibrating the water and ice were not the same as those used with the empty umbrella. The values for the buoyancy of the umbrella as read from the unbroken straight line on fig. 10 are thus subject to a correction of ·012 gramme, which must be subtracted from the values thus found. The results for the buoyancy of the umbrella are thus taken from the broken line in this figure.

Let W = the number of grammes necessary to equilibrate the water at 0° C. ; *i.e.*, the load in the pan less the corrected buoyancy of the umbrella,
I = the number of grammes to equilibrate the ice at $-t$° C.,
i_{-t} = the density of ice at $-t$° C.,
w_0 = the density of water at 0° C.,
h_0 = the density of mercury at 0° C.,
h_{-t} = the density of mercury at $-t$° C.

Then equating the two values obtained from the above for M, the mass in grammes of the material taken, we have

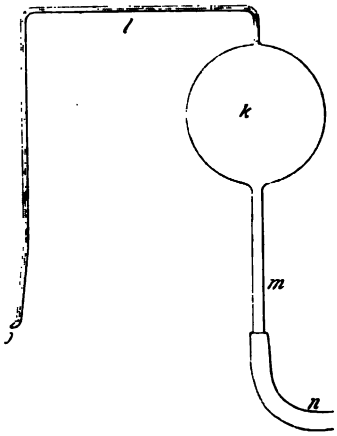
$$\frac{M}{w_0} h_0 - M = W \quad \text{and} \quad \frac{M}{i_{-t}} . h_{-t} - M = I ; \qquad \text{therefore} \; \frac{w_0}{h_0 - w_0} W = \frac{i_{-t}}{h_{-t} - i_{-t}} I .$$

$$\text{Let} \qquad K = \frac{I}{W} \quad \text{and} \quad \frac{w_0}{h_0 - w_0} = q ; \qquad \text{then} \; i_{-t} = \frac{q h_{-t}}{K + q} ,$$

which was the formula used in computing the results. Since the number K only depends on the ratio of the weights, no correction for the effect of the buoyancy of the air is necessary.

The density of water at 0° C. was taken as ·999884, and that of mercury at the same temperature as ·135956, while the density of mercury at lower temperatures was found from the formula of CHAPPUIS (' Procès-verbaux des Séances du Comité International,' 1891, p. 37). The results needed were read off from a plotted curve.

Fig. 11.
(¼ size.)



The funnel having been filled with pure dry air-free mercury in the manner already described, the closing plate was removed and air-free water introduced under the umbrella. This was accomplished by means of the apparatus shown in fig. 11.

This consisted of a glass bulb, *k*, which had a capillary tube, *l*, sealed into it above, and which terminated in a stout tube, *m*, below. The whole could be supported by fixing this tube in a clamp. A flexible rubber tube, *n*, communicated with the bulb and with a somewhat larger reservoir not shown in the figure.

Mercury was poured into this reservoir, and it was raised until the mercury poured out of the orifice, *o*, in the capillary tube, which was plunged into a beaker of water which had been kept boiling for half an hour. The reservoir was then lowered, and the boiled water rushed into the bulb, *k*. On again raising the reservoir, this water was expelled, and boiled water was then introduced in its place. This was repeated seven or eight times, when the mercury reservoir was raised and the beaker removed. The water spurted out of *o* in a rapid stream, so that no air could pass back into the bulb. The point *o* was then dipped into the mercury of the funnel, and as the water was ejected from the capillary tube it floated up and occupied the upper portion of the inside of the umbrella. This was kept in position during filling by a set of three stout iron wires fixed in a wooden board, which was fixed on the rim of the funnel so that the lower ends of the wires pressed on the flat top of the umbrella. This was necessary because the umbrella was only stable when it contained more than a certain quantity of water. The point *o* was so made that although the water was projected upwards, the point offered no projection for the rim of the umbrella to catch upon, so that when sufficient water had been introduced, the tube could be removed.

The water used in these experiments was prepared from ordinary distilled water by re-distilling in a new block-tin still, the earlier products of the second distillation being rejected.

After filling, the funnel and its contents were allowed to cool; the funnel was then closed with the closing plate as already described, and was placed in the apparatus, fig. 9, the boxes having been previously washed out to get rid of the salt from the freezing mixture. The mercury to seal the top was poured into the collar, and the

TABLE IV.

I.	Equilibrations. II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.
Experiment.	Water at 0° C.	Ice at $-t$° C.	Temp. $= -t$° C.	W by subtracting 5·886 from numbers in column II.	Buoyancy of umbrella at $-t$° C., from fig. 10.	I by subtracting numbers in column VI from those in column III.	$K = \frac{I}{W}$.	$K+q$. $q = {\cdot}07938$.	h_{-t} from fig. 11.	$i_{-t} = \frac{qh_{-t}}{K+q}$.
1	(*a*) 646·225	(*b*) 709·350	−1·95	640·354	5·930	703·420	1·09849	1·17787	13·6004	·916603
	(*e*) 646·255	(*c*) 709·675	−10·02	—	6·112	703·563	1·09871	1·17809	13·6204	·917780
	Mean of (*a*) and (*e*) =643·240	(*d*) 709·300	−1·01	—	5·909	703·391	1·09844	1·17782	13·5981	·916487
2	(*a*) 860·055	(*b*) 945·235	−2·90	854·169	5·952	939·283	1·09965	1·17903	13·6028	·915863
		(*c*) 945·100	− ·37	—	5·894	939·206	1·09956	1·17894	13·5965	·915509
3	(*a*) 822·495	(*b*) 903·450	−8·37	816·609	6·074	897·376	1·09890	1·17828	13·6163	·917355
		(*c*) 903·080	−1·02	—	5·909	897·171	1·09865	1·17803	13·5981	·916324
4	(*a*) 792·690	(*b*) 871·060	−2·60	786·804	5·945	865·115	1·09953	1·17891	13·6020	·915903
		(*c*) 870·760	−6·59	—	6·034	864·726	1·09904	1·17842	13·6119	·916950
		(*d*) 870·580	−3·15	—	5·957	864·623	1·09890	1·17828	13·6034	·916487

funnel was then levelled. The jar and boxes were filled with ice, and the equilibration of the water was performed after waiting about eighteen hours.

The ice in the boxes was then exchanged for freezing mixture, the glass jar was emptied of its ice, and the metal vessel *g* was filled with freezing mixture and put into the jar. After leaving for two days, the ice was equilibrated, and this could be done at different temperatures, as the ice very gradually rose in temperature. The ice could then be examined, or a new equilibration of the water could be obtained, by allowing the ice to melt and bringing the whole again to 0° C.

The results of the different equilibrations and the computation of the density of ice for different temperatures below 0° C. are set out in Table IV.

In this table the letters prefixed to the equilibrations in columns II. and III., indicate the order in which these numbers were obtained. In finding the numbers in column IX., the values of *q* is only needed to four significant figures; but to compute the density of ice at different temperatures (column XI.) this same quantity *q* is ·0793829.

The whole of the equilibrations performed with the final form of apparatus are given in the table. An improvement of the filler was introduced subsequently and anomalous results were obtained for the only experiment carried out. This was traced to the fact that the wire had become damaged during the filling, and the results for this experiment were rejected.

In the case of the first experiment the water was equilibrated at the beginning and end, and the mean value was used for computation. The ice in this experiment was not examined. In Experiments 2 and 3, the ice was taken out after the last equilibration and was found to be free from milkiness and air bubbles, but it had fine circular cracks running round it concentric with the central stem of the umbrella.

The fourth experiment was conducted differently from the others, and the values obtained proved unmistakably that the same specimen of water may assume different densities on freezing. After the value *b* had been obtained, the ice was permitted to melt either partially or completely. A fresh freezing mixture was put in the apparatus, and two subsequent equilibrations of the new specimen of ice were performed. The second specimen of ice had a considerably greater density than the first although it was made from identically the same water.

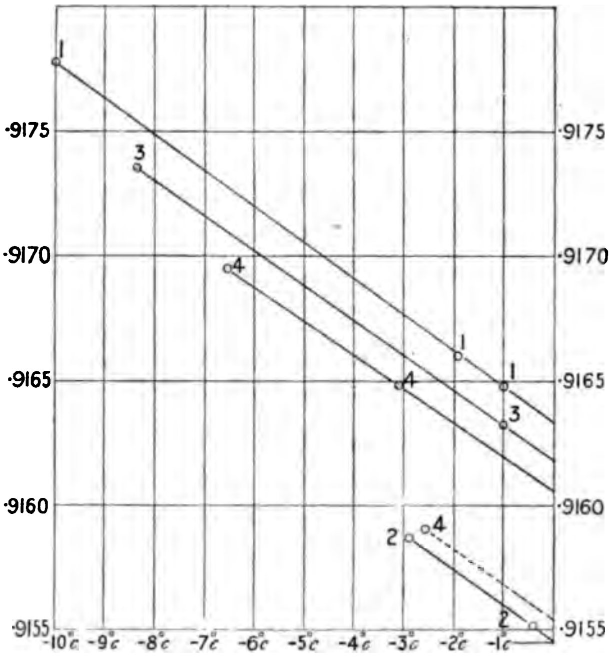
The results given in column IX. are plotted in fig. 12. The points are marked with numbers indicating the experiment. The unbroken straight lines drawn through the points give us, by extra-polation, four values for the density of ice at 0° C., while a broken line drawn parallel to the straight line for Experiment 2 through the point given by the first specimen of ice in Experiment 4 furnishes a fifth value. The numbers thus obtained and the weights to be assigned to them in computing the mean are set out in Table V., the weight assigned in each case being equal to the number of equilibrations of the ice.

TABLE V.

Experiment.	Density of Ice at 0° C.	Weight Assigned.
1	·916335	3
2	·915460	2
3	·916180	2
4	·915540	1
	·916060	2
Weighted mean . . .	·9160	

We thus obtain ·9160 gramme per cub. centim. as the density of the ice at 0° C. This result depends upon the assumption that the density of ice is a linear function of the temperature on a mercury-in-glass thermometer. Systematic error in the thermometer, so long as the zero is correct, is eliminated. The result likewise depends on the values assumed for the density of water and of mercury at zero, but is independent of the absolute mass of the weights employed.

Fig 12.



The Coefficient of Cubical Expansion of Ice.

The errors of the thermometer are, however, involved in computing the coefficient of cubical expansion which also depends upon the particular law of dilatation of mercury used, but these errors are probably such as will not affect the result to the accuracy with which it is given below. The four values which can be found from the data available are set out in Table VI.

TABLE VI.

Experiment.	Coefficient of cubical expansion.	
1	·000155	Mean ·000152
2	·000152	
3	·000153	
4	·000148	

Comparison of Results.

The value ·9160 for the density of ice at 0° C., is lower by two parts in 10,000 than the mean of the results obtained by **PLÜCKER** and **GEISSLER**, **BUNSEN**, and **NICHOLS**. It is 1 part in 10,000 less than the mean of **NICHOLS**'s values, but is 7 parts in 10,000 lower than **BUNSEN**'s value. The value ·000152 for the coefficient of cubical expansion is 4 per cent. lower than that of **PLÜCKER** and **GEISSLER**, the last published value for the directly determined cubical coefficient of artifical ice. It is 5 per cent. lower than the mean value given in Table II.

Conclusion.

The results of this determination of the density and coefficient of cubical expansion of ice are, that **NICHOLS**'s value for the density is confirmed, and that **BUNSEN**'s value is probably too high ; but as the same specimen of water can freeze into specimens of ice having different density, the use of the Bunsen ice calorimeter in absolute determinations must be limited to an accuracy of probably about 1 in 1,000.

The coefficient of cubical expansion seems to be 4 or 5 per cent. less than the mean of previous determinations.

The expenses of this research have been in part defrayed by a Government grant from the Royal Society, and in part by the Cavendish Laboratory. I wish to thank Professor J. J. **THOMSON**, F.R.S., for his kind encouragement, and my thanks are also due to Mr. **GRIFFITHS**, F.R.S., through whom I was led to undertake the investigation.

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